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Chapter 7: Integrals

NCERT Exemplar · Class 12 · Maths

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Question 1

$$\int \frac{2x-1}{2x+3} dx = x - \log |(2x+3)^2| + C$$

Solution

$$\begin{aligned}
\text{L.H.S.} &= \int \frac{2x-1}{2x+3} dx \\
&= \int \left(1 - \frac{4}{2x+3} \right) dx && \text{[Dividing the numerator by the denominator]} \\
&= \int 1 \cdot dx - 4 \int \frac{1}{2x+3} dx = \int 1 \cdot dx - \frac{4}{2} \int \frac{1}{x + \frac{3}{2}} dx \\
&= \int 1 \cdot dx - 2 \int \frac{1}{x + \frac{3}{2}} dx = x - 2 \log \left| x + \frac{3}{2} \right| + C \\
&= x - 2 \log \left| \frac{2x+3}{2} \right| + C = x - \log \left| \left(\frac{2x+3}{2} \right)^2 \right| + C \\
&&& [\because n \log m = \log m^n] \\
&= x - \log \left| (2x+3)^2 \right| - \log 2^2 + C \\
&= x - \log \left| (2x+3)^2 \right| + C_1 \Rightarrow \text{R.H.S.} \quad [\text{where } C_1 = C - \log 2^2] \\
&\text{L.H.S.} = \text{R.H.S.}
\end{aligned}$$

- Hence Proved

Question 2

$$\int \frac{2x+3}{x^2+3x} dx = \log |x^2+3x| + C$$

Solution

$$\text{L.H.S.} = \int \frac{2x+3}{x^2+3x} dx$$

Putting, $x^2+3x = t$

So, $(2x+3) dx = dt$

$$\Rightarrow \int \frac{dt}{t} = \log |t| \Rightarrow \log |x^2+3x| + C = \text{R.H.S.}$$

$$\text{L.H.S.} = \text{R.H.S.}$$

- Hence proved.

Evaluate the following:

Question 3

$$\int \frac{(x^2+2) dx}{x+1}$$

Solution

$$\text{Let } I = \int \frac{x^2 + 2}{x + 1} dx$$

$$\begin{aligned} \text{So, } I &= \int \left[(x - 1) + \frac{3}{x + 1} \right] dx \\ &= \int (x - 1) dx + 3 \int \frac{1}{x + 1} dx \\ &= \frac{x^2}{2} - x + 3 \log |x + 1| + C \end{aligned}$$

Therefore,

$$I = \frac{x^2}{2} - x + 3 \log |x + 1| + C$$

Performing long division of the given integral, we have

$$\begin{array}{r} x + 1 \overline{) x^2 + 2} \quad (x - 1) \\ \underline{(-) \quad x^2 + x} \\ -x + 2 \\ \underline{(-) \quad -x - 1} \\ 3 \end{array}$$

Question 4

$$\int \frac{e^{6 \log x} - e^{5 \log x}}{e^{4 \log x} - e^{3 \log x}} dx$$

Solution

$$\begin{aligned}\text{Let } I &= \int \frac{e^{6 \log x} - e^{5 \log x}}{e^{4 \log x} - e^{3 \log x}} dx = \int \frac{e^{\log x^6} - e^{\log x^5}}{e^{\log x^4} - e^{\log x^3}} dx \\ &= \int \frac{x^6 - x^5}{x^4 - x^3} dx && \text{(Using properties of logarithms)} \\ &= \int \frac{x^2(x^4 - x^3)}{x^4 - x^3} dx = \int x^2 dx = \frac{1}{3}x^3 + C\end{aligned}$$

Therefore,

$$I = \int x^2 dx = \frac{1}{3}x^3 + C$$

Question 5

$$\int \frac{(1 + \cos x)}{x + \sin x} dx$$

Solution

$$I = \int \frac{1 + \cos x}{x + \sin x} dx$$

Putting, $x + \sin x = t \Rightarrow (1 + \cos x) dx = dt$

$$\text{So, } I = \int \frac{dt}{t} = \log |t| = \log |x + \sin x| + C$$

Therefore,

$$I = \log |t| = \log |x + \sin x| + C$$

$$I = \int \frac{1 + \cos x}{x + \sin x} dx$$

Putting, $x + \sin x = t \Rightarrow (1 + \cos x) dx = dt$

$$\text{So, } I = \int \frac{dt}{t} = \log |t| = \log |x + \sin x| + C$$

Therefore,

$$I = \log |t| = \log |x + \sin x| + C$$

Question 6

$$\int \frac{dx}{1 + \cos x}$$

Solution

$$\begin{aligned} \text{Let } I &= \int \frac{dx}{1 + \cos x} = \int \frac{dx}{2 \cos^2 \frac{x}{2}} \quad \left[\because 1 + \cos x = 2 \cos^2 \frac{x}{2} \right] \\ &= \frac{1}{2} \int \sec^2 \frac{x}{2} dx = \frac{1}{2} \cdot 2 \tan \frac{x}{2} + C = \tan \frac{x}{2} + C \end{aligned}$$

Therefore,

$$I = \tan \frac{x}{2} + C$$

Question 7

$$\int \tan^2 x \sec^4 x dx$$

Solution

$$\begin{aligned}\text{Let } I &= \int \tan^2 x \cdot \sec^4 x \, dx \\ &= \int \tan^2 x \sec^2 x \cdot \sec^2 x \, dx = \int \tan^2 x (1 + \tan^2 x) \cdot \sec^2 x \, dx\end{aligned}$$

Putting, $\tan x = t, \Rightarrow \sec^2 x \, dx = dt$

$$\begin{aligned}I &= \int t^2(1 + t^2) \, dt = \int (t^2 + t^4) \, dt = \int t^2 \, dt + \int t^4 \, dt \\ &= \frac{1}{3}t^3 + \frac{1}{5}t^5 = \frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x + C\end{aligned}$$

Therefore,

$$I = \frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x + C$$

Question 8

$$\int \frac{\sin x + \cos x}{\sqrt{1 + \sin 2x}} \, dx$$

Solution

$$\begin{aligned}\text{Let } I &= \int \frac{\sin x + \cos x}{\sqrt{1 + 2 \sin x \cos x}} dx \\&= \int \frac{(\sin x + \cos x)}{\sqrt{\sin^2 x + \cos^2 x + 2 \sin x \cos x}} dx \\&= \int \frac{\sin x + \cos x}{\sqrt{(\sin x + \cos x)^2}} dx = \int \frac{\sin x + \cos x}{\sin x + \cos x} dx \\&= \int 1 dx \\&= x + C, \text{ where } C \text{ is a constant}\end{aligned}$$

Therefore,

$$I = x + C$$

Question 9

$$\int \sqrt{1 + \sin x} dx$$

Solution

$$\begin{aligned}\text{Let } I &= \int \sqrt{1 + \sin x} \, dx \\&= \int \sqrt{\left(\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} + 2 \sin \frac{x}{2} \cos \frac{x}{2}\right)} \, dx \\&= \int \sqrt{\left(\sin \frac{x}{2} + \cos \frac{x}{2}\right)^2} \, dx = \int \left(\sin \frac{x}{2} + \cos \frac{x}{2}\right) \, dx \\&= \int \sin \frac{x}{2} \, dx + \int \cos \frac{x}{2} \, dx = -2 \cos \frac{x}{2} + 2 \sin \frac{x}{2} + C \\&= 2 \left(\sin \frac{x}{2} - \cos \frac{x}{2}\right) + C, \text{ where } C \text{ is a constant}\end{aligned}$$

Therefore,

$$I = 2 \left(\sin \frac{x}{2} - \cos \frac{x}{2}\right) + C$$

Question 10

$$\int \frac{x}{\sqrt{x} + 1} \, dx \quad (\text{Hint : Put } \sqrt{x} = z)$$

Solution

$$\text{Let, } I = \int \frac{x}{\sqrt{x} + 1} dx$$

$$\text{Put } \sqrt{x} = t \Rightarrow x = t^2 \quad \therefore dx = 2t \cdot dt$$

$$\begin{aligned} \text{So, } I &= \int \frac{t^2 \cdot 2t \cdot dt}{t + 1} = 2 \int \frac{t^3}{t + 1} dt = 2 \int \frac{t^3 + 1 - 1}{t + 1} dt \\ &= 2 \int \frac{t^3 + 1}{t + 1} dt - 2 \int \frac{1}{t + 1} dt \\ &= 2 \int \frac{(t + 1)(t^2 - t + 1)}{t + 1} dt - 2 \int \frac{1}{t + 1} dt \\ &= 2 \int (t^2 - t + 1) dt - 2 \int \frac{1}{t + 1} dt \\ &= 2 \left[\frac{t^3}{3} - \frac{t^2}{2} + t \right] - 2 \log |t + 1| \\ &= 2 \left[\frac{x^{3/2}}{3} - \frac{x}{2} + \sqrt{x} \right] - 2 \log |\sqrt{x} + 1| + C \\ &= 2 \left[\frac{x\sqrt{x}}{3} - \frac{x}{2} + \sqrt{x} - \log |\sqrt{x} + 1| \right] + C \end{aligned}$$

Therefore,

$$I = 2 \left[\frac{x\sqrt{x}}{3} - \frac{x}{2} + \sqrt{x} - \log |\sqrt{x} + 1| \right] + C$$

$$\text{Let, } I = \int \frac{x}{\sqrt{x} + 1} dx$$

$$\text{Put } \sqrt{x} = t \Rightarrow x = t^2 \therefore dx = 2t \cdot dt$$

$$\begin{aligned}\text{So, } I &= \int \frac{t^2 \cdot 2t \cdot dt}{t + 1} = 2 \int \frac{t^3}{t + 1} dt = 2 \int \frac{t^3 + 1 - 1}{t + 1} dt \\&= 2 \int \frac{t^3 + 1}{t + 1} dt - 2 \int \frac{1}{t + 1} dt \\&= 2 \int \frac{(t + 1)(t^2 - t + 1)}{t + 1} dt - 2 \int \frac{1}{t + 1} dt \\&= 2 \int (t^2 - t + 1) dt - 2 \int \frac{1}{t + 1} dt \\&= 2 \left[\frac{t^3}{3} - \frac{t^2}{2} + t \right] - 2 \log |t + 1| \\&= 2 \left[\frac{x^{3/2}}{3} - \frac{x}{2} + \sqrt{x} \right] - 2 \log |\sqrt{x} + 1| + C \\&= 2 \left[\frac{x\sqrt{x}}{3} - \frac{x}{2} + \sqrt{x} - \log |\sqrt{x} + 1| \right] + C\end{aligned}$$

Therefore,

$$I = 2 \left[\frac{x\sqrt{x}}{3} - \frac{x}{2} + \sqrt{x} - \log |\sqrt{x} + 1| \right] + C$$

Question 11

$$\int \sqrt{\frac{a+x}{a-x}}$$

Solution

$$\begin{aligned}
 \text{Let, } I &= \int \sqrt{\frac{a+x}{a-x}} dx \\
 &= \int \sqrt{\frac{a+x}{a-x}} \times \frac{a+x}{a+x} dx = \int \frac{a+x}{\sqrt{(a-x)(a+x)}} dx \\
 &= \int \frac{a+x}{\sqrt{a^2-x^2}} dx = \int \frac{a}{\sqrt{a^2-x^2}} dx + \int \frac{x}{\sqrt{a^2-x^2}} dx
 \end{aligned}$$

Let's consider, $I = I_1 + I_2$

$$\text{Now, } I_1 = \int \frac{a}{\sqrt{a^2-x^2}} dx = a \cdot \sin^{-1} \frac{x}{a} + C_1$$

$$\text{And, } I_2 = \int \frac{x}{\sqrt{a^2-x^2}} dx$$

$$\text{Putting, } a^2 - x^2 = t \Rightarrow -2x dx = dt$$

$$x dx = \frac{dt}{-2}$$

$$\text{Hence, } I_2 = -\frac{1}{2} \int \frac{dt}{\sqrt{t}} = -\frac{1}{2} \times 2\sqrt{t} = -\sqrt{a^2-x^2} + C_2$$

$$\text{As, } I = I_1 + I_2$$

$$= a \sin^{-1} \frac{x}{a} + C_1 - \sqrt{a^2-x^2} + C_2$$

$$\therefore I = a \sin^{-1} \frac{x}{a} - \sqrt{a^2-x^2} + (C_1 + C_2)$$

$$\text{Therefore, } I = a \sin^{-1} \frac{x}{a} - \sqrt{a^2-x^2} + C \quad [C = C_1 + C_2]$$

$$\begin{aligned}\text{Let, } I &= \int \sqrt{\frac{a+x}{a-x}} dx \\&= \int \sqrt{\frac{a+x}{a-x}} \times \frac{a+x}{a+x} dx = \int \frac{a+x}{\sqrt{(a-x)(a+x)}} dx \\&= \int \frac{a+x}{\sqrt{a^2-x^2}} dx = \int \frac{a}{\sqrt{a^2-x^2}} dx + \int \frac{x}{\sqrt{a^2-x^2}} dx\end{aligned}$$

Let's consider, $I = I_1 + I_2$

$$\text{Now, } I_1 = \int \frac{a}{\sqrt{a^2-x^2}} dx = a \cdot \sin^{-1} \frac{x}{a} + C_1$$

$$\text{And, } I_2 = \int \frac{x}{\sqrt{a^2-x^2}} dx$$

$$\text{Putting, } a^2 - x^2 = t \Rightarrow -2x dx = dt$$

$$x dx = \frac{dt}{-2}$$

$$\text{Hence, } I_2 = -\frac{1}{2} \int \frac{dt}{\sqrt{t}} = -\frac{1}{2} \times 2\sqrt{t} = -\sqrt{a^2-x^2} + C_2$$

$$\text{As, } I = I_1 + I_2$$

$$= a \sin^{-1} \frac{x}{a} + C_1 - \sqrt{a^2-x^2} + C_2$$

$$\therefore I = a \sin^{-1} \frac{x}{a} - \sqrt{a^2-x^2} + (C_1 + C_2)$$

$$\text{Therefore, } I = a \sin^{-1} \frac{x}{a} - \sqrt{a^2-x^2} + C \quad [C = C_1 + C_2]$$

Question 12

$$\int \frac{x^{\frac{1}{2}}}{1+x^{\frac{3}{4}}} dx \quad (\text{Hint : Put } x = z^4)$$

Solution

$$\text{Let, } I = \int \frac{x^{1/2}}{1+x^{3/4}} dx$$

$$\text{Putting, } x = t^4 \Rightarrow dx = 4t^3 dt$$

$$= \int \frac{t^2 \cdot 4t^3}{1+t^3} dt = 4 \int \frac{t^5}{1+t^3} dt$$

$$= 4 \int \left(t^2 - \frac{t^2}{t^3+1} \right) dt = 4 \int t^2 dt - 4 \int \frac{t^2}{t^3+1} dt$$

$$\begin{array}{r} t^3+1 \bigg) \overline{t^5} \quad \left(t^2 \right. \\ \underline{(-) \quad (-)} \\ -t^2 \end{array}$$

$$\text{Let's consider, } I = I_1 - I_2$$

$$\text{Now, } I_1 = 4 \int t^2 dt = 4 \cdot \frac{t^3}{3} + C_1 = \frac{4}{3} x^{3/4} + C_1$$

$$I_2 = 4 \int \frac{t^2}{t^3+1} dt$$

$$\text{Putting, } t^3+1 = z \Rightarrow 3t^2 dt = dz$$

$$t^2 dt = \frac{1}{3} dz$$

$$\text{So, } I_2 = \frac{4}{3} \int \frac{dz}{z} = \frac{4}{3} \log |z| + C_2 = \frac{4}{3} \log |t^3+1| + C_2$$

$$= \frac{4}{3} \log |(x)^{3/4} + 1| + C_2$$

$$\text{As, } I = I_1 - I_2$$

$$= \frac{4}{3} x^{3/4} + C_1 - \frac{4}{3} \log |(x)^{3/4} + 1| - C_2$$

$$= \frac{4}{3} [x^{3/4} - \log |(x)^{3/4} + 1|] + C_1 - C_2$$

$$\text{Therefore, } I = \frac{4}{3} [x^{3/4} - \log |(x)^{3/4} + 1|] + C \quad [\because C = C_1 - C_2]$$

Question 13

$$\int \frac{\sqrt{1+x^2}}{x^4} dx$$

Solution

$$\begin{aligned}\text{Let, } I &= \int \frac{\sqrt{1+x^2}}{x^4} dx \\ &= \int \sqrt{\frac{1+x^2}{x^2}} \cdot \frac{1}{x^3} dx = \int \sqrt{\frac{1}{x^2} + 1} \cdot \frac{1}{x^3} dx\end{aligned}$$

$$\text{Taking, } \frac{1}{x^2} + 1 = t^2$$

$$\text{So, } \frac{-2}{x^3} dx = 2t dt \Rightarrow \frac{dx}{x^3} = -t dt$$

$$\therefore I = \int t (-t dt) = -\int t^2 dt = -\frac{1}{3}t^3 + C$$

$$\text{Therefore, } I = -\frac{1}{3}\left(\frac{1}{x^2} + 1\right)^{3/2} + C$$

Question 14

$$\int \frac{dx}{\sqrt{16-9x^2}}$$

Solution

$$\begin{aligned}\text{Let, } I &= \int \frac{dx}{\sqrt{16 - 9x^2}} \\&= \frac{1}{3} \int \frac{dx}{\sqrt{\frac{16}{9} - x^2}} = \frac{1}{3} \int \frac{dx}{\sqrt{\left(\frac{4}{3}\right)^2 - x^2}} \\&= \frac{1}{3} \sin^{-1} \frac{x}{4/3} + C \quad \left[\because \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + C \right] \\&= \frac{1}{3} \sin^{-1} \frac{3x}{4} + C\end{aligned}$$

$$\text{Therefore, } I = \frac{1}{3} \sin^{-1} \frac{3x}{4} + C.$$

Question 15

$$\int \frac{dt}{\sqrt{3t - 2t^2}}$$

Solution

$$\begin{aligned}\text{Let, } I &= \int \frac{dt}{\sqrt{3t - 2t^2}} = \int \frac{dt}{\sqrt{-2\left(t^2 - \frac{3}{2}t\right)}} \\&= \frac{1}{\sqrt{2}} \int \frac{dt}{\sqrt{-\left(t^2 - \frac{3}{2}t + \frac{9}{16} - \frac{9}{16}\right)}} \quad [\text{Making perfect square}] \\&= \frac{1}{\sqrt{2}} \int \frac{dt}{\sqrt{-\left[\left(t - \frac{3}{4}\right)^2 - \frac{9}{16}\right]}} = \frac{1}{\sqrt{2}} \int \frac{dt}{\sqrt{\frac{9}{16} - \left(t - \frac{3}{4}\right)^2}} \\&= \frac{1}{\sqrt{2}} \int \frac{dt}{\sqrt{\left(\frac{3}{4}\right)^2 - \left(t - \frac{3}{4}\right)^2}} = \frac{1}{\sqrt{2}} \cdot \sin^{-1} \frac{t - \frac{3}{4}}{\frac{3}{4}} + C \\&= \frac{1}{\sqrt{2}} \sin^{-1} \frac{4t - 3}{3} + C\end{aligned}$$

$$\text{Therefore, } I = \frac{1}{\sqrt{2}} \sin^{-1} \left(\frac{4t - 3}{3} \right) + C.$$

$$\begin{aligned}\text{Let, } I &= \int \frac{dt}{\sqrt{3t - 2t^2}} = \int \frac{dt}{\sqrt{-2\left(t^2 - \frac{3}{2}t\right)}} \\&= \frac{1}{\sqrt{2}} \int \frac{dt}{\sqrt{-\left(t^2 - \frac{3}{2}t + \frac{9}{16} - \frac{9}{16}\right)}} \quad [\text{Making perfect square}] \\&= \frac{1}{\sqrt{2}} \int \frac{dt}{\sqrt{-\left[\left(t - \frac{3}{4}\right)^2 - \frac{9}{16}\right]}} = \frac{1}{\sqrt{2}} \int \frac{dt}{\sqrt{\frac{9}{16} - \left(t - \frac{3}{4}\right)^2}} \\&= \frac{1}{\sqrt{2}} \int \frac{dt}{\sqrt{\left(\frac{3}{4}\right)^2 - \left(t - \frac{3}{4}\right)^2}} = \frac{1}{\sqrt{2}} \cdot \sin^{-1} \frac{t - \frac{3}{4}}{\frac{3}{4}} + C \\&= \frac{1}{\sqrt{2}} \sin^{-1} \frac{4t - 3}{3} + C\end{aligned}$$

$$\text{Therefore, } I = \frac{1}{\sqrt{2}} \sin^{-1} \left(\frac{4t - 3}{3} \right) + C.$$

Question 16

$$\int \frac{3x - 1}{\sqrt{x^2 + 9}} dx$$

Solution

$$\text{Let, } I = \int \frac{3x-1}{\sqrt{x^2+9}} dx = \int \frac{3x}{\sqrt{x^2+9}} dx - \int \frac{1}{\sqrt{x^2+9}} dx$$

$$I = I_1 - I_2$$

$$\text{Now, } I_1 = \int \frac{3x}{\sqrt{x^2+9}} dx$$

$$\text{Putting, } x^2 + 9 = t \Rightarrow 2x dx = dt$$

$$\text{So, } x dx = \frac{1}{2} dt$$

$$I_1 = \frac{3}{2} \int \frac{dt}{\sqrt{t}} = \frac{3}{2} \cdot 2\sqrt{t} + C_1 = 3\sqrt{x^2+9} + C_1$$

And,

$$I_2 = \int \frac{1}{\sqrt{x^2+9}} dx = \int \frac{1}{\sqrt{x^2+(3)^2}} dx = \log |x + \sqrt{x^2+(3)^2}| + C_2$$

$$\left[\because \int \frac{1}{\sqrt{x^2+a^2}} dx = \log |x + \sqrt{x^2+a^2}| + C \right]$$

$$= \log |x + \sqrt{x^2+9}| + C_2$$

$$\therefore I = I_1 - I_2$$

$$= 3\sqrt{x^2+9} + C_1 - \log |x + \sqrt{x^2+9}| - C_2$$

$$= 3\sqrt{x^2+9} - \log |x + \sqrt{x^2+9}| + (C_1 - C_2)$$

$$\text{Therefore, } I = 3\sqrt{x^2+9} - \log |x + \sqrt{x^2+9}| + C$$

$$\text{Let, } I = \int \frac{3x-1}{\sqrt{x^2+9}} dx = \int \frac{3x}{\sqrt{x^2+9}} dx - \int \frac{1}{\sqrt{x^2+9}} dx$$

$$I = I_1 - I_2$$

$$\text{Now, } I_1 = \int \frac{3x}{\sqrt{x^2+9}} dx$$

$$\text{Putting, } x^2 + 9 = t \Rightarrow 2x dx = dt$$

$$\text{So, } x dx = -dt$$

$$I_1 = \frac{3}{2} \int \frac{dt}{\sqrt{t}} = \frac{3}{2} \cdot 2\sqrt{t} + C_1 = 3\sqrt{x^2+9} + C_1$$

And,

$$I_2 = \int \frac{1}{\sqrt{x^2+9}} dx = \int \frac{1}{\sqrt{x^2+(3)^2}} dx = \log |x + \sqrt{x^2+(3)^2}| + C_2$$

$$\left[\because \int \frac{1}{\sqrt{x^2+a^2}} dx = \log |x + \sqrt{x^2+a^2}| + C \right]$$

$$= \log |x + \sqrt{x^2+9}| + C_2$$

$$\therefore I = I_1 - I_2$$

$$= 3\sqrt{x^2+9} + C_1 - \log |x + \sqrt{x^2+9}| - C_2$$

$$= 3\sqrt{x^2+9} - \log |x + \sqrt{x^2+9}| + (C_1 - C_2)$$

$$\text{Therefore, } I = 3\sqrt{x^2+9} - \log |x + \sqrt{x^2+9}| + C$$

Question 17

$$\int \sqrt{5-2x+x^2} dx$$

Solution

$$\begin{aligned}
 \text{Let } I &= \int \sqrt{5 - 2x + x^2} \, dx = \int \sqrt{x^2 - 2x + 5} \, dx \\
 &= \int \sqrt{x^2 - 2x + 1 - 1 + 5} \, dx \quad (\text{Making perfect square}) \\
 &= \int \sqrt{(x-1)^2 + 4} \, dx = \int \sqrt{(x-1)^2 + (2)^2} \, dx \\
 &= \frac{x-1}{2} \sqrt{(x-1)^2 + (2)^2} + \frac{4}{2} \log \left| (x-1) + \sqrt{(x-1)^2 + (2)^2} \right| + C \\
 &\quad \left[\because \int \sqrt{x^2 + a^2} \, dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \left\{ \log \left| x + \sqrt{x^2 + a^2} \right| \right\} + C \right] \\
 &= \frac{x-1}{2} \sqrt{x^2 + 1 - 2x + 4} + 2 \log \left| (x-1) + \sqrt{x^2 + 1 - 2x + 4} \right| + C \\
 &= \frac{x-1}{2} \sqrt{x^2 - 2x + 5} + 2 \log \left| (x-1) + \sqrt{x^2 - 2x + 5} \right| + C
 \end{aligned}$$

Therefore,

$$I = \frac{x-1}{2} \sqrt{x^2 - 2x + 5} + 2 \log \left| (x-1) + \sqrt{x^2 - 2x + 5} \right| + C$$

Question 18

$$\int \frac{x}{x^4 - 1} \, dx$$

Solution

$$\text{Let, } I = \int \frac{x}{x^4 - 1} dx$$

$$\text{Putting, } x^2 = t \Rightarrow 2x dx = dt \Rightarrow x dx = \frac{dt}{2}$$

$$\frac{1}{2} \int \frac{dt}{t^2 - 1} = \frac{1}{2} \int \frac{dt}{t^2 - (1)^2} = \frac{1}{2} \cdot \frac{1}{2 \cdot 1} \log \left| \frac{t-1}{t+1} \right| + C$$

$$\left[\because \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C \right]$$

$$= \frac{1}{4} \log \left| \frac{x^2 - 1}{x^2 + 1} \right| + C$$

$$\text{Therefore, } I = \frac{1}{4} \log \left| \frac{x^2 - 1}{x^2 + 1} \right| + C$$

$$\text{Let, } I = \int \frac{x}{x^4 - 1} dx$$

$$\text{Putting, } x^2 = t \Rightarrow 2x dx = dt \Rightarrow x dx = \frac{dt}{2}$$

$$\frac{1}{2} \int \frac{dt}{t^2 - 1} = \frac{1}{2} \int \frac{dt}{t^2 - (1)^2} = \frac{1}{2} \cdot \frac{1}{2 \cdot 1} \log \left| \frac{t-1}{t+1} \right| + C$$

$$\left[\because \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C \right]$$

$$= \frac{1}{4} \log \left| \frac{x^2 - 1}{x^2 + 1} \right| + C$$

$$\text{Therefore, } I = \frac{1}{4} \log \left| \frac{x^2 - 1}{x^2 + 1} \right| + C$$

Question 19

$$\int \frac{x^2}{1-x^4} dx \text{ put } x^2 = t$$

Solution

$$\text{Let } I = \int \frac{x^2}{1-x^4} dx = \int \frac{x^2}{(1-x^2)(1+x^2)} dx$$

Putting, $x^2 = t$ for the purpose of partial fractions.

$$\text{We get } \frac{t}{(1-t)(1+t)}$$

Resolving into partial fraction, we have

$$\frac{t}{(1-t)(1+t)} = \frac{A}{1-t} + \frac{B}{1+t}$$

[where A and B are arbitrary constants]

$$\text{So, } \frac{t}{(1-t)(1+t)} = \frac{A(1+t) + B(1-t)}{(1-t)(1+t)}$$

$$\Rightarrow t = A + At + B - Bt$$

Comparing the like terms, we get $A - B = 1$ and $A + B = 0$

On solving the above equations, we have

$$A = \frac{1}{2} \text{ and } B = -\frac{1}{2}$$

$$\text{Now, } I = \int \frac{1/2}{1-x^2} dx + \int \frac{-1/2}{1+x^2} dx \quad (\text{Putting } t = x^2)$$

$$= \frac{1}{2} \cdot \frac{1}{2.1} \log \left| \frac{1+x}{1-x} \right| - \frac{1}{2} \tan^{-1} x + C$$

$$= \frac{1}{4} \log \left| \frac{1+x}{1-x} \right| - \frac{1}{2} \tan^{-1} x + C$$

$$\text{Therefore, } I = \frac{1}{4} \log \left| \frac{1+x}{1-x} \right| - \frac{1}{2} \tan^{-1} x + C.$$

Question 20

$$\int \sqrt{2ax - x^2} \, dx$$

Solution

$$\begin{aligned}\text{Let } I &= \int \sqrt{2ax - x^2} \, dx \\&= \int \sqrt{-(x^2 - 2ax)} \, dx = \int \sqrt{-(x^2 - 2ax + a^2 - a^2)} \, dx \\&= \int \sqrt{-(x - a)^2 + a^2} \, dx = \int \sqrt{a^2 - (x - a)^2} \, dx \\&= \frac{x - a}{2} \sqrt{a^2 - (x - a)^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x - a}{a} \right) + C \\&\quad \left[\because \int \sqrt{a^2 - x^2} \, dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + C \right] \\&= \frac{x - a}{2} \sqrt{a^2 - (x^2 - 2ax + a^2)} + \frac{a^2}{2} \sin^{-1} \left(\frac{x - a}{a} \right) + C \\&= \frac{x - a}{2} \sqrt{2ax - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x - a}{a} \right) + C\end{aligned}$$

Therefore, $I = \frac{x - a}{2} \sqrt{2ax - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x - a}{a} \right) + C.$

Question 21

$$\int \frac{\sin^{-1} x}{(1 - x^2)^{\frac{3}{2}}} \, dx$$

Solution

$$\text{Let } I = \int \frac{\sin^{-1} x}{(1-x^2)^{3/2}} dx$$

$$\text{Putting, } x = \sin \theta \Rightarrow dx = \cos \theta d\theta$$

$$\begin{aligned} I &= \int \frac{\sin^{-1}(\sin \theta)}{(1 - \sin^2 \theta)^{3/2}} \cdot \cos \theta d\theta \\ &= \int \frac{\theta \cdot \cos \theta d\theta}{(\cos^2 \theta)^{3/2}} = \int \frac{\theta \cdot \cos \theta}{\cos^3 \theta} d\theta \\ &= \int \frac{\theta}{\cos^2 \theta} d\theta = \int \theta \sec^2 \theta d\theta \\ &= \theta \cdot \int \sec^2 \theta d\theta - \int (D(\theta) \cdot \int \sec^2 \theta d\theta) d\theta \\ &\quad \left[\because \int u \cdot v dx = u \cdot \int v dx - \int (D(u) \int v dv) dv + C \right] \\ &= \theta \cdot \tan \theta - \int 1 \cdot \tan \theta d\theta \\ &= \theta \cdot \tan \theta - \log \sec \theta + C \\ &= \sin^{-1} x \cdot \frac{x}{\sqrt{1-x^2}} - \log \left| \sqrt{1-x^2} \right| + C \\ &\quad \left[\begin{array}{l} \text{when } x = \sin \theta \\ \therefore \tan \theta = \frac{x}{\sqrt{1-x^2}} \text{ and } \sec \theta = \sqrt{1-x^2} \end{array} \right] \end{aligned}$$

$$\text{Therefore, } I = \frac{x \sin^{-1} x}{\sqrt{1-x^2}} - \log \left| \sqrt{1-x^2} \right| + C$$

$$\text{Let } I = \int \frac{\sin^{-1} x}{(1-x^2)^{3/2}} dx$$

$$\text{Putting, } x = \sin \theta \Rightarrow dx = \cos \theta d\theta$$

$$I = \int \frac{\sin^{-1}(\sin \theta)}{(1 - \sin^2 \theta)^{3/2}} \cdot \cos \theta d\theta$$

$$= \int \frac{\theta \cdot \cos \theta d\theta}{(\cos^2 \theta)^{3/2}} = \int \frac{\theta \cdot \cos \theta}{\cos^3 \theta} d\theta$$

$$= \int \frac{\theta}{\cos^2 \theta} d\theta = \int \theta \sec^2 \theta d\theta$$

$$= \theta \cdot \int \sec^2 \theta d\theta - \int (D(\theta) \cdot \int \sec^2 \theta d\theta) d\theta$$

$$\left[\because \int u \cdot v dx = u \cdot \int v dx - \int (D(u) \int v dv) dv + C \right]$$

$$= \theta \cdot \tan \theta - \int 1 \cdot \tan \theta d\theta$$

$$= \theta \cdot \tan \theta - \log \sec \theta + C$$

$$= \sin^{-1} x \cdot \frac{x}{\sqrt{1-x^2}} - \log \left| \sqrt{1-x^2} \right| + C$$

$$\left[\begin{array}{l} \text{when } x = \sin \theta \\ \therefore \tan \theta = \frac{x}{\sqrt{1-x^2}} \text{ and } \sec \theta = \sqrt{1-x^2} \end{array} \right]$$

$$\text{Therefore, } I = \frac{x \sin^{-1} x}{\sqrt{1-x^2}} - \log \left| \sqrt{1-x^2} \right| + C$$

Question 22

$$\int \frac{(\cos 5x + \cos 4x)}{1 - 2 \cos 3x} dx$$

Solution

$$\begin{aligned}
 \text{Let } I &= \int \frac{\cos 5x + \cos 4x}{1 - 2 \cos 3x} dx = \int \frac{2 \cos \frac{5x+4x}{2} \cdot \cos \frac{5x-4x}{2}}{1 - 2 \left(2 \cos^2 \frac{3x}{2} - 1 \right)} dx \\
 &= \int \frac{2 \cos \frac{9x}{2} \cdot \cos \frac{x}{2}}{1 - 4 \cos^2 \frac{3x}{2} + 2} dx = \int \frac{2 \cos \frac{9x}{2} \cdot \cos \frac{x}{2}}{3 - 4 \cos^2 \frac{3x}{2}} dx \quad [\text{Using trigonometric identities}] \\
 &= - \int \frac{2 \cos \frac{9x}{2} \cdot \cos \frac{x}{2}}{4 \cos^2 \frac{3x}{2} - 3} dx = - \int \frac{2 \cos \frac{9x}{2} \cdot \cos \frac{x}{2} \cdot \cos \frac{3x}{2}}{4 \cos^3 \frac{3x}{2} - 3 \cos \frac{3x}{2}} dx \\
 &\quad \left[\text{Multiplying and dividing by } \cos \frac{3x}{2} \right] \\
 &= - \int \frac{2 \cos \frac{9x}{2} \cdot \cos \frac{x}{2} \cdot \cos \frac{3x}{2}}{\cos 3 \cdot \frac{3x}{2}} dx \\
 &\quad [\because \cos 3x = 4 \cos^3 x - 3 \cos x] \\
 &= - \int \frac{2 \cos \frac{9x}{2} \cdot \cos \frac{x}{2} \cdot \cos \frac{3x}{2}}{\cos \frac{9x}{2}} dx = - \int 2 \cos \frac{3x}{2} \cdot \cos \frac{x}{2} dx \\
 &= - \int \left[\cos \left(\frac{3x}{2} + \frac{x}{2} \right) + \cos \left(\frac{3x}{2} - \frac{x}{2} \right) \right] dx \\
 &= - \int (\cos 2x + \cos x) dx \\
 &\quad [\because 2 \cos A \cos B = \cos (A+B) + \cos (A-B)] \\
 &= - \int \cos 2x dx - \int \cos x dx = -\frac{1}{2} \sin 2x - \sin x + C \\
 \text{Therefore, } I &= -\left[\frac{1}{2} \sin 2x + \sin x \right] + C.
 \end{aligned}$$

$$\begin{aligned}
 \text{Let } I &= \int \frac{\cos 5x + \cos 4x}{1 - 2 \cos 3x} dx = \int \frac{2 \cos \frac{5x+4x}{2} \cdot \cos \frac{5x-4x}{2}}{1 - 2 \left(2 \cos^2 \frac{3x}{2} - 1 \right)} dx \\
 &= \int \frac{2 \cos \frac{9x}{2} \cdot \cos \frac{x}{2}}{1 - 4 \cos^2 \frac{3x}{2} + 2} dx = \int \frac{2 \cos \frac{9x}{2} \cdot \cos \frac{x}{2}}{3 - 4 \cos^2 \frac{3x}{2}} dx \quad [\text{Using trigonometric identities}] \\
 &= - \int \frac{2 \cos \frac{9x}{2} \cdot \cos \frac{x}{2}}{4 \cos^2 \frac{3x}{2} - 3} dx = - \int \frac{2 \cos \frac{9x}{2} \cdot \cos \frac{x}{2} \cdot \cos \frac{3x}{2}}{4 \cos^3 \frac{3x}{2} - 3 \cos \frac{3x}{2}} dx \\
 &\quad \left[\text{Multiplying and dividing by } \cos \frac{3x}{2} \right] \\
 &= - \int \frac{2 \cos \frac{9x}{2} \cdot \cos \frac{x}{2} \cdot \cos \frac{3x}{2}}{\cos 3 \cdot \frac{3x}{2}} dx \\
 &\quad [\because \cos 3x = 4 \cos^3 x - 3 \cos x] \\
 &= - \int \frac{2 \cos \frac{9x}{2} \cdot \cos \frac{x}{2} \cdot \cos \frac{3x}{2}}{\cos \frac{9x}{2}} dx = - \int 2 \cos \frac{3x}{2} \cdot \cos \frac{x}{2} dx \\
 &= - \int \left[\cos \left(\frac{3x}{2} + \frac{x}{2} \right) + \cos \left(\frac{3x}{2} - \frac{x}{2} \right) \right] dx \\
 &= - \int (\cos 2x + \cos x) dx \\
 &\quad [\because 2 \cos A \cos B = \cos (A+B) + \cos (A-B)] \\
 &= - \int \cos 2x dx - \int \cos x dx = -\frac{1}{2} \sin 2x - \sin x + C
 \end{aligned}$$

$$\text{Therefore, } I = -\left[\frac{1}{2} \sin 2x + \sin x \right] + C.$$

Question 23

$$\int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x} dx$$

Solution

$$\begin{aligned} \text{Let } I &= \int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x} dx = \int \frac{(\sin^2 x)^3 + (\cos^2 x)^3}{\sin^2 x \cos^2 x} dx \\ &= \int \frac{(\sin^2 x + \cos^2 x)^3 - 3 \sin^2 x \cos^2 x (\sin^2 x + \cos^2 x)}{\sin^2 x \cos^2 x} dx \\ &\quad [\because a^3 + b^3 = (a + b)^3 - 3ab(a + b)] \\ &= \int \frac{(1)^3 - 3 \sin^2 x \cos^2 x \cdot (1)}{\sin^2 x \cos^2 x} dx \\ &= \int \frac{1 - 3 \sin^2 x \cos^2 x}{\sin^2 x \cos^2 x} dx \\ &= \int \left(\frac{1}{\sin^2 x \cos^2 x} - \frac{3 \sin^2 x \cos^2 x}{\sin^2 x \cos^2 x} \right) dx \\ &= \int \left(\frac{1}{\sin^2 x \cos^2 x} - 3 \right) dx = \int \left(\frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} - 3 \right) dx \\ &= \int \left[\left(\frac{1}{\cos^2 x} + \frac{1}{\sin^2 x} \right) - 3 \right] dx \\ &= \int (\sec^2 x + \operatorname{cosec}^2 x - 3) dx \\ &= \int \sec^2 x dx + \int \operatorname{cosec}^2 x dx - 3 \int 1 dx \\ &= \tan x - \cot x - 3x + C \end{aligned}$$

Therefore, $I = \tan x - \cot x - 3x + C$.

Question 24

$$\int \frac{\sqrt{x}}{\sqrt{a^3 - x^3}} dx$$

Solution

$$\text{Let } I = \int \frac{\sqrt{x}}{\sqrt{a^3 - x^3}} dx = \int \frac{x^{1/2}}{\sqrt{(a^{3/2})^2 - (x^{3/2})^2}} dx$$

$$\text{Putting, } x^{3/2} = t \Rightarrow \frac{3}{2} x^{1/2} dx = dt \Rightarrow x^{1/2} dx = \frac{2}{3} dt$$

$$\begin{aligned} \text{Now, } I &= \frac{2}{3} \int \frac{dt}{\sqrt{(a^{3/2})^2 - (t)^2}} \\ &= \frac{2}{3} \sin^{-1} \frac{t}{a^{3/2}} + C = \frac{2}{3} \sin^{-1} \left(\frac{x^{3/2}}{a^{3/2}} \right) + C \end{aligned}$$

$$\text{Therefore, } I = \frac{2}{3} \sin^{-1} \left(\frac{x}{a} \right)^{3/2} + C.$$

$$\text{Let } I = \int \frac{\sqrt{x}}{\sqrt{a^3 - x^3}} dx = \int \frac{x^{1/2}}{\sqrt{(a^{3/2})^2 - (x^{3/2})^2}} dx$$

$$\text{Putting, } x^{3/2} = t \Rightarrow \frac{3}{2} x^{1/2} dx = dt \Rightarrow x^{1/2} dx = \frac{2}{3} dt$$

$$\begin{aligned} \text{Now, } I &= \frac{2}{3} \int \frac{dt}{\sqrt{(a^{3/2})^2 - (t)^2}} \\ &= \frac{2}{3} \sin^{-1} \frac{t}{a^{3/2}} + C = \frac{2}{3} \sin^{-1} \left(\frac{x^{3/2}}{a^{3/2}} \right) + C \end{aligned}$$

$$\text{Therefore, } I = \frac{2}{3} \sin^{-1} \left(\frac{x}{a} \right)^{3/2} + C.$$

Question 25

$$\int \frac{\cos x - \cos 2x}{1 - \cos x} dx$$

Solution

$$\begin{aligned} \text{Let } I &= \int \frac{\cos x - \cos 2x}{1 - \cos x} dx \\ &= \int \frac{2 \sin \frac{x+2x}{2} \cdot \sin \left(\frac{2x-x}{2} \right)}{2 \sin^2 x/2} dx \\ &\quad \left[\because \cos C - \cos D = 2 \sin \frac{C+D}{2} \cdot \sin \frac{D-C}{2} \right] \\ &= \int \frac{2 \sin \frac{3x}{2} \cdot \sin \frac{x}{2}}{2 \sin^2 \frac{x}{2}} dx = \int \frac{\sin \frac{3x}{2}}{\sin \frac{x}{2}} dx = \int \frac{\sin 3\left(\frac{x}{2}\right)}{\sin \frac{x}{2}} dx \\ &= \int \frac{3 \sin \frac{x}{2} - 4 \sin^3 \frac{x}{2}}{\sin \frac{x}{2}} dx \quad \text{Since, } [\sin 3x = 3 \sin x - 4 \sin^3 x] \\ &= \int \frac{\sin \frac{x}{2} \left(3 - 4 \sin^2 \frac{x}{2} \right)}{\sin \frac{x}{2}} dx = \int \left(3 - 4 \sin^2 \frac{x}{2} \right) dx \\ &= \int [3 - 2(1 - \cos x)] dx \quad \left[\because 2 \sin^2 \frac{x}{2} = 1 - \cos x \right] \\ &= \int (3 - 2 + 2 \cos x) dx = \int (1 + 2 \cos x) dx \\ &= x + 2 \sin x + C \end{aligned}$$

Therefore, $I = x + 2 \sin x + C$.

Question 26

$$\int \frac{dx}{x\sqrt{x^4-1}} \quad (\text{Hint : Put } x^2 = \sec \theta)$$

Solution

$$\text{Let } I = \int \frac{dx}{x\sqrt{x^4-1}} = \int \frac{x dx}{x^2\sqrt{x^4-1}}$$

Putting, $x^2 = \sec \theta$

$$\text{So, } 2x dx = \sec \theta \tan \theta d\theta$$

$$x dx = \frac{1}{2} \sec \theta \tan \theta d\theta$$

$$\text{Now, } I = \frac{1}{2} \int \frac{\sec \theta \tan \theta}{\sec \theta \sqrt{\sec^2 \theta - 1}} d\theta$$

$$= \frac{1}{2} \int \frac{\sec \theta \tan \theta}{\sec \theta \cdot \tan \theta} d\theta = \frac{1}{2} \int 1 d\theta = \frac{1}{2} \theta + C$$

$$I = \frac{1}{2} \sec^{-1} x^2 + C$$

$$\text{Therefore, } I = \frac{1}{2} \sec^{-1} x^2 + C.$$

Evaluate the following as limit of sums:

Question 27

$$\int_0^2 (x^2 + 3) dx$$

Solution

Let $I = \int_0^2 (x^2 + 3) dx$

Using the formula,

$$\int_a^b f(x) dx = \lim_{h \rightarrow 0} h [f(a) + f(a+h) + f(a+2h) + \dots + f(a+(n-1)h)]$$

where $h = \frac{b-a}{n}$

Here, we have $a = 0$ and $b = 2$

So,

$$h = \frac{2-0}{n} \quad \therefore nh = 2$$

Here, $f(x) = x^2 + 3$

$$f(0) = 0 + 3 = 3$$

$$f(0+h) = (0+h)^2 + 3 = h^2 + 3$$

$$f(0+2h) = (0+2h)^2 + 3 = 4h^2 + 3$$

.....

.....

$$f(0+(n-1)h) = (0+(n-1)h)^2 + 3(n-1)h^2 + 3$$

Now

$$\int_0^2 (x^2 + 3) dx$$

$$= \lim_{h \rightarrow 0} h [3 + h^2 + 3 + 4h^2 + 3 + \dots + (n-1)^2 h^2 + 3]$$

$$= \lim_{h \rightarrow 0} h [(3+3+3+\dots+n) + \{h^2 + 4h^2 + \dots + (n-1)^2 h^2\}]$$

$$= \lim_{h \rightarrow 0} h [3n + h^2 \{1 + 4 + \dots + (n-1)^2\}]$$

$$= \lim_{h \rightarrow 0} h \left[3n + h^2 \frac{n(n-1)(2n-1)}{6} \right]$$

$$\left[\because 1 + 4 + 9 + \dots + (n-1)^2 = \frac{n(n-1)(2n-1)}{6} \right]$$

$$\left[\therefore 1^2 + 2^2 + 3^2 + \dots + (n-1)^2 = \frac{n(n-1)(2n-1)}{6} \right]$$

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \left[3nh + \frac{h^2 n(n-1)(2n-1)}{6} \right] \\
 &= \lim_{h \rightarrow 0} \left[3nh + \frac{nh(nh-h)(2nh-h)}{6} \right] \\
 &= \left[3 \times 2 + \frac{2(2-0)(2 \times 2-0)}{6} \right] \quad \left[\because nh = 2 \right. \\
 &\quad \left. h = 0 \right] \\
 &= \left[6 + \frac{2 \times 2 \times 4}{6} \right] = 6 + \frac{8}{3} = \frac{26}{3}
 \end{aligned}$$

Therefore, $\int_0^2 (x^2 + 3) dx = \frac{26}{3}.$

Let $I = \int_0^2 (x^2 + 3) dx$

Using the formula,

$$\int_a^b f(x) dx = \lim_{h \rightarrow 0} h [f(a) + f(a+h) + f(a+2h) + \dots + f(a+(n-1)h)]$$

where $h = \frac{b-a}{n}$

Here, we have $a = 0$ and $b = 2$

So,

$$h = \frac{2-0}{n} \quad \therefore nh = 2$$

Here,

$$f(x) = x^2 + 3$$

$$f(0) = 0 + 3 = 3$$

$$f(0+h) = (0+h)^2 + 3 = h^2 + 3$$

$$f(0+2h) = (0+2h)^2 + 3 = 4h^2 + 3$$

.....

.....

$$f(0+(n-1)h) = (0+(n-1)h)^2 + 3(n-1)h^2 + 3$$

Now

$$\int_0^2 (x^2 + 3) dx$$

$$= \lim_{h \rightarrow 0} h [3 + h^2 + 3 + 4h^2 + 3 + \dots + (n-1)^2 h^2 + 3]$$

$$= \lim_{h \rightarrow 0} h [(3 + 3 + 3 + \dots + n) + \{h^2 + 4h^2 + \dots + (n-1)^2 h^2\}]$$

$$= \lim_{h \rightarrow 0} h [3n + h^2 \{1 + 4 + \dots + (n-1)^2\}]$$

$$= \lim_{h \rightarrow 0} h \left[3n + h^2 \frac{n(n-1)(2n-1)}{6} \right]$$

$$\left[\because 1 + 4 + 9 + \dots + (n-1)^2 = \frac{n(n-1)(2n-1)}{6} \right]$$

$$\left[\therefore 1^2 + 2^2 + 3^2 + \dots + (n-1)^2 = \frac{n(n-1)(2n-1)}{6} \right]$$

$$\begin{aligned}
&= \lim_{h \rightarrow 0} \left[3nh + \frac{h^2 n(n-1)(2n-1)}{6} \right] \\
&= \lim_{h \rightarrow 0} \left[3nh + \frac{nh(nh-h)(2nh-h)}{6} \right] \\
&= \left[3 \times 2 + \frac{2(2-0)(2 \times 2-0)}{6} \right] \quad \left[\because nh = 2 \right. \\
&\quad \left. h = 0 \right] \\
&= \left[6 + \frac{2 \times 2 \times 4}{6} \right] = 6 + \frac{8}{3} = \frac{26}{3}
\end{aligned}$$

Therefore, $\int_0^2 (x^2 + 3) dx = \frac{26}{3}.$

Question 28

$$\int_0^2 e^x dx$$

Solution

$$\text{Let, } I = \int_0^2 e^x dx$$

$$\text{Here, } a = 0 \text{ and } b = 2 \therefore h = \frac{b-a}{n} \Rightarrow h = \frac{2-0}{n} \therefore nh = 2$$

$$\begin{aligned}\text{Now, } f(x) &= e^x \\ f(0) &= e^0 = 1 \\ f(0+h) &= e^{0+h} = e^h \\ f(0+2h) &= e^{0+2h} = e^{2h} \\ &\dots\dots\dots \\ &\dots\dots\dots \\ f(0+\overline{n-1}h) &= e^{0+(n-1)h} = e^{(n-1)h}\end{aligned}$$

$$\begin{aligned}\text{Using } \int_a^b f(x) dx \\ &= \lim_{h \rightarrow 0} h [f(a) + f(a+h) + f(a+2h) + \dots + f(a+\overline{n-1}h)]\end{aligned}$$

$$\begin{aligned}\therefore \int_0^2 e^x dx &= \lim_{h \rightarrow 0} h [1 + e^h + e^{2h} + \dots + e^{(n-1)h}] \\ &= \lim_{h \rightarrow 0} h \left[\frac{1(e^{nh} - 1)}{e^h - 1} \right] \\ &\quad \left[\because a + ar + ar^2 + \dots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1} \right] \\ &= \lim_{h \rightarrow 0} \frac{e^{nh} - 1}{e^h - 1} = \frac{e^2 - 1}{1} = e^2 - 1 \left[\because \lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1 \right]\end{aligned}$$

$$\text{Therefore, } I = e^2 - 1.$$

$$\text{Let, } I = \int_0^2 e^x dx$$

$$\text{Here, } a = 0 \text{ and } b = 2 \therefore h = \frac{b-a}{n} \Rightarrow h = \frac{2-0}{n} \therefore nh = 2$$

$$\begin{aligned} \text{Now, } f(x) &= e^x \\ f(0) &= e^0 = 1 \\ f(0+h) &= e^{0+h} = e^h \\ f(0+2h) &= e^{0+2h} = e^{2h} \end{aligned}$$

.....

.....

$$f(0 + \overline{n-1}h) = e^{0 + (n-1)h} = e^{(n-1)h}$$

$$\text{Using } \int_a^b f(x) dx$$

$$= \lim_{h \rightarrow 0} h \left[f(a) + f(a+h) + f(a+2h) + \dots + f(a + \overline{n-1}h) \right]$$

$$\therefore \int_0^2 e^x dx = \lim_{h \rightarrow 0} h \left[1 + e^h + e^{2h} + \dots + e^{(n-1)h} \right]$$

$$= \lim_{h \rightarrow 0} h \left[\frac{1(e^{nh} - 1)}{e^h - 1} \right]$$

$$\left[\because a + ar + ar^2 + \dots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1} \right]$$

$$= \lim_{h \rightarrow 0} \frac{e^{nh} - 1}{e^h - 1} = \frac{e^2 - 1}{1} = e^2 - 1 \left[\because \lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1 \right]$$

$$\text{Therefore, } I = e^2 - 1.$$

Evaluate the following:

Question 29

$$\int_0^1 \frac{dx}{e^x + e^{-x}}$$

Solution

$$\begin{aligned}\text{Let } I &= \int_0^1 \frac{dx}{e^x + e^{-x}} \\ &= \int_0^1 \frac{dx}{e^x + \frac{1}{e^x}} = \int_0^1 \frac{dx}{\frac{e^{2x} + 1}{e^x}} = \int_0^1 \frac{e^x dx}{e^{2x} + 1}\end{aligned}$$

Putting, $e^x = t \Rightarrow e^x dx = dt$

Changing the limit, we have

When $x = 0 \quad \therefore t = e^0 = 1$

When $x = 1 \quad \therefore t = e^1 = e$

Now,

$$\begin{aligned}I &= \int_1^e \frac{dt}{t^2 + 1} = [\tan^{-1} t]_1^e \\ &= [\tan^{-1} e - \tan^{-1}(1)] = \tan^{-1} e - \frac{\pi}{4}\end{aligned}$$

Therefore, $I = \tan^{-1} e - \frac{\pi}{4}$

Question 30

$$\int_0^{\frac{\pi}{2}} \frac{\tan x dx}{1 + m^2 \tan^2 x}$$

Solution

$$\begin{aligned}
 \text{Let, } I &= \int_0^{\pi/2} \frac{\tan x}{1 + m^2 \tan^2 x} dx \\
 &= \int_0^{\pi/2} \frac{\frac{\sin x}{\cos x}}{1 + m^2 \frac{\sin^2 x}{\cos^2 x}} dx = \int_0^{\pi/2} \frac{\frac{\sin x}{\cos x}}{\frac{\cos^2 x + m^2 \sin^2 x}{\cos^2 x}} dx \\
 &= \int_0^{\pi/2} \frac{\sin x \cos x}{\cos^2 x + m^2 \sin^2 x} dx = \int_0^{\pi/2} \frac{\sin x \cos x}{1 - \sin^2 x + m^2 \sin^2 x} dx \\
 &= \int_0^{\pi/2} \frac{\sin x \cos x}{1 - \sin^2 x (1 - m^2)} dx
 \end{aligned}$$

Putting, $\sin^2 x = t$

$$2 \sin x \cos x dx = dt$$

$$\sin x \cos x dx = \frac{dt}{2}$$

Changing the limits we get,

$$\text{When } x = 0 \quad \therefore t = \sin^2 0 = 0; \quad \text{When } x = \frac{\pi}{2} \quad \therefore t = \sin^2 \frac{\pi}{2} = 1$$

$$\text{Now, } I = \frac{1}{2} \int_0^1 \frac{dt}{1 - (1 - m^2)t}$$

$$\begin{aligned}
 I &= \frac{1}{2} \int_0^1 \frac{dt}{1 + (m^2 - 1)t} = \frac{1}{2} \left[\frac{\log [1 + (m^2 - 1)t]}{m^2 - 1} \right]_0^1 \\
 &= \frac{1}{2(m^2 - 1)} [\log (1 + m^2 - 1) - \log (1)] = \frac{\log |m^2|}{2(m^2 - 1)}
 \end{aligned}$$

$$\text{Therefore, } I = \frac{\log |m^2|}{2(m^2 - 1)} = \frac{\log |m|}{m^2 - 1}.$$

Question 31

$$\int_1^2 \frac{dx}{\sqrt{(x-1)(2-x)}}$$

Solution

$$\begin{aligned}
 \text{Let } I &= \int_1^2 \frac{dx}{\sqrt{(x-1)(2-x)}} \\
 &= \int_1^2 \frac{dx}{\sqrt{2x - x^2 - 2 + x}} = \int_1^2 \frac{dx}{\sqrt{-x^2 + 3x - 2}} \\
 &= \int_1^2 \frac{dx}{\sqrt{-(x^2 - 3x + 2)}} \\
 &= \int_1^2 \frac{dx}{\sqrt{-\left(x^2 - 3x + \frac{9}{4} - \frac{9}{4} + 2\right)}} \quad \text{[Making perfect square]} \\
 &= \int_1^2 \frac{dx}{\sqrt{-\left[\left(x - \frac{3}{2}\right)^2 - \frac{1}{4}\right]}} = \int_1^2 \frac{dx}{\sqrt{\frac{1}{4} - \left(x - \frac{3}{2}\right)^2}} \\
 &= \int_1^2 \frac{dx}{\sqrt{\left(\frac{1}{2}\right)^2 - \left(x - \frac{3}{2}\right)^2}} = \left[\sin^{-1} \left(\frac{x - \frac{3}{2}}{\frac{1}{2}} \right) \right]_1^2 \\
 &= \left[\sin^{-1} \left(\frac{2x - 3}{1} \right) \right]_1^2 = \sin^{-1}(4 - 3) - \sin^{-1}(2 - 3) \\
 &= \sin^{-1}(1) - \sin^{-1}(-1) = \sin^{-1}(1) + \sin^{-1}(1) \\
 &= 2 \sin^{-1}(1) = 2 \times \frac{\pi}{2} = \pi
 \end{aligned}$$

Therefore, $I = \pi$