

Chapter 8: Application of Integrals

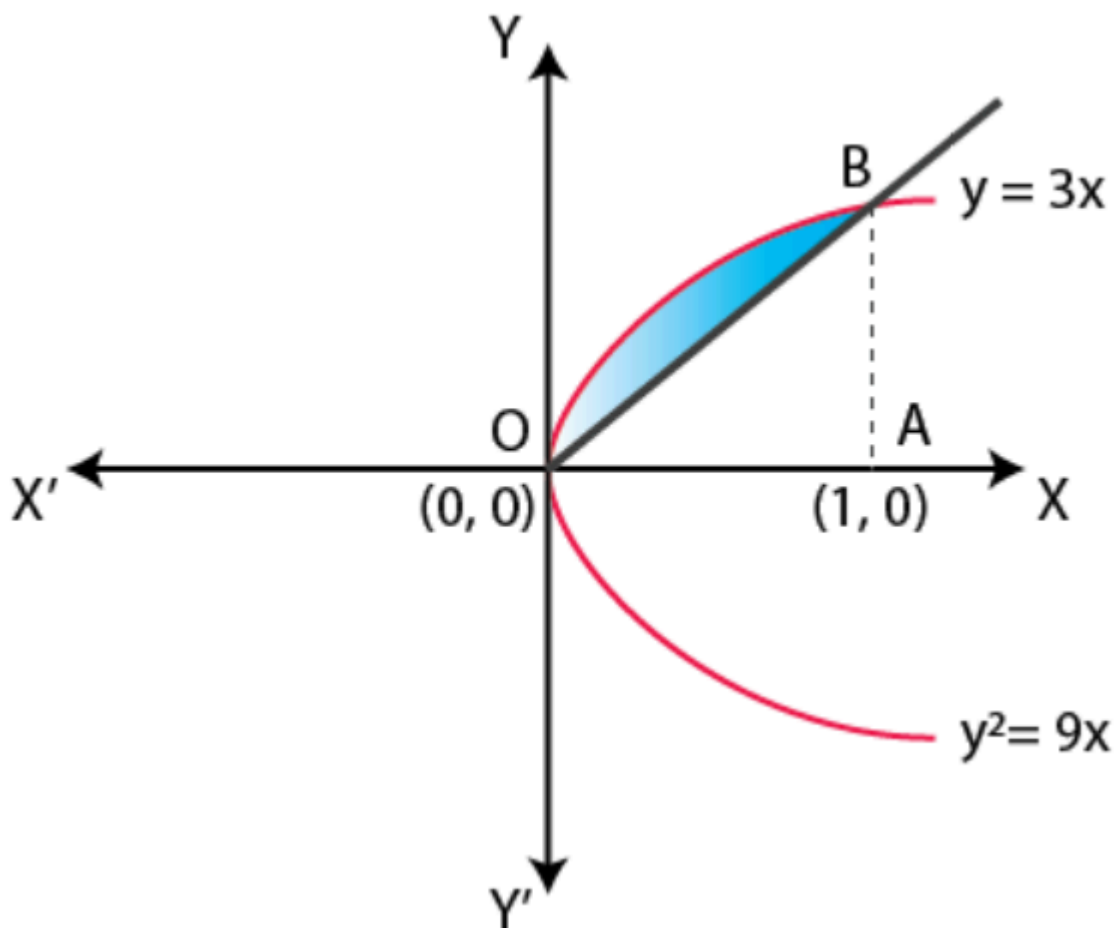
NCERT Exemplar · Class 12 · Maths

Questions

Question 1

Find the area of the region bounded by the curves $y^2 = 9x$, $y = 3x$.

Solution



Given curves are $y^2 = 9x$ and $y = 3x$

Now, solving the two equations we have

$$(3x)^2 = 9x$$

$$9x^2 = 9x$$

$$9x^2 - 9x = 0 \Rightarrow 9x(x - 1) = 0$$

Thus, $x = 0, 1$

So, the area of the shaded region is given by

$$= \text{ar}(\text{region OAB}) - \text{ar}(\Delta \text{OAB})$$

$$= \text{ar}(\text{region OAB}) - \text{ar}(\Delta \text{OAB})$$

$$= - \int_0^1 y_1 \cdot dx = \int_0^1 \sqrt{9x} \, dx - \int_0^1 3x \, dx$$

$$= 3 \int_0^1 \sqrt{x} \, dx - 3 \int_0^1 x \, dx = 3 \times \frac{2}{3} [x^{3/2}]_0^1 - 3 \left[\frac{x^2}{2} \right]_0^1$$

$$= 2 \left[(1)^{3/2} - 0 \right] - \frac{3}{2} [(1)^2 - 0] = 2(1) - \frac{3}{2}(1) = 2 - \frac{3}{2} = \frac{1}{2} \text{ sq. units}$$

Therefore, the required area is $\frac{1}{2}$ sq.units.

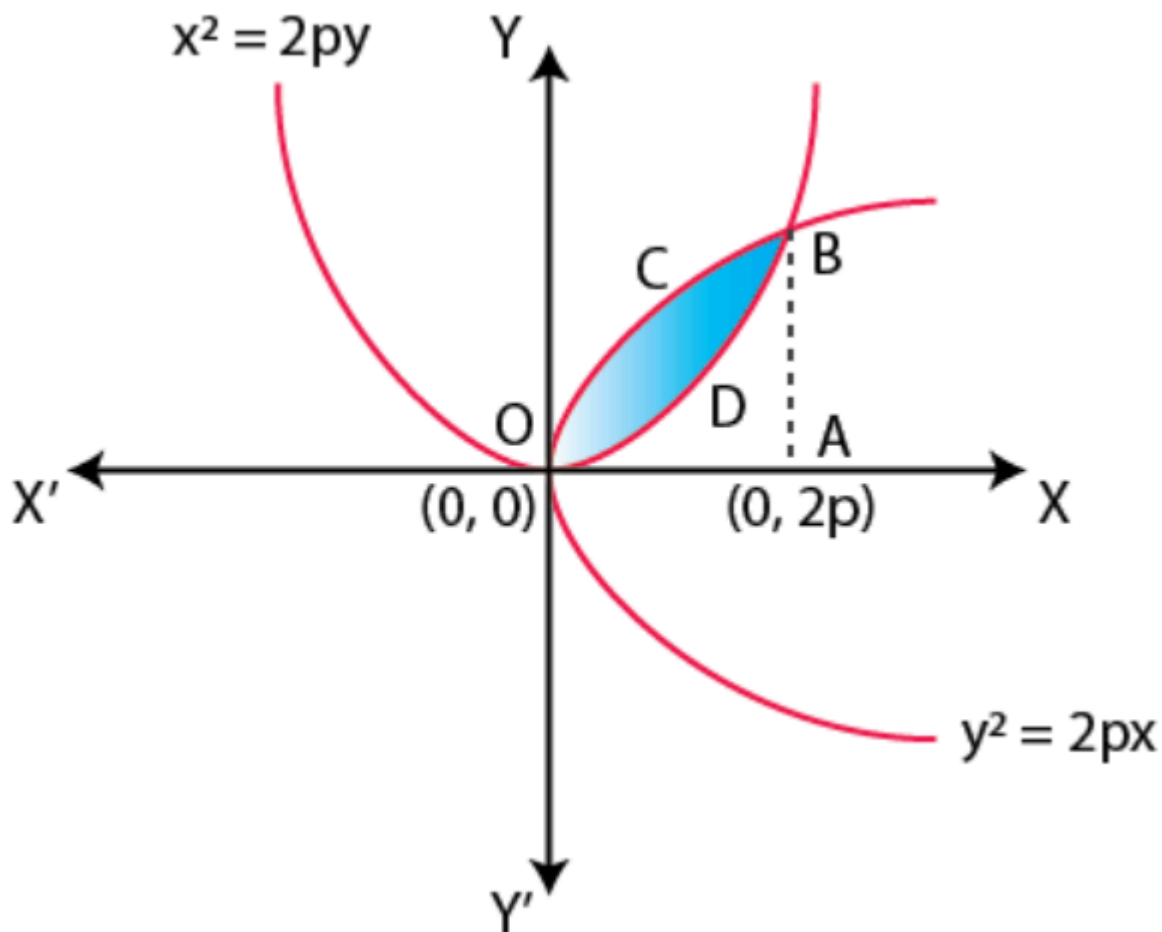
Question 2

Find the area of the region bounded by the parabola $y^2 = 2px$, $x^2 = 2py$.

Solution

Given parabolas are $y^2 = 2px$ (i) and $x^2 = 2py$ (ii)

Now, from equation (ii) we have



$$y = x^2/2p$$

Putting the value of y in equation (i), we have

$$(x^2/2p)^2 = 2px$$

$$x^4/4p^2 = 2px$$

$$x^4 = 8p^3x$$

$$x^4 - 8p^3x = 0$$

$$x(x^3 - 8p^3) = 0$$

$$\text{So, } x = 0 \text{ or } x^3 - 8p^3 = 0 \Rightarrow x = 2p$$

Now, the required area is

$$\begin{aligned}
&= \text{Area of the region (OCBA - ODBA)} \\
&= \int_0^{2p} \sqrt{2px} \, dx - \int_0^{2p} \frac{x^2}{2p} \, dx = \sqrt{2p} \int_0^{2p} \sqrt{x} \, dx - \frac{1}{2p} \int_0^{2p} x^2 \, dx \\
&= \sqrt{2p} \cdot \frac{2}{3} [x^{3/2}]_0^{2p} - \frac{1}{2p} \cdot \frac{1}{3} [x^3]_0^{2p} \\
&= \frac{2\sqrt{2}}{3} \sqrt{p} [(2p)^{3/2} - 0] - \frac{1}{6p} [(2p)^3 - 0] \\
&= \frac{2\sqrt{2}}{3} \sqrt{p} \cdot 2\sqrt{2} p^{3/2} - \frac{1}{6p} \cdot 8p^3 \\
&= \frac{8}{3} \cdot p^2 - \frac{8}{6} p^2 = \frac{8}{6} p^2 = \frac{4}{3} p^2 \text{ sq. units}
\end{aligned}$$

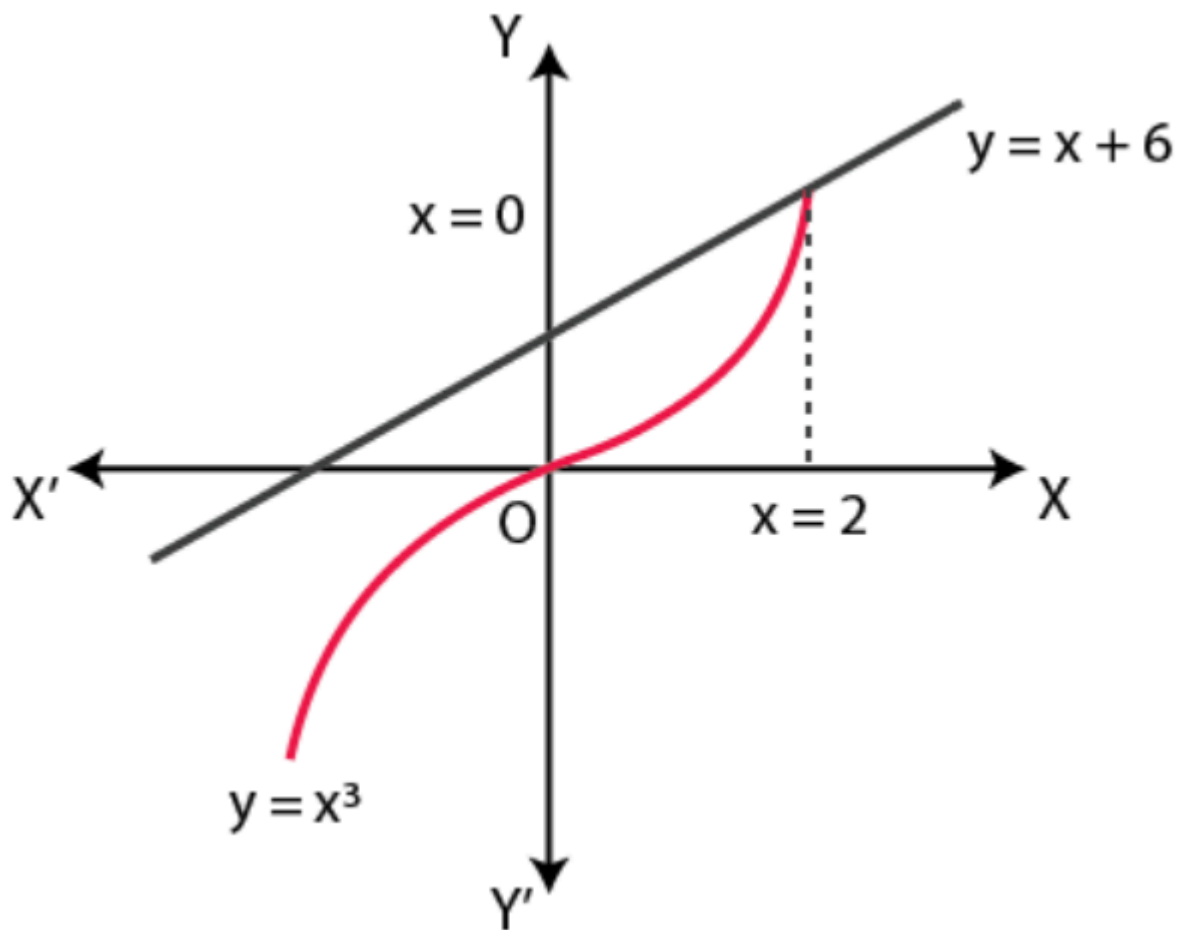
$$\begin{aligned}
&= \text{Area of the region (OCBA - ODBA)} \\
&= \int_0^{2p} \sqrt{2px} \, dx - \int_0^{2p} \frac{x^2}{2p} \, dx = \sqrt{2p} \int_0^{2p} \sqrt{x} \, dx - \frac{1}{2p} \int_0^{2p} x^2 \, dx \\
&= \sqrt{2p} \cdot \frac{2}{3} [x^{3/2}]_0^{2p} - \frac{1}{2p} \cdot \frac{1}{3} [x^3]_0^{2p} \\
&= \frac{2\sqrt{2}}{3} \sqrt{p} [(2p)^{3/2} - 0] - \frac{1}{6p} [(2p)^3 - 0] \\
&= \frac{2\sqrt{2}}{3} \sqrt{p} \cdot 2\sqrt{2} p^{3/2} - \frac{1}{6p} \cdot 8p^3 \\
&= \frac{8}{3} \cdot p^2 - \frac{8}{6} p^2 = \frac{8}{6} p^2 = \frac{4}{3} p^2 \text{ sq. units}
\end{aligned}$$

Therefore, the required area is $\frac{4}{3} p^2$ sq. units.

Question 3

Find the area of the region bounded by the curve $y = x^3$ and $y = x + 6$ and $x = 0$.

Solution



Given curves

are $y = x^3$, $y = x + 6$ and $x = 0$

On solving $y = x^3$ and $y = x + 6$, we have

$$x^3 = x + 6$$

$$x^3 - x - 6 = 0$$

$$x^2(x - 2) + 2x(x - 2) + 3(x - 2) = 0$$

$$(x - 2)(x^2 + 2x + 3) = 0$$

It's seen that $x^2 + 2x + 3 = 0$ has no real roots

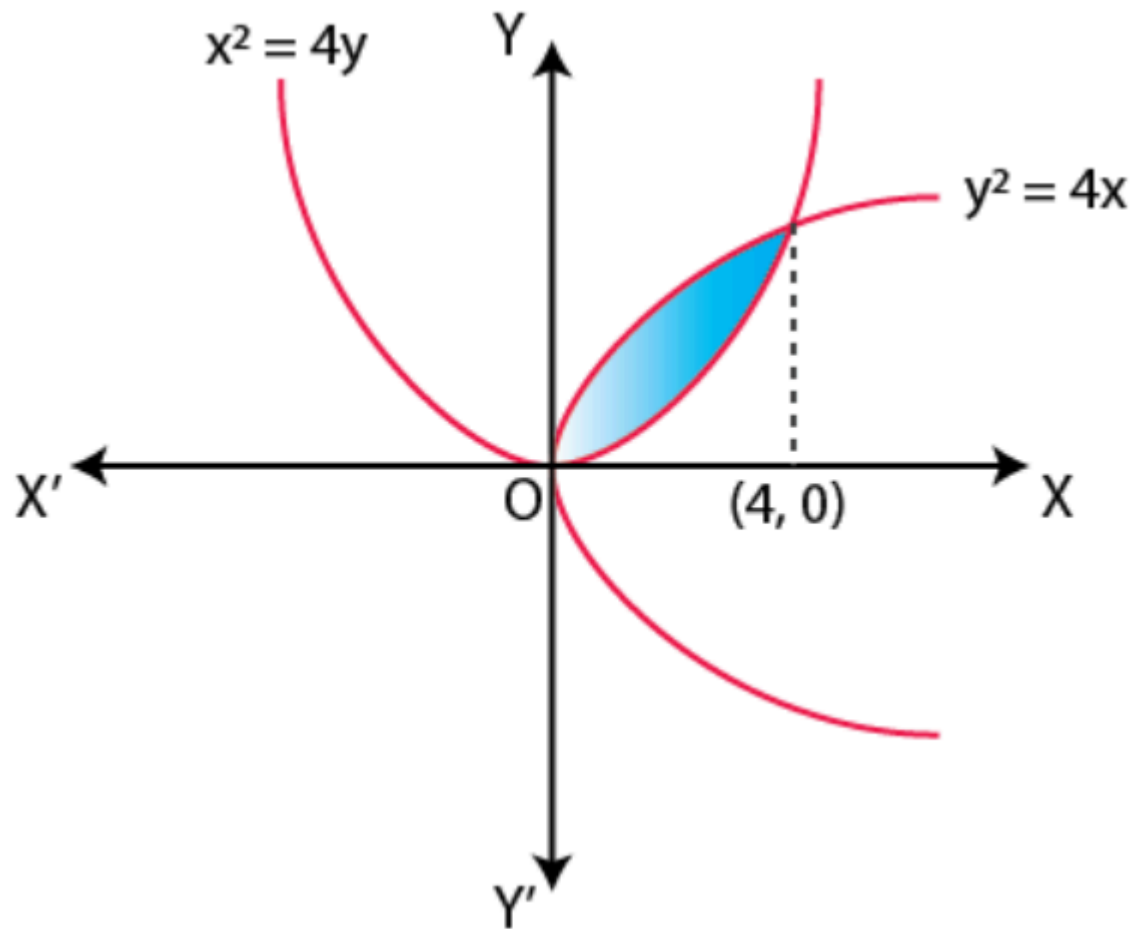
So, $x = 2$ is the only root for the above equation.

Now, the required area of the shaded region is given by

$$\begin{aligned} &= \int_0^2 (x+6) dx - \int_0^2 x^3 dx \\ &= \left[\frac{x^2}{2} + 6x \right]_0^2 - \frac{1}{4} [x^4]_0^2 \\ &= \left(\frac{4}{2} + 12 \right) - (0+0) - \frac{1}{4} [(2)^4 - 0] \\ &= 14 - \frac{1}{4} \times 16 = 14 - 4 = 10 \text{ sq. units.} \end{aligned}$$

Question 4

Find the area of the region bounded by the curve $y^2 = 4x$ and $x^2 = 4y$.

Solution

Given curves are

$$y^2 = 4x \dots (i) \text{ and } x^2 = 4y \dots (ii)$$

On solving the equations, we get

From (ii),

$$y = x^2/4$$

Putting value of y in (i), we have

$$(x^2/4)^2 = 4x$$

$$x^4/16 = 4x$$

$$x^4 = 64x$$

$$x^4 - 64x = 0$$

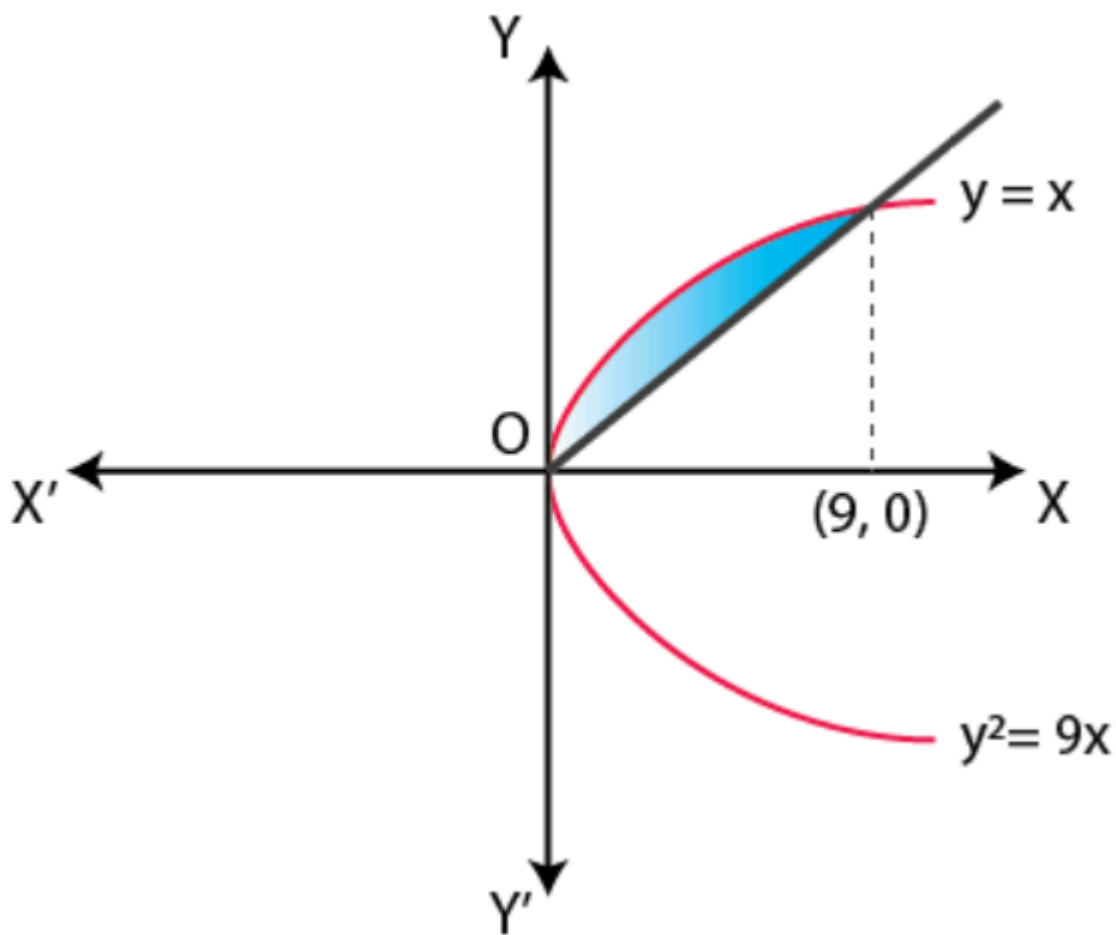
$$\text{So, } x = 0, 4$$

Now, the required area is the shaded region

$$\begin{aligned}\text{Required area} &= \int_0^4 \sqrt{4x} \, dx - \int_0^4 \frac{x^2}{4} \, dx = 2 \int_0^4 \sqrt{x} \, dx - \frac{1}{4} \int_0^4 x^2 \, dx \\&= 2 \cdot \frac{2}{3} [x^{3/2}]_0^4 - \frac{1}{4} \cdot \frac{1}{3} [x^3]_0^4 \\&= \frac{4}{3} [(4)^{3/2} - 0] - \frac{1}{12} [(4)^3 - 0] = \frac{4}{3} [8] - \frac{1}{12} [64] \\&= \frac{32}{3} - \frac{16}{3} = \frac{16}{3} \text{ sq. units}\end{aligned}$$

Question 5

Find the area of the region included between $y^2 = 9x$ and $y = x$

Solution

Given curves

are $y^2 = 9x$ and $y = x$

Solving the above equations, we have

$$x^2 = 9x \Rightarrow x^2 - 9x = 0$$

$$x(x - 9) = 0$$

$$\text{So, } x = 0, 9$$

Now, the required area is

$$\begin{aligned} &= \int_0^9 \sqrt{9x} \, dx - \int_0^9 x \, dx = 3 \int_0^9 \sqrt{x} \, dx - \int_0^9 x \, dx \\ &= 3 \cdot \frac{2}{3} [x^{3/2}]_0^9 - \frac{1}{2} [x^2]_0^9 \\ &= 2[(9)^{3/2} - 0] - \frac{1}{2}[(9)^2 - 0] \\ &= 2(27) - \frac{1}{2}(81) = 54 - \frac{81}{2} = \frac{108 - 81}{2} \\ &= \frac{27}{2} \text{ sq. units} \end{aligned}$$

Therefore, the required area is $27/2$ sq. units.

Question 6

Find the area of the region enclosed by the parabola $x^2 = y$ and the line $y = x + 2$

Solution

Given, equation of parabola $x^2 = y$ and line $y = x + 2$

Solving the above equations, we get

$$x^2 = x + 2$$

$$x^2 - x - 2 = 0$$

$$x^2 - 2x + x - 2 = 0$$

$$x(x - 2) + 1(x - 2) = 0$$

$$(x + 1)(x - 2) = 0$$

So, $x = -1, 2$

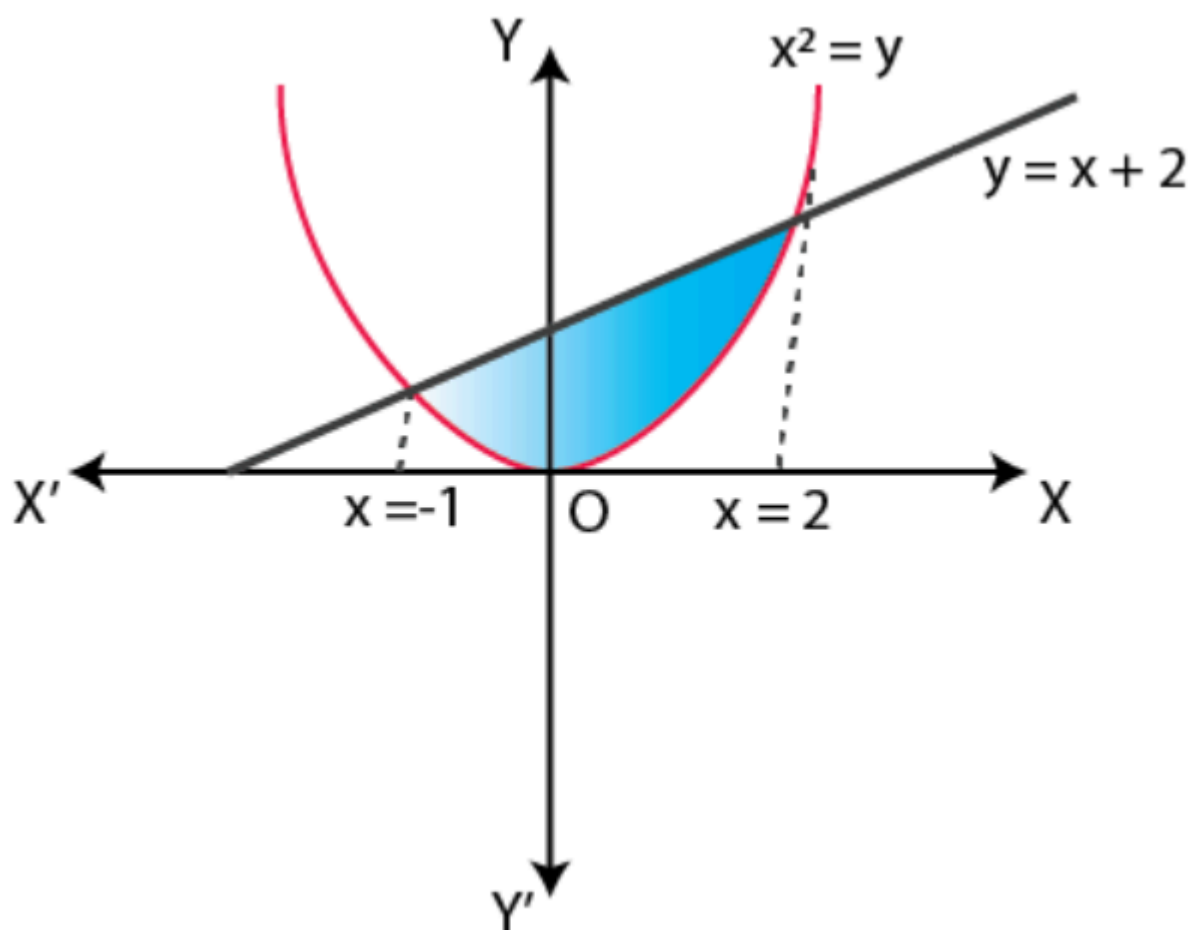
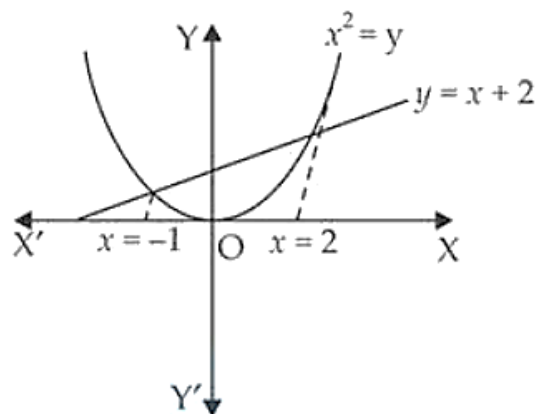
Now,

Graph of $y = x + 2$

x	0	-2
y	2	0

Area of the required region

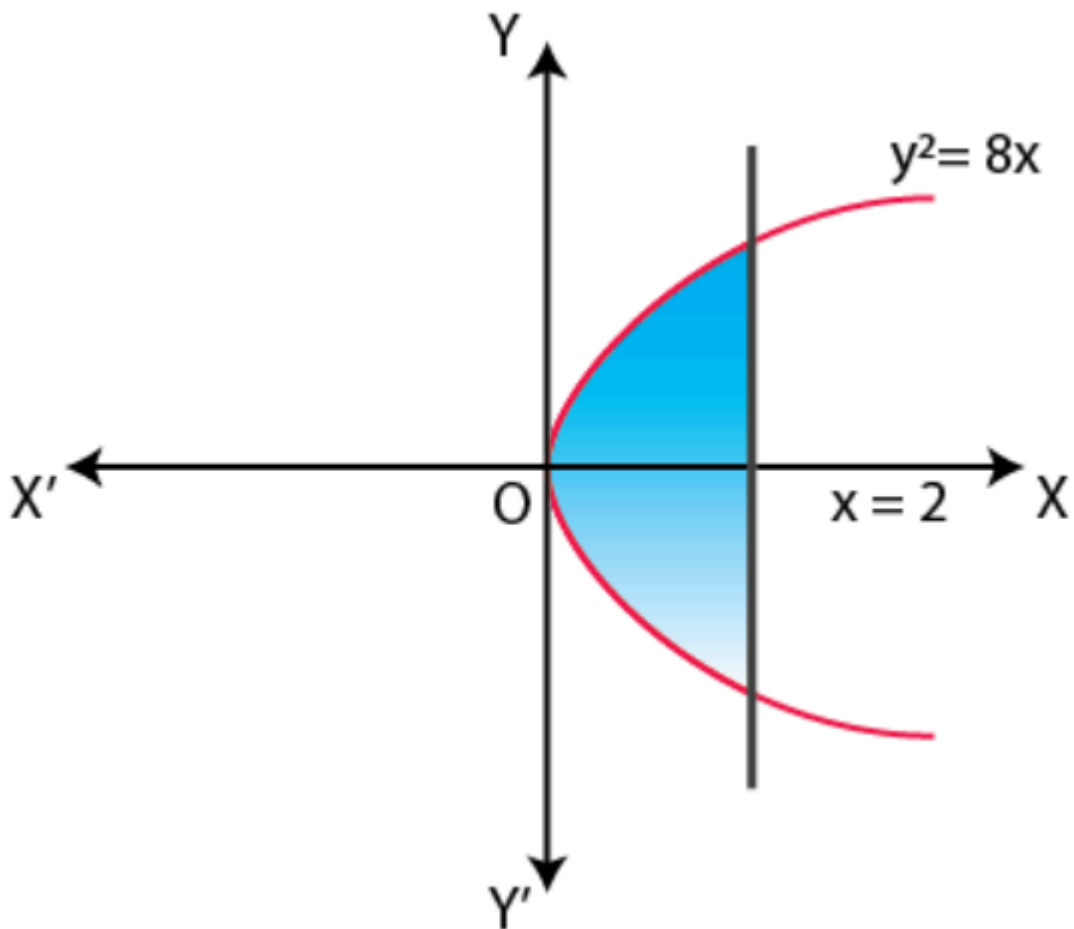
$$\begin{aligned}
 &= \int_{-1}^2 (x+2) dx - \int_{-1}^2 x^2 dx = \left[\frac{x^2}{2} + 2x \right]_{-1}^2 - \frac{1}{3} [x^3]_{-1}^2 \\
 &= \left[\left(\frac{4}{2} + 4 \right) - \left(\frac{1}{2} - 2 \right) \right] - \frac{1}{3} [8 - (-1)] \\
 &= \left(6 + \frac{3}{2} \right) - \frac{1}{3} (9) = \frac{15}{2} - 3 = \frac{9}{2} \text{ sq. units}
 \end{aligned}$$



Therefore, the area of the required region is $9/2$ sq. units.

Question 7

Find the area of region bounded by the line $x = 2$ and the parabola $y^2 = 8x$

Solution

Given, equation of line $x = 2$ and parabola $y^2 = 8x$

Putting value of x in the other equation, we have

$$y^2 = 8(2)$$

$$y^2 = 16$$

$$\text{So, } y = \pm 4$$

Now, the required area is

$$\begin{aligned} &= 2 \int_0^2 \sqrt{8x} \, dx = 2 \times 2\sqrt{2} \int_0^2 \sqrt{x} \, dx \\ &= 4\sqrt{2} \times \frac{2}{3} [x^{3/2}]_0^2 \\ &= \frac{8\sqrt{2}}{3} [(2)^{3/2}] = \frac{8\sqrt{2}}{3} \times 2\sqrt{2} = \frac{32}{3} \text{ sq. units} \end{aligned}$$

Therefore, the area of the region = $32/3$ sq. units

Question 8

Sketch the region $\{(x, 0) : y = \sqrt{4 - x^2}\}$ and x-axis. Find the area of the region using integration.

Solution

Given, $\{(x, 0) : y = \sqrt{4 - x^2}\}$

$$\text{So, } y^2 = 4 - x^2$$

$x^2 + y^2 = 4$ which is a circle.

Now, the required area

$$= 2 \cdot \int_0^2 \sqrt{4 - x^2} \, dx$$

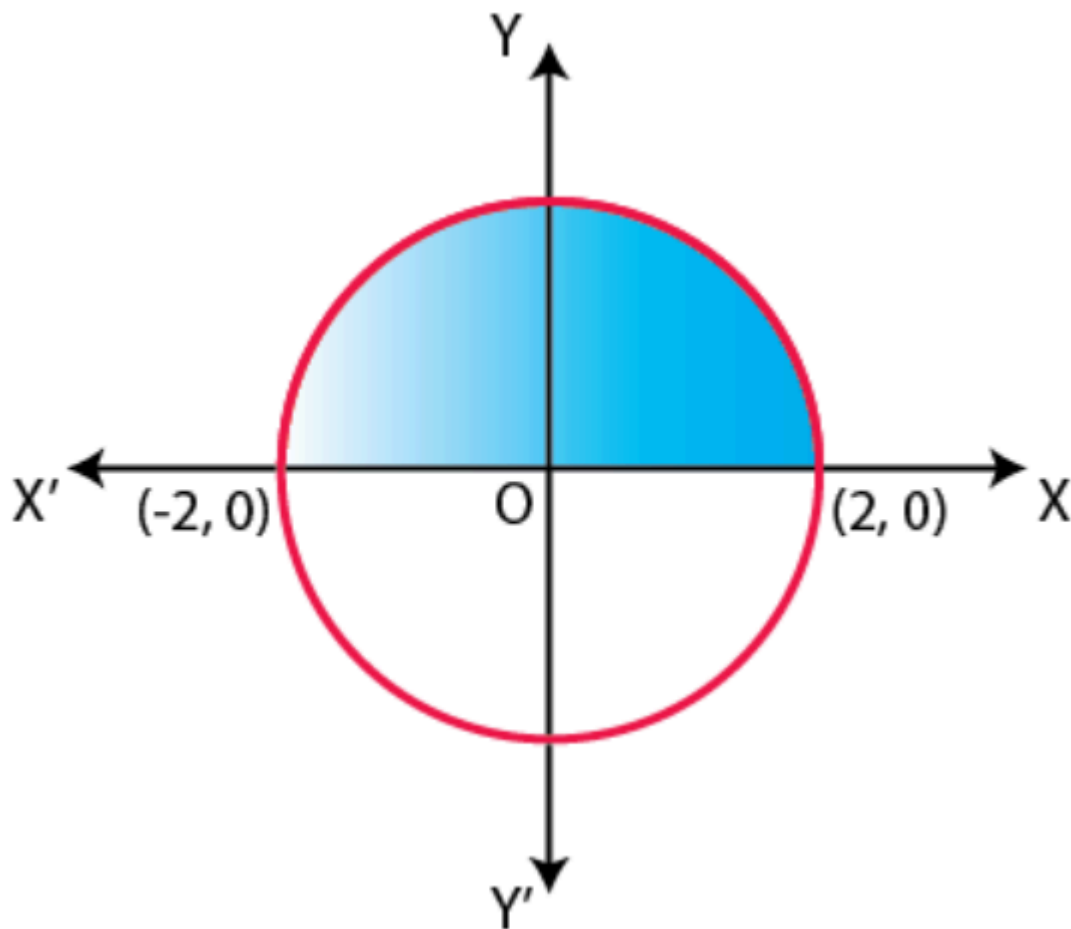
[Since circle is symmetrical about y -axis]

$$= 2 \cdot \int_0^2 \sqrt{(2)^2 - x^2} \, dx$$

$$= 2 \cdot \left[\frac{x}{2} \sqrt{4 - x^2} + \frac{4}{2} \sin^{-1} \frac{x}{2} \right]_0^2$$

$$= 2 \left[\left(\frac{2}{2} \sqrt{4 - 4} + 2 \sin^{-1}(1) \right) - (0 + 0) \right]$$

$$= 2 \left[2 \cdot \frac{\pi}{2} \right] = 2\pi \text{ sq. units}$$



Question 9

Calculate the area under the curve $y = 2\sqrt{x}$ included between the lines $x = 0$ and $x = 1$.

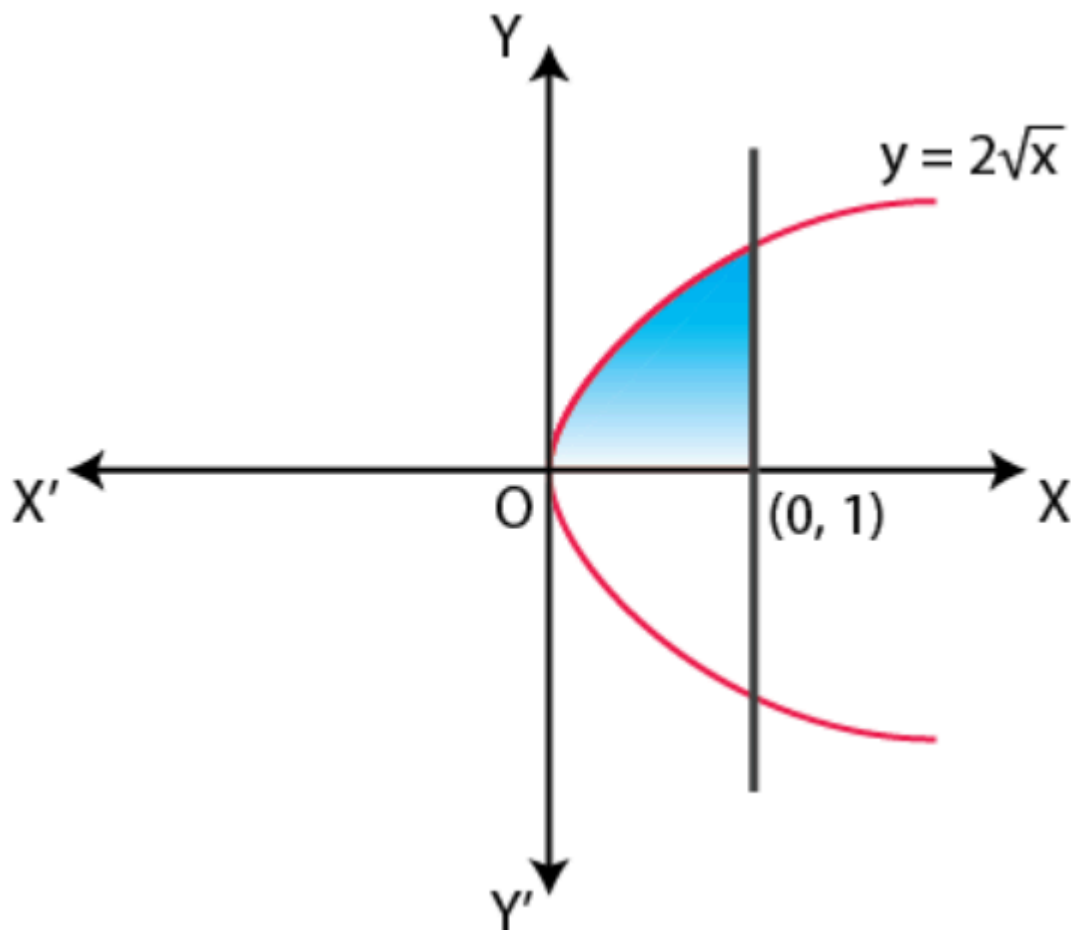
Solution

Given the curves $y = 2\sqrt{x}$, $x = 0$ and $x = 1$.

$$y = 2\sqrt{x} \Rightarrow y^2 = 4x \text{ (Parabola)}$$

$$\begin{aligned}\text{Required area} &= \int_0^1 (2\sqrt{x}) dx \\ &= 2 \times \frac{2}{3} [x^{3/2}]_0^1 \\ &= \frac{4}{3} [(1)^{3/2} - 0] \\ &= \frac{4}{3} \text{ sq. units}\end{aligned}$$

Thus, required area = $\frac{4}{3}$ sq. units.



Question 10

Using integration, find the area of the region bounded by the line $2y = 5x + 7$, x -axis and the lines $x = 2$ and $x = 8$.

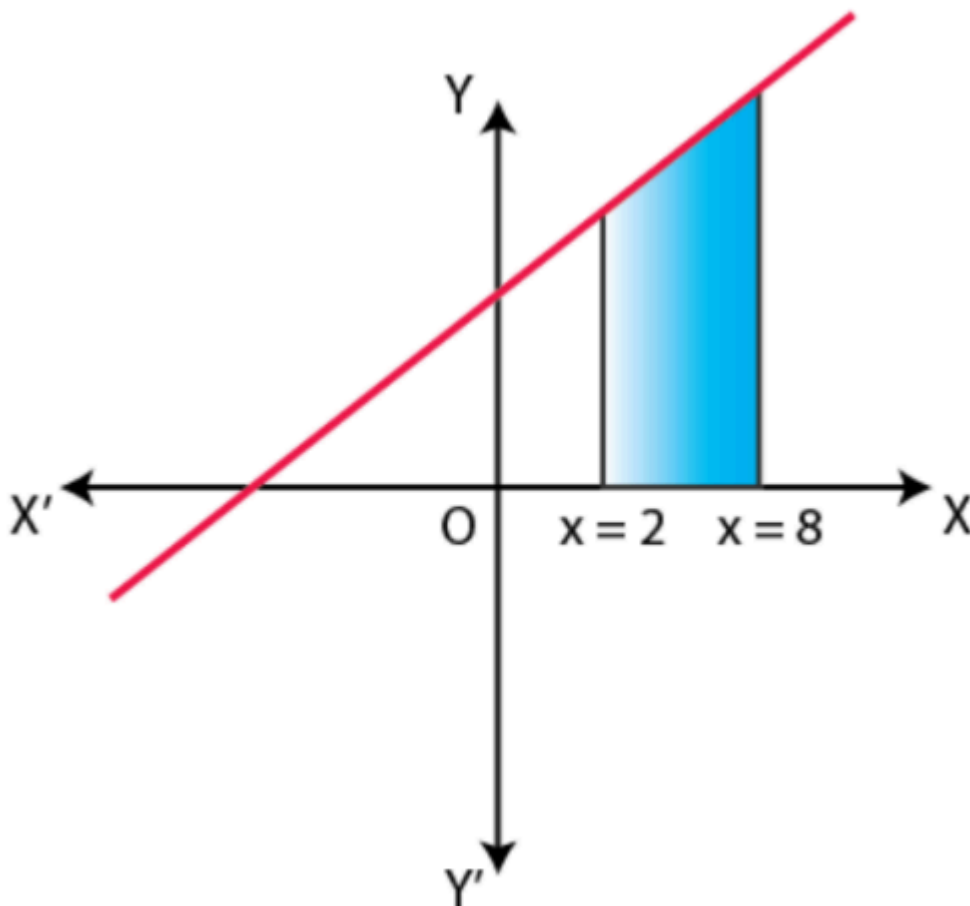
Solution

Given, $2y = 5x + 7$, x -axis, $x = 2$ and $x = 8$

Let's draw the graph of $2y = 5x + 7 \Rightarrow y = (5x + 7)/2$

x	1	-1
y	6	1

Now, let's plot the straight line on a graph with other lines.



region is

The area of the required

$$\begin{aligned} &= \int_2^8 \left(\frac{5x+7}{2} \right) dx = \frac{1}{2} \left[\frac{5}{2} x^2 + 7x \right]_2^8 \\ &= \frac{1}{2} \left[\frac{5}{2} (64 - 4) + 7 (8 - 2) \right] \\ &= \frac{1}{2} \left[\frac{5}{2} \times 60 + 7 \times 6 \right] = \frac{1}{2} [150 + 42] \\ &= \frac{1}{2} \times 192 = 96 \text{ sq. units} \end{aligned}$$

Therefore, the required area the region = 96 sq. units

Question 11

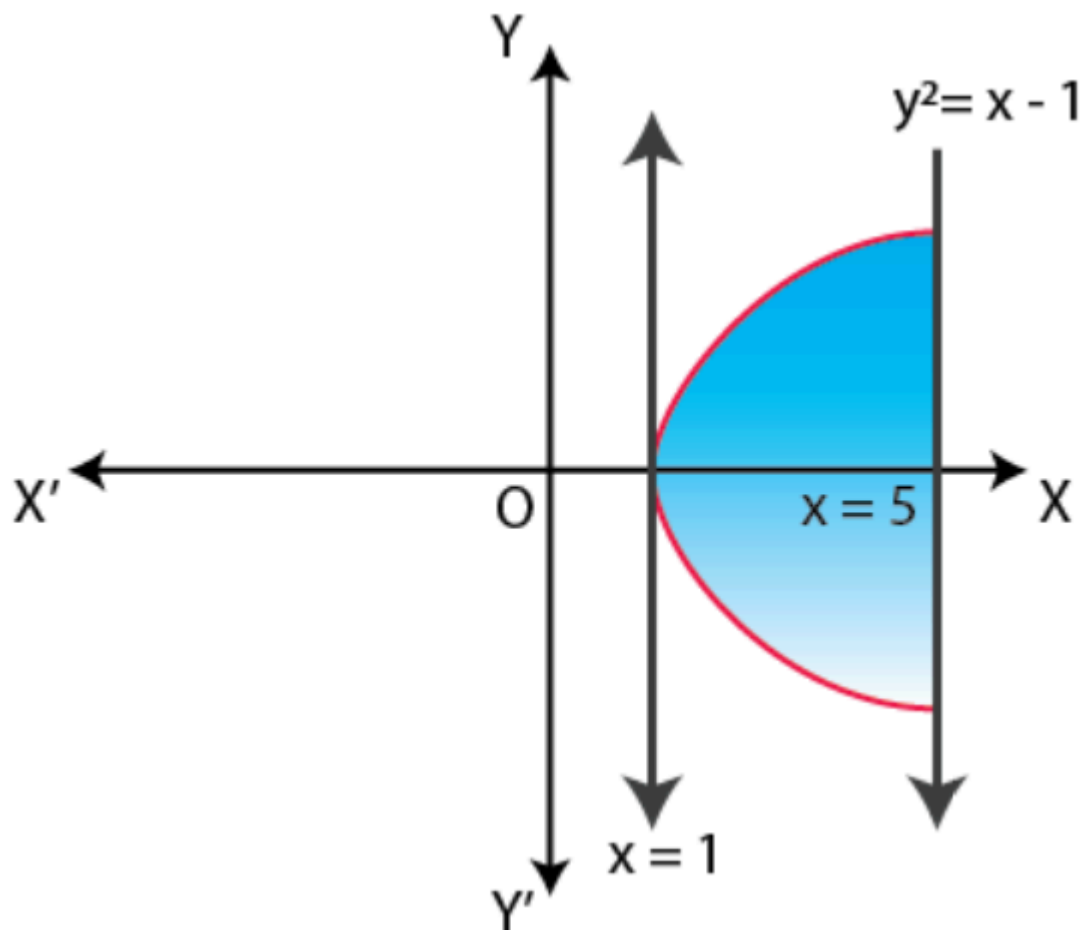
Draw a rough sketch of the curve $y = \sqrt{x-1}$ in the interval $[1, 5]$. Find the area under the curve and between the lines $x = 1$ and $x = 5$.

Solution

Given curve $y = \sqrt{x-1}$

$$\Rightarrow y^2 = x - 1$$

Plotting the curve and finding the area of the shaded region between the lines $x = 1$ and $x = 5$, we have



Area of the required region

$$\begin{aligned}
 &= \int_1^5 \sqrt{x-1} \, dx \\
 &= \frac{2}{3} [(x-1)^{3/2}]_1^5 \\
 &= \frac{2}{3} [(5-1)^{3/2} - 0] = \frac{2}{3} \times (4)^{3/2} \\
 &= \frac{2}{3} \times 8 = \frac{16}{3} \text{ sq. units}
 \end{aligned}$$

Therefore, the area of the required region = $\frac{16}{3}$ sq.units

Question 12

Determine the area under the curve $y = \sqrt{a^2 - x^2}$ included between the lines $x = 0$ and $x = a$.

Solution

Given curve $y = \sqrt{a^2 - x^2}$ and lines $x = 0$ and $x = a$

$$y = \sqrt{a^2 - x^2} \Rightarrow y^2 = a^2 - x^2$$

$x^2 + y^2 = a^2$ which is equation of a circle.

Now, the required region is found by plotting the curve and lines.

So, the area of the shaded region is

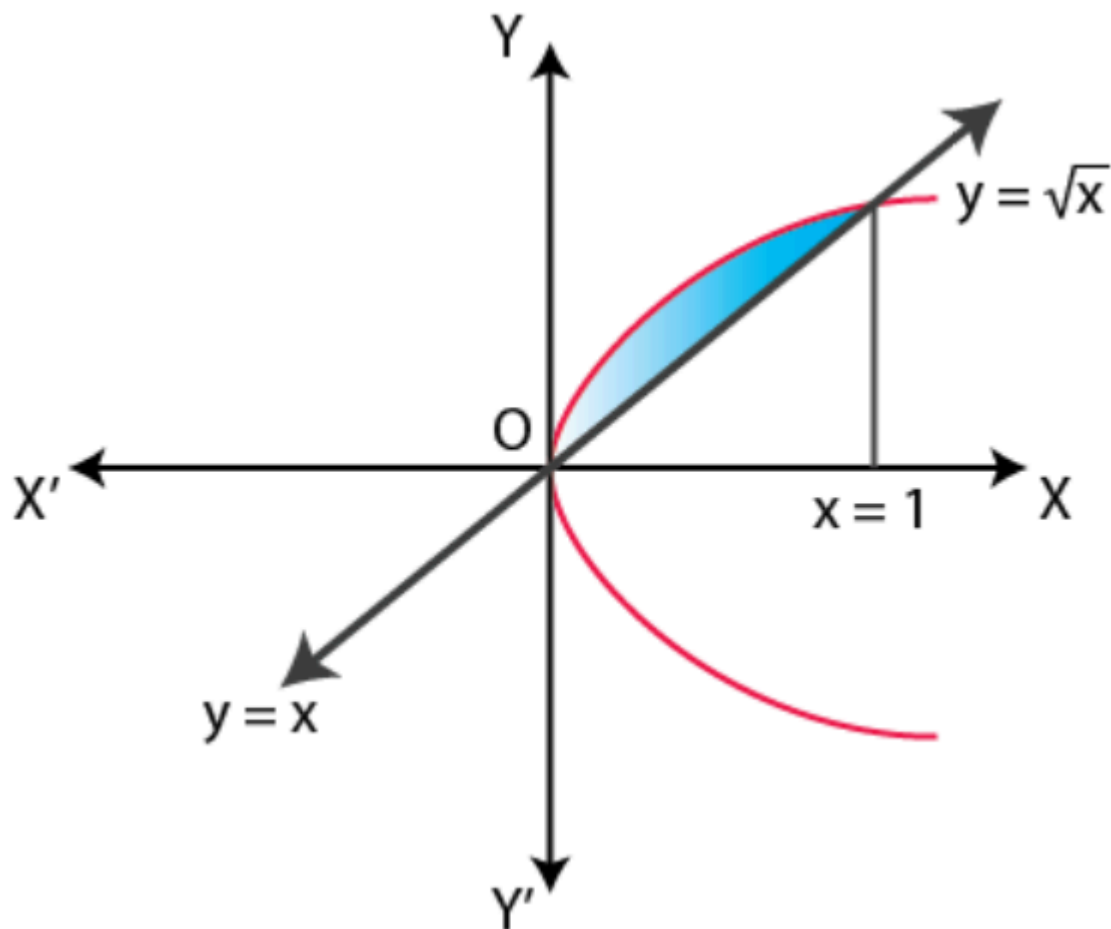
$$\begin{aligned} &= 2 \left[(1)^{3/2} - 0 \right] - \frac{3}{2} [(1)^2 - 0] \\ &= \left[\frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \right]_0^a \\ &= \left[\frac{a}{2} \sqrt{a^2 - a^2} + \frac{a^2}{2} \sin^{-1} \frac{a}{a} - 0 - 0 \right] \\ &= \frac{a^2}{2} \sin^{-1}(1) = \frac{a^2}{2} \cdot \frac{\pi}{2} = \frac{\pi a^2}{4} \end{aligned}$$

$$\begin{aligned} &= 2 \left[(1)^{3/2} - 0 \right] - \frac{3}{2} [(1)^2 - 0] \\ &= \left[\frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \right]_0^a \\ &= \left[\frac{a}{2} \sqrt{a^2 - a^2} + \frac{a^2}{2} \sin^{-1} \frac{a}{a} - 0 - 0 \right] \\ &= \frac{a^2}{2} \sin^{-1}(1) = \frac{a^2}{2} \cdot \frac{\pi}{2} = \frac{\pi a^2}{4} \end{aligned}$$

Therefore, the required area = $\pi a^2/4$

Question 13

Find the area of the region bounded by $y = \sqrt{x}$ and $y = x$.

Solution

Given equations

of curve $y = \sqrt{x}$ and line $y = x$

Solving the equations $y = \sqrt{x} \Rightarrow y^2 = x$ and $y = x$, we get

$$x^2 = x$$

$$x^2 - x = 0$$

$$x(x - 1) = 0$$

$$\text{So, } x = 0, 1$$

Now, the required area of the shaded region

$$\begin{aligned} &= \int_0^1 \sqrt{x} \, dx - \int_0^1 x \, dx \\ &= \frac{2}{3} [x^{3/2}]_0^1 - \frac{1}{2} [x^2]_0^1 \\ &= \frac{2}{3} [(1)^{3/2} - 0] - \frac{1}{2} [(1)^2 - 0] \\ &= \frac{2}{3} - \frac{1}{2} \Rightarrow \frac{4-3}{6} \Rightarrow \frac{1}{6} \text{ sq. units} \end{aligned}$$

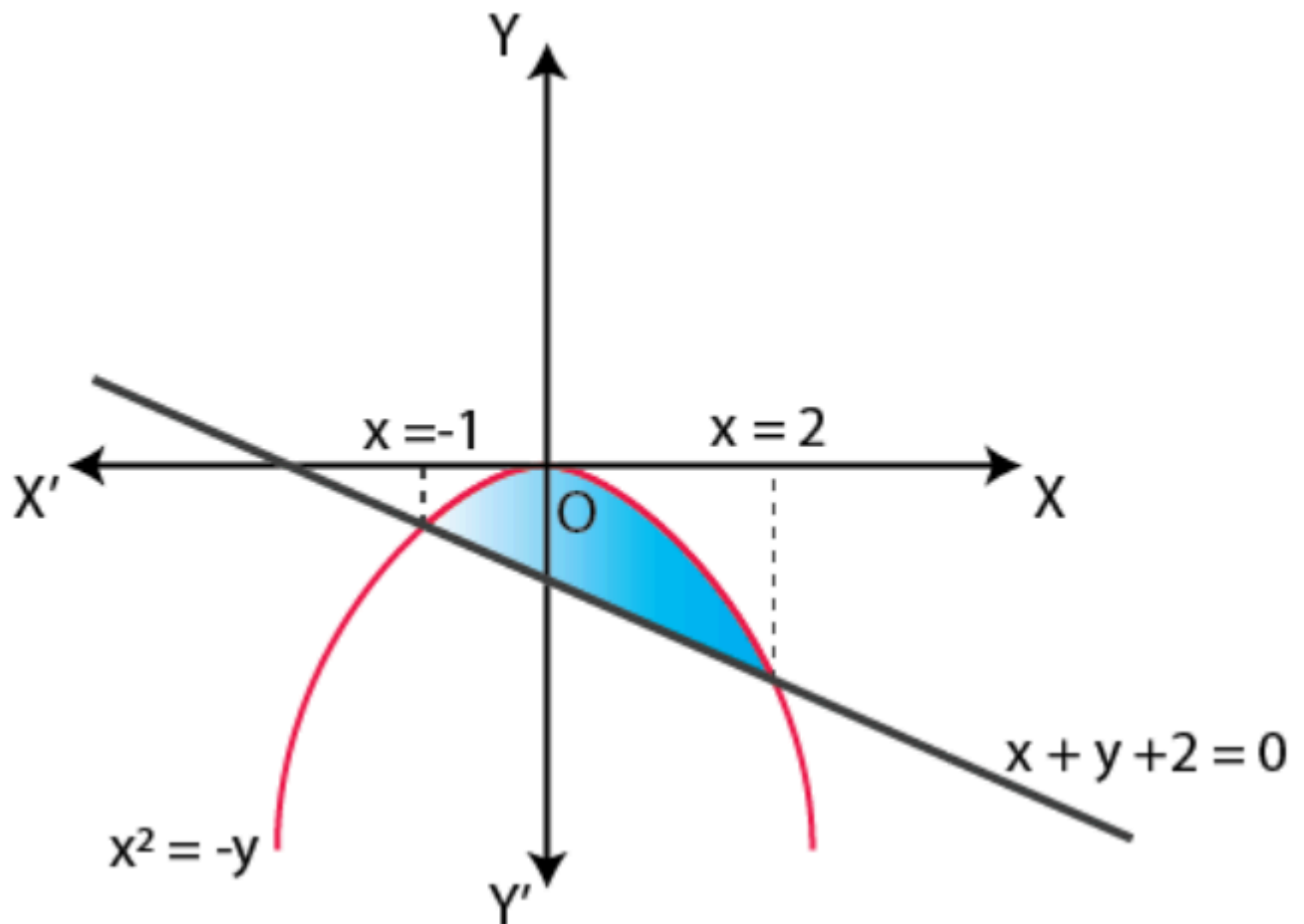
Therefore, the required area = $\frac{1}{6}$ sq.units.

Question 14

Find the area enclosed by the curve $y = -x^2$ and the straight-line $x + y + 2 = 0$.

Solution

Given curve $y = -x^2$ or $x^2 = -y$ and the line $x + y + 2 = 0$



Solving the two equation, we get

$$x - x^2 + 2 = 0$$

$$x^2 - x - 2 = 0$$

$$x^2 - 2x + x - 2 = 0$$

$$x(x - 2) + 1(x - 2) = 0$$

$$(x - 2)(x + 1) = 0$$

$$\text{So, } x = -1, 2$$

Now,

The area of the required shaded region

$$\begin{aligned}
&= \left| \int_{-1}^2 (-x-2) dx - \int_{-1}^2 -x^2 dx \right| \\
&= \left| -\left[\frac{x^2}{2} + 2x\right]_{-1}^2 + \frac{1}{3}[x^3]_{-1}^2 \right| \\
&= \left| -\left[\left(\frac{4}{2} + 4\right) - \left(\frac{1}{2} - 2\right)\right] + \frac{1}{3}(8+1) \right| \\
&= \left| -\left(6 + \frac{3}{2}\right) + \frac{1}{3}(9) \right| \Rightarrow \left| -\frac{15}{2} + 3 \right| \\
&= \left| \frac{-15+6}{2} \right| = \left| \frac{-9}{2} \right| = \frac{9}{2} \text{ sq. units}
\end{aligned}$$

$$\begin{aligned}
&= \left| \int_{-1}^2 (-x-2) dx - \int_{-1}^2 -x^2 dx \right| \\
&= \left| -\left[\frac{x^2}{2} + 2x\right]_{-1}^2 + \frac{1}{3}[x^3]_{-1}^2 \right| \\
&= \left| -\left[\left(\frac{4}{2} + 4\right) - \left(\frac{1}{2} - 2\right)\right] + \frac{1}{3}(8+1) \right| \\
&= \left| -\left(6 + \frac{3}{2}\right) + \frac{1}{3}(9) \right| \Rightarrow \left| -\frac{15}{2} + 3 \right| \\
&= \left| \frac{-15+6}{2} \right| = \left| \frac{-9}{2} \right| = \frac{9}{2} \text{ sq. units}
\end{aligned}$$

Therefore, the required area = $\frac{9}{2}$ sq.units

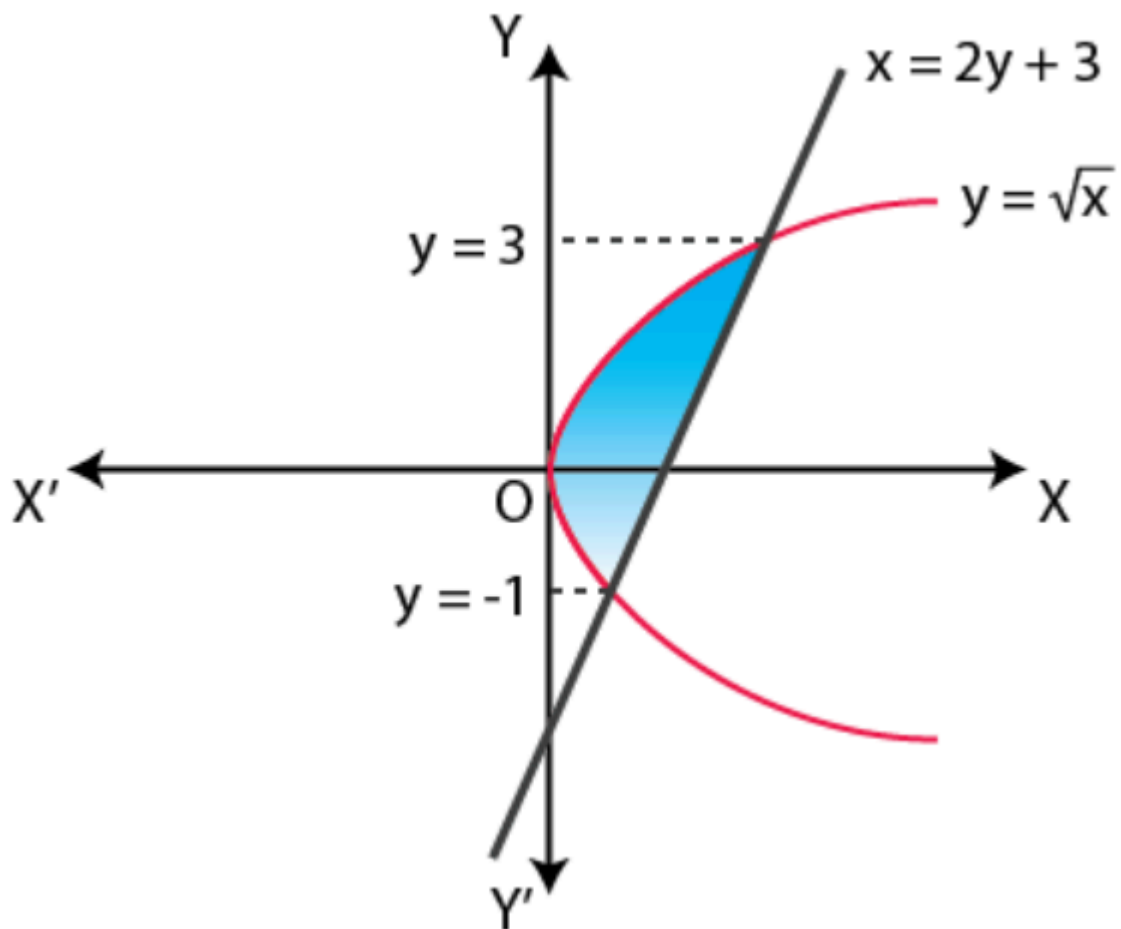
Question 15

Find the area bounded by the curve $y = \sqrt{x}$, $x = 2y + 3$ in the first quadrant and x-axis.

Solution

Given curve $y = \sqrt{x}$ and line $x = 2y + 3$, first quadrant and x-axis.

Solving $y = \sqrt{x}$ and $x = 2y + 3$, we get



$$\begin{aligned}
 y &= \sqrt{2y+3} \Rightarrow y^2 = 2y+3 \\
 y^2 - 2y - 3 &= 0 \Rightarrow y^2 - 3y + y - 3 = 0 \\
 y(y-3) + 1(y-3) &= 0 \\
 (y+1)(y-3) &= 0 \\
 \therefore y &= -1, 3
 \end{aligned}$$

The area of shaded region

$$\begin{aligned}
 &= \int_0^3 (2y+3) dy - \int_0^3 y^2 dy \\
 &= \left[2\frac{y^2}{2} + 3y \right]_0^3 - \frac{1}{3} [y^3]_0^3 \\
 &= [(9+9) - (0+0)] - \frac{1}{3} [27-0] \\
 &= 18 - 9 = 9 \text{ sq. units}
 \end{aligned}$$

Thus, the required area = 9 sq. units.

Long Answer (L.A.)

Question 16

Find the area of the region bounded by the curve $y^2 = 2x$ and $x^2 + y^2 = 4x$.

Solution

Given equation of curves are $y^2 = 2x$ and $x^2 + y^2 = 4x$

Solving the equations, we have

$$x^2 - 4x + y^2 = 0$$

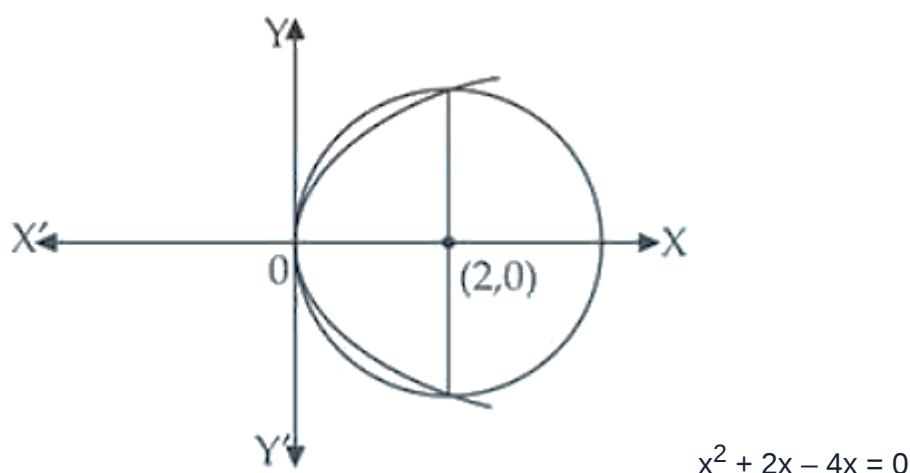
$$x^2 - 4x + 4 - 4 + y^2 = 0$$

$$(x-2)^2 + y^2 = 4$$

It's clearly seen that the equation of the circle having its centre (2, 0) and radius 2.

Solving $x^2 + y^2 = 4x$ and $y^2 = 2x$

$$x^2 + 2x = 4x$$



$$x^2 - 2x = 0$$

$$x(x - 2) = 0$$

$$\text{So, } x = 0, 2$$

Now, the area of the required region is given as

Area of the required region

$$= 2 \left[\int_0^2 \sqrt{4 - (x - 2)^2} dx - \int_0^2 \sqrt{2x} dx \right]$$

[$\therefore$ Parabola and circle both are symmetrical about x-axis.]

$$= 2 \left[\frac{x - 2}{2} \sqrt{4 - (x - 2)^2} + \frac{4}{2} \sin^{-1} \frac{x - 2}{2} \right]_0^2 - 2 \cdot \sqrt{2} \cdot \frac{2}{3} [x^{3/2}]_0^2$$

$$= 2 \left[(0 + 0) - (0 + 2 \sin^{-1}(-1)) \right] - \frac{4\sqrt{2}}{3} [2^{3/2} - 0]$$

$$= -2 \times 2 \cdot \left(-\frac{\pi}{2} \right) - \frac{4\sqrt{2}}{3} \cdot 2\sqrt{2}$$

$$= 2\pi - \frac{16}{3} = 2 \left(\pi - \frac{8}{3} \right) \text{ sq. units}$$

Thus, the required area = $2 \left(\pi - \frac{8}{3} \right)$ sq. units.

Question 17

Find the area bounded by the curve $y = \sin x$ between $x = 0$ and $x = 2\pi$.

Solution

$$\begin{aligned}\text{Required area} &= \int_0^{\pi} \sin x \, dx + \int_{\pi}^{2\pi} |\sin x| \, dx \\ &= -[\cos x]_0^{\pi} + |(-\cos x)|_{\pi}^{2\pi} = -[\cos \pi - \cos 0] + [\cos 2\pi - \cos \pi] \\ &= -[-1 - 1] + [1 + 1] = 2 + 2 = 4 \text{ sq. units}\end{aligned}$$

