

Chapter 6: Application of Derivatives

NCERT Exemplar · Class 12 · Maths

Questions

Question 1

A spherical ball of salt is dissolving in water in such a manner that the rate of decrease of the volume at any instant is proportional to the surface. Prove that the radius is decreasing at a constant rate.

Solution

Given, a spherical ball of salt

Then, the volume of ball $V = \frac{4}{3} \pi r^3$ where r = radius of the ball

Now, according to the question we have

$dV/dt \propto S$, where S = surface area of the ball

$$\begin{aligned}\frac{d}{dt} \left(\frac{4}{3} \pi r^3 \right) &\propto 4\pi r^2 & [\because S = 4\pi r^2] \\ \frac{4}{3} \pi \cdot 3r^2 \cdot \frac{dr}{dt} &\propto 4\pi r^2 \\ 4\pi r^2 \cdot \frac{dr}{dt} &= K \cdot 4\pi r^2 \quad (K = \text{Constant of proportionality}) \\ \frac{dr}{dt} &= K \cdot \frac{4\pi r^2}{4\pi r^2} \\ \frac{dr}{dt} &= K \cdot 1 = K\end{aligned}$$

Therefore, the radius of the ball is decreasing at constant rate.

Question 2

If the area of a circle increases at a uniform rate, then prove that perimeter varies inversely as the radius.

Solution

We know that the area of circle, $A = \pi r^2$, where r = radius of the circle

And, perimeter = $2\pi r$

According to the question, we have

$$\begin{aligned}\frac{dA}{dt} &= K, \text{ where } K = \text{constant} \\ \Rightarrow \frac{d}{dt}(\pi r^2) &= K \Rightarrow \pi \cdot 2r \cdot \frac{dr}{dt} = K \\ \text{So, } \frac{dr}{dt} &= \frac{K}{2\pi r}\end{aligned}$$

Now, perimeter $c = 2\pi r$

Differentiating w.r.t t , we get

$$\begin{aligned}\frac{dc}{dt} &= \frac{d}{dt}(2\pi r) \Rightarrow \frac{dc}{dt} = 2\pi \cdot \frac{dr}{dt} \\ \frac{dc}{dt} &= 2\pi \cdot \frac{K}{2\pi r} = \frac{K}{r} && [\text{From (1)}] \\ \frac{dc}{dt} &\propto \frac{1}{r}\end{aligned}$$

Therefore, it's seen that the perimeter of the circle varies inversely as the radius of the circle.

Question 3

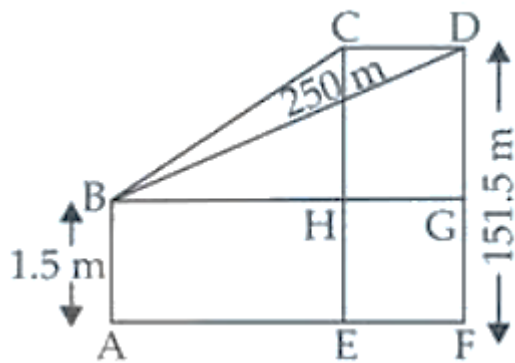
A kite is moving horizontally at a height of 151.5 meters. If the speed of kite is

10 m/s, how fast is the string being let out; when the kite is 250 m away from

the boy who is flying the kite? The height of boy is 1.5 m.

Solution

Given,



Height of the kite(h) = 151.5 m

Speed of the kite(V) = 10 m/s

Let FD be the height of the kite and AB be the height of the kite and AB be the height of the boy.

Now, let $AF = x$ m

So, $BG = AF = x$

And, $dx/dt = 10$ m/s

From the figure, it's seen that

$$GD = DF - GF = DF - AB$$

$$= (151.5 - 1.5) \text{ m} = 150 \text{ m [As } AB = GF]$$

Now, in $\triangle BDG$

$$BG^2 + GD^2 = BD^2 \text{ (By Pythagoras Theorem)}$$

$$x^2 + (150)^2 = (250)^2$$

$$x^2 + 22500 = 62500$$

$$x^2 = 62500 - 22500 = 40000$$

$$x = 200 \text{ m}$$

Let initially the length of the string be y m

So, in $\triangle BDG$

$$BG^2 + GD^2 = BD^2$$

$$x^2 + (150)^2 = y^2$$

Differentiating both sides w.r.t., t , we have

$$2x \cdot \frac{dx}{dt} + 0 = 2y \cdot \frac{dy}{dt} \quad \left[\because \frac{dx}{dt} = 10 \text{ m/s} \right]$$

$$2 \times 200 \times 10 = 2 \times 250 \times \frac{dy}{dt}$$

$$\frac{dy}{dt} = \frac{2 \times 200 \times 10}{2 \times 250} = 8 \text{ m/s}$$

Therefore, the rate of change of the length of the string is 8 m/s.

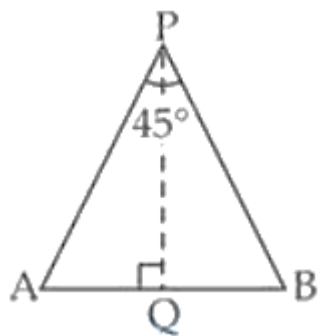
Question 4

Two men A and B start with velocities v at the same time from the junction of two roads inclined at 45° to each other. If they travel by different roads, find the rate at which they are being separated.

Solution

Let's consider P to be any point at which the two roads are inclined at an angle of 45° .

Now, two men A and B are moving along the roads PA and PB respectively with same speed 'V'.



And, let A and B be their final positions such that $AB = y$

$\angle APB = 45^\circ$ and they move with the same speed.

So, $\triangle APB$ is an isosceles triangle.

Now, draw $PQ \perp AB$.

We have, $AB = y$

So, $AQ = y/2$ and $PA = PB = x$ (assumption)

And, $\angle APQ = \angle BPQ = 45^\circ/2 = 22.5^\circ$

Now, in right $\triangle APQ$

$$\sin 22.5^\circ = AQ/AP$$

$$\Rightarrow \sin 22\frac{1}{2}^\circ = \frac{\frac{y}{2}}{x} = \frac{y}{2x} \Rightarrow y = 2x \cdot \sin 22\frac{1}{2}^\circ$$

Differentiating both sides w.r.t, t , we get

$$\begin{aligned} \frac{dy}{dt} &= 2 \cdot \frac{dx}{dt} \cdot \sin 22\frac{1}{2}^\circ \\ &= 2 \cdot V \cdot \frac{\sqrt{2-\sqrt{2}}}{2} \quad \left[\because \sin 22\frac{1}{2}^\circ = \frac{\sqrt{2-\sqrt{2}}}{2} \right] \\ &= \sqrt{2-\sqrt{2}} \text{ V unit/s} \end{aligned}$$

Therefore, the rate of their separation is

$$\sqrt{2-\sqrt{2}} \text{ V m/s}$$

Question 5

Find an angle θ , $0 < \theta < \pi/2$, which increases twice as fast as its sine.

Solution

According to the question, we have

$$\begin{aligned} \frac{d\theta}{dt} &= 2 \frac{d}{dt}(\sin \theta) \\ \Rightarrow \frac{d\theta}{dt} &= 2 \cos \theta \cdot \frac{d\theta}{dt} \Rightarrow 1 = 2 \cos \theta \\ \text{So, } \cos \theta &= \frac{1}{2} \Rightarrow \cos \theta = \cos \frac{\pi}{3} \Rightarrow \theta = \frac{\pi}{3} \end{aligned}$$

Therefore, the required angle is $\pi/3$.

Question 6

Find the approximate value of $(1.999)^5$.

Solution

$$(1.999)^5 = (2 - 0.001)^5$$

Let $x = 2$ and $\Delta x = -0.001$

Also, let $y = x^5$

Differentiating both sides w.r.t, x , we get

$$dy/dx = 5x^4 = 5(2)^4 = 80$$

Now, $\Delta y = (dy/dx) \cdot \Delta x = 80 \cdot (-0.001) = -0.080$

And, $(1.999)^5 = y + \Delta y$

$$= x^5 - 0.080 = (2)^5 - 0.080 = 32 - 0.080 = 31.92$$

Therefore, approximate value of $(1.999)^5$ is 31.92

Question 7

Find the approximate volume of metal in a hollow spherical shell whose internal and external radii are 3 cm and 3.0005 cm, respectively.

Solution

Given,

The internal radius $r = 3$ cm

And, external radius $R = r + \Delta r = 3.0005$ cm

$$\Delta r = 3.0005 - 3 = 0.0005 \text{ cm}$$

$$\text{Let } y = r^3 \Rightarrow y + \Delta y = (r + \Delta r)^3 = R^3 = (3.0005)^3$$

Differentiating both sides w.r.t., r , we get

$$\frac{dy}{dr} = 3r^2$$

$$\text{So, } \Delta y = \frac{dy}{dr} \times \Delta r = 3r^2 \times 0.0005$$

$$= 3 \times (3)^2 \times 0.0005 = 27 \times 0.0005 = 0.0135$$

$$\therefore (3.0005)^3 = y + \Delta y \quad [\text{From eq. (i)}]$$

$$= (3)^3 + 0.0135 = 27 + 0.0135 = 27.0135$$

$$\text{Volume of the shell} = \frac{4}{3}\pi[R^3 - r^3]$$

$$= \frac{4}{3}\pi[27.0135 - 27] = \frac{4}{3}\pi \times 0.0135$$

$$= 4\pi \times 0.0045 = 4 \times 3.14 \times 0.0045 = 0.018\pi \text{ cm}^3$$

Therefore, the approximate volume of the metal in the shell is $0.018\pi \text{ cm}^3$

$$1\frac{2}{3}$$

$$5\frac{1}{3}$$

Question 8

A man, 2m tall, walks at the rate of m/s towards a street light which is m above

the ground. At what rate is the tip of his shadow moving? At what rate is the length of the

$$3\frac{1}{3}$$

shadow changing when he is m from the base of the light?

Solution

Let AB is the height of street light post and CD is the height of the man such that

$$AB = 5(1/3) = 16/3 \text{ m and } CD = 2 \text{ m}$$

Let $BC = x$ length (the distance of the man from the lamp post)

And $CE = y$ is the length of the shadow of the man at any instant.

It's seen from the figure that,

$\triangle ABE \sim \triangle DCE$ [by AAA similarity criterion]

Now, taking ratio of their corresponding sides, we have

$$\begin{aligned}\frac{AB}{CD} &= \frac{BE}{CE} \Rightarrow \frac{AB}{CD} = \frac{BC + CE}{CE} \\ \frac{16/3}{2} &= \frac{x + y}{y} \Rightarrow \frac{8}{3} = \frac{x + y}{y} \\ 8y &= 3x + 3y \Rightarrow 8y - 3y = 3x \Rightarrow 5y = 3x\end{aligned}$$

Differentiating both sides w.r.t, t , we have

$$\begin{aligned}5 \cdot \frac{dy}{dt} &= 3 \cdot \frac{dx}{dt} \\ \Rightarrow \frac{dy}{dt} &= \frac{3}{5} \cdot \frac{dx}{dt} \Rightarrow \frac{dy}{dt} = \frac{3}{5} \cdot \left(-1 \frac{2}{3}\right) = \frac{3}{5} \cdot \left(\frac{-5}{3}\right) \\ &= -1 \text{ m/s} \quad [\because \text{man is moving in opposite direction}]\end{aligned}$$

So, the length of shadow is decreasing at the rate of 1 m/s.

Now, let $u = x + y$

(where, u = distance of the tip of shadow from the light post)

On differentiating both sides w.r.t. t , we get

$$\begin{aligned}2 \frac{2}{3} \\ \frac{du}{dt} &= \frac{dx}{dt} + \frac{dy}{dt} \\ &= \left(-1 \frac{2}{3} - 1\right) = -\left(\frac{5}{3} + 1\right) = -\frac{8}{3} = -2 \frac{2}{3} \text{ m/s}\end{aligned}$$

Therefore, the tip of the shadow is moving at the rate of m/s towards the light post and the length of shadow decreasing at the rate of 1 m/s.

Question 9

A swimming pool is to be drained for cleaning. If L represents the number of litres of water in the pool t seconds after the pool has been plugged off to drain and $L = 200(10 - t)^2$. How fast is the water running out at the end of 5 seconds? What is the average rate at which the water flows out during the first 5 seconds?

Solution

Given, $L = 200(10 - t)^2$ where L represents the number of liters of water in the pool.

On differentiating both the sides w.r.t, t , we get

$$dL/dt = 200 \times 2(10 - t) (-1) = -400(10 - t)$$

But, the rate at which the water is running out

$$= -dL/dt = 400(10 - t)$$

Now, rate at which the water is running after 5 seconds will be

$$= 400 \times (10 - 5) = 2000 \text{ L/s (final rate)}$$

$T = 0$ for initial rate

$$= 400 (10 - 0) = 4000 \text{ L/s}$$

So, the average rate at which the water is running out is given by

$$= (\text{Initial rate} + \text{Final rate})/2 = (4000 + 2000)/2 = 6000/2 = 3000 \text{ L/s}$$

Therefore, the required rate = 3000 L/s

Question 10

The volume of a cube increases at a constant rate. Prove that the increase in its surface area varies inversely as the length of the side.

Solution

Let's assume x to be the length of the cube.

So, the volume of the cube $V = x^3$ (1)

Given that, $dV/dt = K$

Now, on differentiating the equation (1) w.r.t. t , we get

$$dV/dt = 3x^2 \cdot dx/dt = K \text{ (constant)}$$

$$\text{So, } dx/dt = K/3x^2$$

Now,

$$\text{Surface area of the cube, } S = 6x^2$$

Differentiating both sides w.r.t. t , we get

$$\begin{aligned} \frac{ds}{dt} &= 6 \cdot 2 \cdot x \cdot \frac{dx}{dt} = 12x \cdot \frac{K}{3x^2} \\ \frac{ds}{dt} &= \frac{4K}{x} \Rightarrow \frac{ds}{dt} \propto \frac{1}{x} \quad (4K = \text{constant}) \end{aligned}$$

Therefore, the surface area of the cube varies inversely as the length of the side.

Question 11

x and y are the sides of two squares such that $y = x - x^2$. Find the rate of change of the area of second square with respect to the area of first square.

Solution

Let's consider the area of the first square $A_1 = x^2$

And, area of the second square be $A_2 = y^2$

Now, $A_1 = x^2$ and $A_2 = y^2 = (x - x^2)^2$

Differentiating both A_1 and A_2 w.r.t. t , we get

$$\begin{aligned}\frac{dA_1}{dt} &= 2x \cdot \frac{dx}{dt} \text{ and } \frac{dA_2}{dt} = 2(x - x^2)(1 - 2x) \cdot \frac{dx}{dt} \\ \text{Thus, } \frac{dA_2}{dA_1} &= \frac{\frac{dA_2}{dt}}{\frac{dA_1}{dt}} = \frac{2(x - x^2)(1 - 2x) \cdot \frac{dx}{dt}}{2x \cdot \frac{dx}{dt}} \\ &= \frac{x(1 - x)(1 - 2x)}{x} = (1 - x)(1 - 2x) \\ &= 1 - 2x - x + 2x^2 = 2x^2 - 3x + 1\end{aligned}$$

Therefore, the rate of change of area of the second square with respect to first is $2x^2 - 3x + 1$.

Question 12

Find the condition that the curves $2x = y^2$ and $2xy = k$ intersect orthogonally.

Solution

It's seen that the given curves are equation of two circles.

$$2x = y^2 \dots\dots (1) \text{ and}$$

$$2xy = k \dots\dots (2)$$

We know that, two circles intersect orthogonally if the angle between the tangents drawn to the two circles at the point of their intersection is 90° .

Now, differentiating equations (1) and (2) w.r.t. t , we get

$$2.1 = 2y \cdot \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{y} \Rightarrow m_1 = \frac{1}{y}$$

(m_1 = slope of the tangent)

$$\Rightarrow 2xy = k$$

$$\Rightarrow 2 \left[x \cdot \frac{dy}{dx} + y \cdot 1 \right] = 0$$

$$\therefore \frac{dy}{dx} = -\frac{y}{x} \Rightarrow m_2 = -\frac{y}{x}$$

(m_2 = slope of the other tangent)

If the two tangents are perpendicular to each other,

then $m_1 \times m_2 = -1$

$$\Rightarrow \frac{1}{y} \times \left(-\frac{y}{x} \right) = -1 \Rightarrow \frac{1}{x} = 1 \Rightarrow x = 1$$

$$2.1 = 2y \cdot \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{y} \Rightarrow m_1 = \frac{1}{y}$$

(m_1 = slope of the tangent)

$$\Rightarrow 2xy = k$$

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(m_2 = slope of the other tangent)

If the two tangents are perpendicular to each other,

then $m_1 \times m_2 = -1$

$$\Rightarrow \frac{1}{y} \times \left(-\frac{y}{x} \right) = -1 \Rightarrow \frac{1}{x} = 1 \Rightarrow x = 1$$

Now, solving equations (1) and (2), we have

$$y = k/2x \text{ [From (2)]}$$

Putting the value of y in equation (1),

$$2x = (k/2x)^2 \Rightarrow 2x = k^2/4x^2$$

$$8x^3 = k^2$$

$$8(1)^3 = k^2$$

$$k^2 = 8$$

Therefore, the required condition is $k^2 = 8$.

Question 13

Prove that the curves $xy = 4$ and $x^2 + y^2 = 8$ touch each other.

Solution

Given curves are equations of two circles,

$$xy = 4 \dots (i) \text{ and}$$

$$x^2 + y^2 = 8 \dots (ii)$$

Different equation (i) w.r.t., x

$$\begin{aligned}
 x \cdot \frac{dy}{dx} + y \cdot 1 &= 0 \\
 \Rightarrow \frac{dy}{dx} &= -\frac{y}{x} \Rightarrow m_1 = -\frac{y}{x} \quad \dots(iii)
 \end{aligned}$$

where, m_1 is the slope of the tangent to the curve.

Differentiating eq. (ii) w.r.t. x

$$2x + 2y \cdot \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y} \Rightarrow m_2 = -\frac{x}{y}$$

where, m_2 is the slope of the tangent to the circle.

To find the point of contact of the two circles

$$m_1 = m_2 \Rightarrow -\frac{y}{x} = -\frac{x}{y} \Rightarrow x^2 = y^2$$

Putting the value of y^2 in eq. (ii)

$$x^2 + x^2 = 8 \Rightarrow 2x^2 = 8 \Rightarrow x^2 = 4$$

Thus, $x = \pm 2$

And, $x^2 = y^2 \Rightarrow y = \pm 2$

Therefore, the point of contact of the two circles are (2, 2) and (-2, 2).

Question 14

Find the co-ordinates of the point on the curve $\sqrt{x} + \sqrt{y} = 4$ at which tangent is equally inclined to the axes.

Solution

Equation of the curve is given by, $\sqrt{x} + \sqrt{y} = 4$

Now, let (x_1, y_1) be the required point on the curve

So, $\sqrt{x_1} + \sqrt{y_1} = 4$

On differentiating on both the sides w.r.t. x_1 , we get

$$\begin{aligned}\frac{d}{dx_1} \sqrt{x_1} + \frac{d}{dx_1} \sqrt{y_1} &= \frac{d}{dx_1} (4) \\ \frac{1}{2\sqrt{x_1}} + \frac{1}{2\sqrt{y_1}} \cdot \frac{dy_1}{dx_1} &= 0 \\ \Rightarrow \frac{1}{\sqrt{x_1}} + \frac{1}{\sqrt{y_1}} \cdot \frac{dy_1}{dx_1} &= 0 \Rightarrow \frac{dy_1}{dx_1} = -\frac{\sqrt{y_1}}{\sqrt{x_1}} \quad \dots(i)\end{aligned}$$

Since the tangent to the given curve at (x_1, y_1) is equally inclined to the axes.

$\therefore$ Slope of the tangent $\frac{dy_1}{dx_1} = \pm \tan \frac{\pi}{4} = \pm 1$
So, from eq. (i) we get

$$-\frac{\sqrt{y_1}}{\sqrt{x_1}} = \pm 1$$

On squaring on both the sides, we get

$$y_1/x_1 = 1 \Rightarrow y_1 = x_1$$

Now, putting the value of y_1 in the given equation of the curve.

$$\sqrt{x_1} + \sqrt{y_1} = 4$$

$$\sqrt{x_1} + \sqrt{x_1} = 4 \Rightarrow 2\sqrt{x_1} = 4 \Rightarrow \sqrt{x_1} = 2 \Rightarrow x_1 = 4$$

$$\text{As } y_1 = x_1$$

$$\Rightarrow y_1 = 4$$

Therefore, the required point is $(4, 4)$.

Question 15

Find the angle of intersection of the curves $y = 4 - x^2$ and $y = x^2$.

Solution

The given curves are $y = 4 - x^2$ (i) and $y = x^2$ (ii)

And, we know that the angle of intersection of two curves is equal to the angle between the tangents drawn to the curves at their point of intersection.

Now, differentiating equations (i) and (ii) w.r.t x , we have

$$dy/dx = -2x \Rightarrow m_1 = -2x$$

m_1 is the slope of the tangent to the curve (i).

$$\text{and } dy/dx = 2x \Rightarrow m_2 = 2x$$

m_2 is the slope of the tangent to the curve (ii).

$$\text{So, } m_1 = -2x \text{ and } m_2 = 2x$$

On solving equation (i) and (ii), we get

$$4 - x^2 = x^2 \Rightarrow 2x^2 = 4 \Rightarrow x^2 = 2 \Rightarrow x = \pm\sqrt{2}$$

$$\text{So, } m_1 = -2x = -2\sqrt{2} \text{ and } m_2 = 2x = 2\sqrt{2}$$

Let θ be the angle of intersection of two curves

$$\begin{aligned} \text{So, } \tan \theta &= \left| \frac{m_2 - m_1}{1 + m_1 m_2} \right| \\ &= \left| \frac{2\sqrt{2} + 2\sqrt{2}}{1 - (2\sqrt{2})(2\sqrt{2})} \right| = \left| \frac{4\sqrt{2}}{1 - 8} \right| = \left| \frac{4\sqrt{2}}{-7} \right| = \frac{4\sqrt{2}}{7} \\ \therefore \theta &= \tan^{-1} \left(\frac{4\sqrt{2}}{7} \right) \end{aligned}$$

Therefore, the required angle is $\tan^{-1} \left(\frac{4\sqrt{2}}{7} \right)$.

Question 16

Prove that the curves $y^2 = 4x$ and $x^2 + y^2 - 6x + 1 = 0$ touch each other at the point (1, 2).

Solution

Given curve equations are: $y^2 = 4x$ (1) and $x^2 + y^2 - 6x + 1 = 0$ (2)

Now, differentiating (i) w.r.t. x , we get

$$2y \cdot (dy/dx) = 4 \Rightarrow dy/dx = 2/y$$

Slope of tangent at $(1, 2)$, $m_1 = 2/2 = 1$

Differentiating (ii) w.r.t. x , we get

$$2x + 2y \cdot (dy/dx) - 6 = 0$$

$$2y \cdot dy/dx = 6 - 2x \Rightarrow dy/dx = (6 - 2x)/2y$$

Hence, the slope of the tangent at the same point $(1, 2)$

$$\Rightarrow m_2 = (6 - 2 \times 1)/(2 \times 2) = 4/4 = 1$$

It's seen that $m_1 = m_2 = 1$ at the point $(1, 2)$.

Therefore, the given circles touch each other at the same point $(1, 2)$.

Question 17

Find the equation of the normal lines to the curve $3x^2 - y^2 = 8$ which are parallel to the line $x + 3y = 4$.

Solution

Given curve, $3x^2 - y^2 = 8$

Differentiating both sides w.r.t. x , we get

$$6x - 2y \cdot dy/dx = 0 \Rightarrow -2y(dy/dx) = -6x \Rightarrow dy/dx = 3x/y$$

So, slope of the tangent to the given curve $= 3x/y$

Thus, the normal to the curve $= -1/(3x/y) = -y/3x$

Now, differentiating both sides of the given line $x + 3y = 4$, we have

$$1 + 3 \cdot (dy/dx) = 0$$

$$dy/dx = -1/3$$

As the normal to the curve is parallel to the given line $x + 3y = 4$

We have, $-y/3x = -1/3 \Rightarrow y = x$

On putting the value of y in $3x^2 - y^2 = 8$, we get

$$3x^2 - x^2 = 8$$

$$2x^2 = 8 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$$

So, $y = \pm 2$

Thus, the points on the curve are $(2, 2)$ and $(-2, -2)$.

Now, the equation of the normal to the curve at $(2, 2)$ is given by

$$\begin{aligned} y - 2 &= -\frac{1}{3}(x - 2) \\ \Rightarrow 3y - 6 &= -x + 2 \Rightarrow x + 3y = 8 \\ \text{at } (-2, -2) \quad y + 2 &= -\frac{1}{3}(x + 2) \\ \Rightarrow 3y + 6 &= -x - 2 \Rightarrow x + 3y = -8 \end{aligned}$$

Therefore, the required equations are $x + 3y = 8$ and $x + 3y = -8$.

Question 18

At what points on the curve $x^2 + y^2 - 2x - 4y + 1 = 0$, the tangents are parallel to the y -axis?

Solution

Given, the equation of the curve is $x^2 + y^2 - 2x - 4y + 1 = 0$ (i)

Differentiating both the sides w.r.t. x , we get

$$2x + 2y \cdot \frac{dy}{dx} - 2 - 4 \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow (2y - 4) \frac{dy}{dx} = 2 - 2x \Rightarrow \frac{dy}{dx} = \frac{2 - 2x}{2y - 4} \dots(ii)$$

Since the tangent to the curve is parallel to the y -axis.

$$\therefore \text{Slope } \frac{dy}{dx} = \tan \frac{\pi}{2} = \infty = \frac{1}{0}$$

So, from eq. (ii) we get

$$\frac{2 - 2x}{2y - 4} = \frac{1}{0} \Rightarrow 2y - 4 = 0 \Rightarrow y = 2$$

Now putting the value of y in equation (i), we get

$$x^2 + (2)^2 - 2x - 8 + 1 = 0$$

$$x^2 - 2x + 4 - 8 + 1 = 0$$

$$x^2 - 2x - 3 = 0 \Rightarrow x^2 - 3x + x - 3 = 0$$

$$x(x - 3) + 1(x - 3) = 0 \Rightarrow (x - 3)(x + 1) = 0$$

$$x = -1 \text{ or } 3$$

Therefore, the required points are $(-1, 2)$ and $(3, 2)$.

Question 19

Show that the line $x/a + y/b = 1$, touches the curve $y = b \cdot e^{-x/a}$ at the point where the curve intersects the axis of y .

Solution

Given curve equation, $y = b \cdot e^{-x/a}$ and line equation $x/a + y/b = 1$

Now, let the coordinates of the point where the curve intersects the y -axis be $(0, y_1)$

Now differentiating $y = b \cdot e^{-x/a}$ both sides w.r.t. x , we get

$$\frac{dy}{dx} = b \cdot e^{-x/a} \left(-\frac{1}{a} \right) = -\frac{b}{a} \cdot e^{-x/a}$$

So, the slope of the tangent, $m_1 = -\frac{b}{a} e^{-x/a}$.

Differentiating $\frac{x}{a} + \frac{y}{b} = 1$ both sides w.r.t. x , we get

$$\frac{1}{a} + \frac{1}{b} \cdot \frac{dy}{dx} = 0$$

So, the slope of the line, $m_2 = \frac{-b}{a}$.

If the line touches the curve, then $m_1 = m_2$

$$\frac{-b}{a} \cdot e^{-x/a} = \frac{-b}{a} \Rightarrow e^{-x/a} = 1$$

$$\frac{-x}{a} \log e = \log 1 \quad (\text{Taking log on both sides})$$

$$\frac{-x}{a} = 0 \Rightarrow x = 0$$

Putting $x = 0$ in equation $y = b \cdot e^{-x/a}$

$$\Rightarrow y = b \cdot e^0 = b$$

$$\frac{dy}{dx} = b \cdot e^{-x/a} \left(-\frac{1}{a} \right) = -\frac{b}{a} \cdot e^{-x/a}$$

So, the slope of the tangent, $m_1 = -\frac{b}{a} e^{-x/a}$.

Differentiating $\frac{x}{a} + \frac{y}{b} = 1$ both sides w.r.t. x , we get

$$\frac{1}{a} + \frac{1}{b} \cdot \frac{dy}{dx} = 0$$

So, the slope of the line, $m_2 = \frac{-b}{a}$.

If the line touches the curve, then $m_1 = m_2$

$$\frac{-b}{a} \cdot e^{-x/a} = \frac{-b}{a} \Rightarrow e^{-x/a} = 1$$

$$\frac{-x}{a} \log e = \log 1 \quad \text{(Taking log on both sides)}$$

$$\frac{-x}{a} = 0 \Rightarrow x = 0$$

Putting $x = 0$ in equation $y = b \cdot e^{-x/a}$

$$\Rightarrow y = b \cdot e^0 = b$$

Therefore, the equation of the curve intersects at $(0, b)$ which is on the y -axis.

Question 20

Show that $f(x) = 2x + \cot^{-1} x + \log [\sqrt{(1+x^2)} - x]$ is increasing in $\mathbb{R}$.

Solution

Given,

$$f(x) = 2x + \cot^{-1} x + \log [\sqrt{(1+x^2)} - x]$$

Differentiating both sides w.r.t. x , we get

$$\begin{aligned}
 f'(x) &= 2 - \frac{1}{1+x^2} + \frac{1}{\sqrt{1+x^2}-x} \times \frac{d}{dx} (\sqrt{1+x^2}-x) \\
 &= 2 - \frac{1}{1+x^2} + \frac{\left(\frac{1}{2\sqrt{1+x^2}} \times (2x-1) \right)}{\sqrt{1+x^2}-x} \\
 &= 2 - \frac{1}{1+x^2} + \frac{x - \sqrt{1+x^2}}{\sqrt{1+x^2} (\sqrt{1+x^2}-x)} \\
 &= 2 - \frac{1}{1+x^2} - \frac{(\sqrt{1+x^2}-x)}{\sqrt{1+x^2} (\sqrt{1+x^2}-x)} \\
 &= 2 - \frac{1}{1+x^2} - \frac{1}{\sqrt{1+x^2}}
 \end{aligned}$$

For increasing function, $f'(x) \geq 0$

$$\begin{aligned}
 2 - \frac{1}{1+x^2} - \frac{1}{\sqrt{1+x^2}} &\geq 0 \\
 \frac{2(1+x^2) - 1 + \sqrt{1+x^2}}{(1+x^2)} &\geq 0 \Rightarrow 2 + 2x^2 - 1 + \sqrt{1+x^2} \geq 0 \\
 2x^2 + 1 + \sqrt{1+x^2} &\geq 0 \Rightarrow 2x^2 + 1 \geq -\sqrt{1+x^2}
 \end{aligned}$$

On squaring both the sides, we get

$$4x^4 + 1 + 4x^2 \geq 1 + x^2$$

$$4x^4 + 4x^2 - x^2 \geq 0$$

$$4x^4 + 3x^2 \geq 0$$

$$x^2(4x^2 + 3) \geq 0$$

The above is true for any value of $x \in \mathbb{R}$.

Therefore, the given function is an increasing function over $\mathbb{R}$.

Question 21

Show that for $a \geq 1$, $f(x) = \sqrt{3} \sin x - \cos x - 2ax + b$ is decreasing in $\mathbb{R}$.

Solution

Given,

$$f(x) = \sqrt{3} \sin x - \cos x - 2ax + b, \quad a \geq 1$$

On differentiating both sides w.r.t. x , we get

$$f'(x) = \sqrt{3} \cos x + \sin x - 2a$$

For increasing function, $f'(x) < 0$

$$\text{So, } \sqrt{3} \cos x + \sin x - 2a < 0$$

$$2 \left(\frac{\sqrt{3}}{2} \cos x + \frac{1}{2} \sin x \right) - 2a < 0$$

$$\frac{\sqrt{3}}{2} \cos x + \frac{1}{2} \sin x - a < 0$$

$$\left(\cos \frac{\pi}{6} \cos x + \sin \frac{\pi}{6} \sin x \right) - a < 0$$

$$\cos \left(x - \frac{\pi}{6} \right) - a < 0$$

As $\cos x \in [-1, 1]$ and $a \geq 1$

$$\text{Hence, } f'(x) < 0$$

Therefore, the given function is decreasing in $\mathbb{R}$.

Question 22

Show that $f(x) = \tan^{-1}(\sin x + \cos x)$ is an increasing function in $(0, \pi/4)$.

Solution

Given, $f(x) = \tan^{-1}(\sin x + \cos x)$ in $(0, \pi/4)$.

Differentiating both sides w.r.t. x , we got

$$f'(x) = \frac{1}{1 + (\sin x + \cos x)^2} \cdot \frac{d}{dx} (\sin x + \cos x)$$

$$f'(x) = \frac{1 \times (\cos x - \sin x)}{1 + (\sin x + \cos x)^2}$$

$$f'(x) = \frac{\cos x - \sin x}{1 + \sin^2 x + \cos^2 x + 2 \sin x \cos x}$$

$$\Rightarrow f'(x) = \frac{\cos x - \sin x}{1 + 1 + 2 \sin x \cos x} \Rightarrow f'(x) = \frac{\cos x - \sin x}{2 + 2 \sin x \cos x}$$

For an increasing function $f'(x) \geq 0$

$$\text{So, } \frac{\cos x - \sin x}{2 + 2 \sin x \cos x} \geq 0$$

$$\cos x - \sin x \geq 0 \quad \left[\because (2 + \sin 2x) \geq 0 \text{ in } \left(0, \frac{\pi}{4}\right) \right]$$

$$\Rightarrow \cos x \geq \sin x, \text{ which is true for } \left(0, \frac{\pi}{4}\right)$$

$$f'(x) = \frac{1}{1 + (\sin x + \cos x)^2} \cdot \frac{d}{dx} (\sin x + \cos x)$$

$$f'(x) = \frac{1 \times (\cos x - \sin x)}{1 + (\sin x + \cos x)^2}$$

$$f'(x) = \frac{\cos x - \sin x}{1 + \sin^2 x + \cos^2 x + 2 \sin x \cos x}$$

$$\Rightarrow f'(x) = \frac{\cos x - \sin x}{1 + 1 + 2 \sin x \cos x} \Rightarrow f'(x) = \frac{\cos x - \sin x}{2 + 2 \sin x \cos x}$$

For an increasing function $f'(x) \geq 0$

$$\text{So, } \frac{\cos x - \sin x}{2 + 2 \sin x \cos x} \geq 0$$

$$\cos x - \sin x \geq 0 \quad \left[\because (2 + \sin 2x) \geq 0 \text{ in } \left(0, \frac{\pi}{4}\right) \right]$$

$$\Rightarrow \cos x \geq \sin x, \text{ which is true for } \left(0, \frac{\pi}{4}\right)$$

Therefore, the given function $f(x)$ is an increasing function in $(0, \pi/4)$.

Question 23

At what point, the slope of the curve $y = -x^3 + 3x^2 + 9x - 27$ is maximum? Also find the maximum slope.

Solution

Given, curve $y = -x^3 + 3x^2 + 9x - 27$

Differentiating both sides w.r.t. x , we get

$$dy/dx = -3x^2 + 6x + 9$$

Let slope of the curve $dy/dx = z$

$$\text{So, } z = -3x^2 + 6x + 9$$

Differentiating both sides w.r.t. x , we get

$$dz/dx = -6x + 6$$

For local maxima and local minima,

$$dz/dx = 0$$

$$-6x + 6 = 0 \Rightarrow x = 1$$

$$d^2z/dx^2 = -6 < 0 \text{ Maxima}$$

Putting $x = 1$ in equation of the curve $y = (-1)^3 + 3(1)^2 + 9(1) - 27$

$$= -1 + 3 + 9 - 27 = -16$$

$$\text{Maximum slope} = -3(1)^2 + 6(1) + 9 = 12$$

Therefore, $(1, -16)$ is the point at which the slope of the given curve is maximum and maximum slope = 12.

Question 24

Prove that $f(x) = \sin x + \sqrt{3} \cos x$ has maximum value at $x = \pi/6$.

Solution

$$\begin{aligned}\text{Given, } f(x) &= \sin x + \sqrt{3} \cos x = 2 \left(\frac{1}{2} \sin x + \frac{\sqrt{3}}{2} \cos x \right) \\ &= 2 \left(\cos \frac{\pi}{3} \sin x + \sin \frac{\pi}{3} \cos x \right) = 2 \sin \left(x + \frac{\pi}{3} \right)\end{aligned}$$

$$f'(x) = 2 \cos \left(x + \frac{\pi}{3} \right); f''(x) = -2 \sin \left(x + \frac{\pi}{3} \right)$$

$$\begin{aligned}f''(x)_{x=\frac{\pi}{6}} &= -2 \sin \left(\frac{\pi}{6} + \frac{\pi}{3} \right) \\ &= -2 \sin \frac{\pi}{2} = -2.1 = -2 < 0 \text{ (Maxima)} \\ &= -2 \times \frac{\sqrt{3}}{2} = -\sqrt{3} < 0 \text{ (Maxima)}\end{aligned}$$

Maximum value of the function at $x = \frac{\pi}{6}$ is

$$\sin \frac{\pi}{6} + \sqrt{3} \cos \frac{\pi}{6} = \frac{1}{2} + \sqrt{3} \cdot \frac{\sqrt{3}}{2} = 2$$

$$\begin{aligned}\text{Given, } f(x) &= \sin x + \sqrt{3} \cos x = 2 \left(\frac{1}{2} \sin x + \frac{\sqrt{3}}{2} \cos x \right) \\ &= 2 \left(\cos \frac{\pi}{3} \sin x + \sin \frac{\pi}{3} \cos x \right) = 2 \sin \left(x + \frac{\pi}{3} \right)\end{aligned}$$

$$f'(x) = 2 \cos \left(x + \frac{\pi}{3} \right); f''(x) = -2 \sin \left(x + \frac{\pi}{3} \right)$$

$$\begin{aligned}f''(x)_{x=\frac{\pi}{6}} &= -2 \sin \left(\frac{\pi}{6} + \frac{\pi}{3} \right) \\ &= -2 \sin \frac{\pi}{2} = -2.1 = -2 < 0 \text{ (Maxima)} \\ &= -2 \times \frac{\sqrt{3}}{2} = -\sqrt{3} < 0 \text{ (Maxima)}\end{aligned}$$

Maximum value of the function at $x = \frac{\pi}{6}$ is

$$\sin \frac{\pi}{6} + \sqrt{3} \cos \frac{\pi}{6} = \frac{1}{2} + \sqrt{3} \cdot \frac{\sqrt{3}}{2} = 2$$

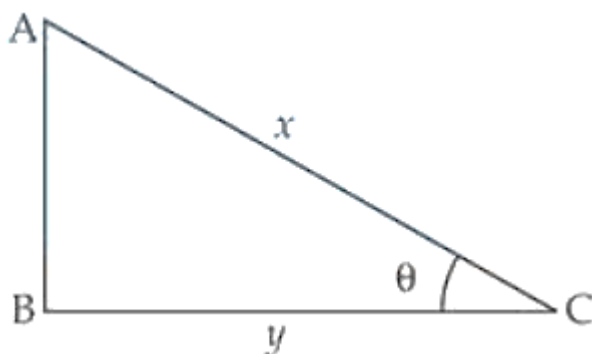
Therefore, the given function has maximum value at $x = \pi/6$ and the maximum value is 2.

Long Answer (L.A.)

Question 25

If the sum of the lengths of the hypotenuse and a side of a right angled triangle is given, show that the area of the triangle is maximum when the angle between them is $\pi/3$.

Solution



Let $\triangle ABC$ be the right-angled triangle in which $\angle B = 90^\circ$

Let $AC = x$, $BC = y$

So, $AB = \sqrt{x^2 + y^2}$

$\angle ACB = \theta$

Let $z = x + y$ (given)

Now, the area of $\triangle ABC = \frac{1}{2} \times AB \times BC$

$$\Rightarrow A = \frac{1}{2} y \cdot \sqrt{x^2 - y^2} \Rightarrow A = \frac{1}{2} y \cdot \sqrt{(Z - y)^2 - y^2}$$

Squaring both sides, we get

$$A^2 = \frac{1}{4} y^2 [(Z - y)^2 - y^2] \Rightarrow A^2 = \frac{1}{4} y^2 [Z^2 + y^2 - 2Zy - y^2]$$

$$\text{So, } P = \frac{1}{4} y^2 [Z^2 - 2Zy] \Rightarrow P = \frac{1}{4} [y^2 Z^2 - 2Zy^3] \quad [A^2 = P]$$

Differentiating both sides w.r.t. y we get

$$\frac{dP}{dy} = \frac{1}{4} [2yZ^2 - 6Zy^2] \quad \dots(i)$$

For local maxima and local minima, $\frac{dP}{dy} = 0$

$$\therefore \frac{1}{4} (2yZ^2 - 6Zy^2) = 0$$

$$\frac{2yZ}{4} (Z - 3y) = 0 \Rightarrow yZ(Z - 3y) = 0$$

$$yZ \neq 0 \quad (\because y \neq 0 \text{ and } Z \neq 0)$$

$$\therefore Z - 3y = 0$$

$$y = \frac{Z}{3} \Rightarrow y = \frac{x + y}{3} \quad (\because Z = x + y)$$

$$3y = x + y \Rightarrow 3y - y = x \Rightarrow 2y = x$$

$$\frac{y}{x} = \frac{1}{2} \Rightarrow \cos \theta = \frac{1}{2}$$

$$\text{Thus, } \theta = \frac{\pi}{3}$$

Differentiating eq. (i) w.r.t. y , we have $\frac{d^2P}{dy^2} = \frac{1}{4} [2Z^2 - 12Zy]$

$$\begin{aligned} \frac{d^2P}{dy^2} \text{ at } y = \frac{Z}{3} &= \frac{1}{4} \left[2Z^2 - 12Z \cdot \frac{Z}{3} \right] \\ &= \frac{1}{4} [2Z^2 - 4Z^2] = \frac{-Z^2}{2} < 0 \text{ Maxima} \end{aligned}$$

$$\Rightarrow A = \frac{1}{2} y \cdot \sqrt{x^2 - y^2} \Rightarrow A = \frac{1}{2} y \cdot \sqrt{(Z - y)^2 - y^2}$$

Squaring both sides, we get

$$A^2 = \frac{1}{4} y^2 [(Z - y)^2 - y^2] \Rightarrow A^2 = \frac{1}{4} y^2 [Z^2 + y^2 - 2Zy - y^2]$$

$$\text{So, } P = \frac{1}{4} y^2 [Z^2 - 2Zy] \Rightarrow P = \frac{1}{4} [y^2 Z^2 - 2Zy^3] \quad [A^2 = P]$$

Differentiating both sides w.r.t. y we get

$$\frac{dP}{dy} = \frac{1}{4} [2yZ^2 - 6Zy^2] \quad \dots(i)$$

For local maxima and local minima, $\frac{dP}{dy} = 0$

$$\therefore \frac{1}{4} (2yZ^2 - 6Zy^2) = 0$$

$$\frac{2yZ}{4} (Z - 3y) = 0 \Rightarrow yZ(Z - 3y) = 0$$

$$yZ \neq 0 \quad (\because y \neq 0 \text{ and } Z \neq 0)$$

$$\therefore Z - 3y = 0$$

$$y = \frac{Z}{3} \Rightarrow y = \frac{x + y}{3} \quad (\because Z = x + y)$$

$$3y = x + y \Rightarrow 3y - y = x \Rightarrow 2y = x$$

$$\frac{y}{x} = \frac{1}{2} \Rightarrow \cos \theta = \frac{1}{2}$$

$$\text{Thus, } \theta = \frac{\pi}{3}$$

Differentiating eq. (i) w.r.t. y , we have $\frac{d^2P}{dy^2} = \frac{1}{4} [2Z^2 - 12Zy]$

$$\begin{aligned} \frac{d^2P}{dy^2} \text{ at } y = \frac{Z}{3} &= \frac{1}{4} \left[2Z^2 - 12Z \cdot \frac{Z}{3} \right] \\ &= \frac{1}{4} [2Z^2 - 4Z^2] = \frac{-Z^2}{2} < 0 \text{ Maxima} \end{aligned}$$

Therefore, the area of the given triangle is maximum when the angle between its hypotenuse and a side is $\pi/3$.

Question 26

Find the points of local maxima, local minima and the points of inflection of the function $f(x) = x^5 - 5x^4 + 5x^3 - 1$. Also find the corresponding local maximum and local minimum values.

Solution

Given, $f(x) = x^5 - 5x^4 + 5x^3 - 1$

Differentiating the function,

$$f'(x) = 5x^4 - 20x^3 + 15x^2$$

For local maxima and local minima, $f'(x) = 0$

$$\begin{aligned} 5x^4 - 20x^3 + 15x^2 &= 0 \Rightarrow 5x^2(x^2 - 4x + 3) = 0 \\ \Rightarrow 5x^2(x^2 - 3x - x + 3) &= 0 \Rightarrow x^2(x - 3)(x - 1) = 0 \\ \therefore x &= 0, x = 1 \text{ and } x = 3 \end{aligned}$$

Now $f''(x) = 20x^3 - 60x^2 + 30x$

$\Rightarrow f''(x)_{\text{at } x=0} = 20(0)^3 - 60(0)^2 + 30(0) = 0$ which is neither maxima nor minima.

$\therefore f(x)$ has the point of inflection at $x = 0$

$$\begin{aligned} f''(x)_{\text{at } x=1} &= 20(1)^3 - 60(1)^2 + 30(1) \\ &= 20 - 60 + 30 = -10 < 0 \text{ Maxima} \end{aligned}$$

$$\begin{aligned} f''(x)_{\text{at } x=3} &= 20(3)^3 - 60(3)^2 + 30(3) \\ &= 540 - 540 + 90 = 90 > 0 \text{ Minima} \end{aligned}$$

The maximum value of the function at $x = 1$

$$f(x) = (1)^5 - 5(1)^4 + 5(1)^3 - 1$$

$$= 1 - 5 + 5 - 1 = 0$$

The minimum value at $x = 3$ is

$$f(x) = (3)^5 - 5(3)^4 + 5(3)^3 - 1$$

$$= 243 - 405 + 135 - 1$$

$$= 378 - 406 = -28$$

Therefore, the function has its maxima at $x = 1$ and the maximum value = 0 and its has minimum value at $x = 3$ and its minimum value is -28.

Question 27

A telephone company in a town has 500 subscribers on its list and collects fixed charges of Rs 300/- per subscriber per year. The company proposes to increase the annual subscription and it is believed that for every increase of Rs 1/- one subscriber will discontinue the service. Find what increase will bring maximum profit?

Solution

Let's consider that the company increases the annual subscription by Rs x .

So, x is the number of subscribers who discontinue the services.

$$\text{Total revenue, } R(x) = (500 - x)(300 + x)$$

$$= 150000 + 500x - 300x - x^2$$

$$= -x^2 + 200x + 150000$$

Differentiating both sides w.r.t. x , we get $R'(x) = -2x + 200$

For local maxima and local minima, $R'(x) = 0$

$$-2x + 200 = 0 \Rightarrow x = 100$$

$$R''(x) = -2 < 0 \text{ Maxima}$$

So, $R(x)$ is maximum at $x = 100$

Therefore, in order to get maximum profit, the company should increase its annual subscription by Rs 100.

Question 28

If the straight-line $x \cos \alpha + y \sin \alpha = p$ touches the curve $x^2/a^2 + y^2/b^2 = 1$, then prove that $a^2 \cos^2 \alpha + b^2 \sin^2 \alpha = p^2$.

Solution

The given curve is $x^2/a^2 + y^2/b^2 = 1$ and the straight-line $x \cos \alpha + y \sin \alpha = p$

Differentiating equation (i) w.r.t. x , we get

$$\frac{1}{a^2} \cdot 2x + \frac{1}{b^2} \cdot 2y \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{x}{a^2} + \frac{y}{b^2} \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{b^2}{a^2} \cdot \frac{x}{y}$$

So the slope of the curve = $-\frac{b^2}{a^2} \cdot \frac{x}{y}$

Now differentiating eq. (ii) w.r.t. x , we have

$$\cos \alpha + \sin \alpha \cdot \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = \frac{-\cos \alpha}{\sin \alpha} = -\cot \alpha$$

So, the slope of the straight line = $-\cot \alpha$

If the line is the tangent to the curve, then

$$-\frac{b^2}{a^2} \cdot \frac{x}{y} = -\cot \alpha \Rightarrow \frac{x}{y} = \frac{a^2}{b^2} \cdot \cot \alpha \Rightarrow x = \frac{a^2}{b^2} \cot \alpha \cdot y$$

Now from eq. (ii) we have $x \cos \alpha + y \sin \alpha = p$

$$\frac{a^2}{b^2} \cdot \cot \alpha \cdot y \cdot \cos \alpha + y \sin \alpha = p$$

$$a^2 \cot \alpha \cdot \cos \alpha y + b^2 \sin \alpha y = b^2 p$$

$$a^2 \frac{\cos \alpha}{\sin \alpha} \cdot \cos \alpha y + b^2 \sin \alpha y = b^2 p$$

$$a^2 \cos^2 \alpha y + b^2 \sin^2 \alpha y = b^2 \sin \alpha p$$

$$a^2 \cos^2 \alpha + b^2 \sin^2 \alpha = \frac{b^2}{y} \cdot \sin \alpha \cdot p$$

$$\Rightarrow a^2 \cos^2 \alpha + b^2 \sin^2 \alpha = p \cdot p \quad \left[\because \frac{b^2}{y} \sin \alpha = p \right]$$

$$\frac{1}{a^2} \cdot 2x + \frac{1}{b^2} \cdot 2y \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{x}{a^2} + \frac{y}{b^2} \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{b^2}{a^2} \cdot \frac{x}{y}$$

So the slope of the curve = $-\frac{b^2}{a^2} \cdot \frac{x}{y}$

Now differentiating eq. (ii) w.r.t. x , we have

$$\cos \alpha + \sin \alpha \cdot \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = \frac{-\cos \alpha}{\sin \alpha} = -\cot \alpha$$

So, the slope of the straight line = $-\cot \alpha$

If the line is the tangent to the curve, then

$$-\frac{b^2}{a^2} \cdot \frac{x}{y} = -\cot \alpha \Rightarrow \frac{x}{y} = \frac{a^2}{b^2} \cdot \cot \alpha \Rightarrow x = \frac{a^2}{b^2} \cot \alpha \cdot y$$

Now from eq. (ii) we have $x \cos \alpha + y \sin \alpha = p$

$$\frac{a^2}{b^2} \cdot \cot \alpha \cdot y \cdot \cos \alpha + y \sin \alpha = p$$

$$a^2 \cot \alpha \cdot \cos \alpha y + b^2 \sin \alpha y = b^2 p$$

$$a^2 \frac{\cos \alpha}{\sin \alpha} \cdot \cos \alpha y + b^2 \sin \alpha y = b^2 p$$

$$a^2 \cos^2 \alpha y + b^2 \sin^2 \alpha y = b^2 \sin \alpha p$$

$$a^2 \cos^2 \alpha + b^2 \sin^2 \alpha = \frac{b^2}{y} \cdot \sin \alpha \cdot p$$

$$\Rightarrow a^2 \cos^2 \alpha + b^2 \sin^2 \alpha = p \cdot p \quad \left[\because \frac{b^2}{y} \sin \alpha = p \right]$$

Therefore, $a^2 \cos^2 \alpha + b^2 \sin^2 \alpha = p^2$.

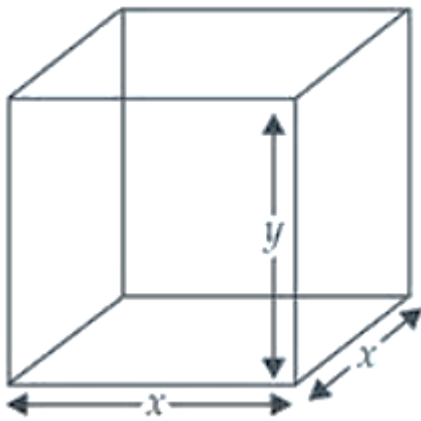
Question 29

An open box with square base is to be made of a given quantity of card board of area c^2 . Show that the maximum volume of the box is $c^3/6\sqrt{3}$ cubic units.

Solution

Let x be the length of the side of the square base of the cubical open box and y be its height.

So, the surface area of the open box



$$c^2 = x^2 + 4xy \Rightarrow y = \frac{c^2 - x^2}{4x} \quad \dots(i)$$

Now volume of the box, $V = x \times x \times y$

$$V = x^2 y$$

$$V = x^2 \left(\frac{c^2 - x^2}{4x} \right)$$

$$\Rightarrow V = \frac{1}{4}(c^2 x - x^3)$$

Differentiating both sides w.r.t. x , we get

$$\frac{dV}{dx} = \frac{1}{4}(c^2 - 3x^2) \quad \dots(ii)$$

For local maxima and local minima, $\frac{dV}{dx} = 0$

$$\therefore \frac{1}{4}(c^2 - 3x^2) = 0 \Rightarrow c^2 - 3x^2 = 0$$

$$\Rightarrow x^2 = \frac{c^2}{3}$$

$$\therefore x = \sqrt{\frac{c^2}{3}} = \frac{c}{\sqrt{3}}$$

Now again differentiating eq. (ii) w.r.t. x , we get

$$\frac{d^2V}{dx^2} = \frac{1}{4}(-6x) = \frac{-3}{2} \cdot \frac{c}{\sqrt{3}} < 0 \quad (\text{maxima})$$

Volume of the cubical box (V) = $x^2 y$

$$= x^2 \left(\frac{c^2 - x^2}{4x} \right) = \frac{c}{\sqrt{3}} \left[\frac{c^2 - \frac{c^2}{3}}{4} \right] = \frac{c}{\sqrt{3}} \times \frac{2c^2}{3 \times 4} = \frac{c^3}{6\sqrt{3}}$$

Therefore, the maximum volume of the open box is

$$\frac{c^3}{6\sqrt{3}} \text{ cubic units.}$$

Question 30

Find the dimensions of the rectangle of perimeter 36 cm which will sweep out a volume as large as possible, when revolved about one of its sides. Also find the maximum volume.

Solution

Let's consider x and y to be the length and breadth of given rectangle ABCD.

According to the question, the rectangle will be resolved about side AD which making a cylinder with radius x and height y .

So, the volume of the cylinder $V = \pi r^2 h = \pi x^2 y$ (1)



Now, perimeter of rectangle $P = 2(x + y)$

$$36 = 2(x + y)$$

$$18 = x + y$$

$$y = 18 - x \text{ (ii)}$$

Putting the value of y in the equation (i), we get

$$V = \pi x^2 (18 - x) = \pi (18x^2 - x^3)$$

Differentiating both sides w.r.t. x , we get

$$dV/dx = \pi(36x - 3x^2) \text{ (iii)}$$

For local maxima and local minima $dV/dx = 0$

$$\pi(36x - 3x^2) = 0$$

$$36x - 3x^2 = 0$$

$$3x(12 - x) = 0$$

$$x \neq 0 \text{ and } 12 - x = 0 \Rightarrow x = 12$$

From equation (ii), we have

$$y = 18 - 12 = 6$$

Differentiating equation (iii) w.r.t. x , we get

$$d^2V/dx^2 = \pi(36 - 6x)$$

At $x = 12$,

$$d^2V/dx^2 = \pi(36 - 6 \times 12) = \pi(36 - 72)$$

$$= -36\pi < 0 \text{ maxima}$$

Now, volume of the cylinder so formed $= \pi x^2 y$

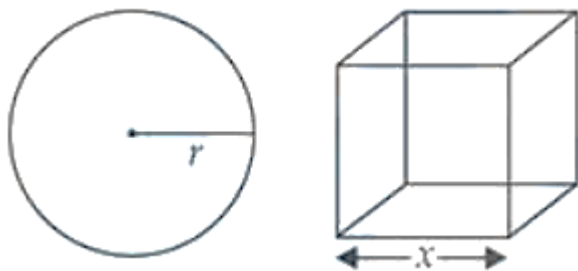
$$= \pi \times (12)^2 \times (6) = \pi(144) \times 6 = 864\pi \text{ cm}^3$$

Therefore, the required dimension are 12 cm and 6 cm and the maximum volume is $864\pi \text{ cm}^3$.

Question 31

If the sum of the surface areas of cube and a sphere is constant, what is the ratio of an edge of the cube to the diameter of the sphere, when the sum of their volumes is minimum?

Solution



Let's assume x be to the edge and r be the radius of the sphere.

$$\text{Surface area of cube} = 6x^2$$

$$\text{And, surface area of the sphere} = 4\pi r^2$$

Now, their sum is

$$6x^2 + 4\pi r^2 = K(\text{constant}) \Rightarrow r = \sqrt{\frac{K - 6x^2}{4\pi}} \quad \dots(i)$$

Volume of the cube = x^3 and the volume of sphere = $\frac{4}{3}\pi r^3$

Now,

Sum of their volumes (V) = Volume of cube
+ Volume of sphere

$$V = x^3 + \frac{4}{3}\pi r^3$$

$$\Rightarrow V = x^3 + \frac{4}{3}\pi \times \left(\frac{K - 6x^2}{4\pi} \right)^{3/2}$$

Differentiating both sides w.r.t. x , we get

$$\begin{aligned} \frac{dV}{dx} &= 3x^2 + \frac{4\pi}{3} \times \frac{3}{2} (K - 6x^2)^{1/2} (-12x) \times \frac{1}{(4\pi)^{3/2}} \\ &= 3x^2 + \frac{2\pi}{(4\pi)^{3/2}} \times (-12x) (K - 6x^2)^{1/2} \\ &= 3x^2 + \frac{1}{4\pi^{1/2}} \times (-12x) (K - 6x^2)^{1/2} \end{aligned}$$

$$\text{Thus, } \frac{dV}{dx} = 3x^2 - \frac{3x}{\sqrt{\pi}} (K - 6x^2)^{1/2} \quad \dots(ii)$$

For local maxima and local minima, $\frac{dV}{dx} = 0$

$$3x^2 - \frac{3x}{\sqrt{\pi}} (K - 6x^2)^{1/2} = 0$$

$$3x \left[x - \frac{(K - 6x^2)^{1/2}}{\sqrt{\pi}} \right] = 0$$

$$x \neq 0 \quad \text{and} \quad x - \frac{(K - 6x^2)^{1/2}}{\sqrt{\pi}} = 0$$

$$\Rightarrow x = \frac{(K - 6x^2)^{1/2}}{\sqrt{\pi}}$$

Squaring both sides, we get

$$x^2 = \frac{K - 6x^2}{\pi} \quad \Rightarrow \pi x^2 = K - 6x^2$$

$$\text{So, } \pi x^2 + 6x^2 = K \quad \Rightarrow x^2(\pi + 6) = K \quad \Rightarrow x^2 = \frac{K}{\pi + 6}$$

$$\sqrt{K}$$

Thus, $x = \sqrt{\frac{K}{\pi + 6}}$

Now putting the value of K in eq. (i), we get

$$6x^2 + 4\pi r^2 = x^2(\pi + 6)$$

$$6x^2 + 4\pi r^2 = \pi x^2 + 6x^2 \Rightarrow 4\pi r^2 = \pi x^2 \Rightarrow 4r^2 = x^2$$

$$2r = x$$

$$\therefore x:2r = 1:1$$

Now differentiating eq. (ii) w.r.t x , we have

$$\frac{d^2V}{dx^2} = 6x - \frac{3}{\sqrt{\pi}} \frac{d}{dx} [x(K - 6x^2)^{1/2}]$$

$$\begin{aligned}
&= 6x - \frac{3}{\sqrt{\pi}} \left[x \cdot \frac{1}{2\sqrt{K-6x^2}} \times (-12x) + (K-6x^2)^{1/2} \cdot 1 \right] \\
&= 6x - \frac{3}{\sqrt{\pi}} \left[\frac{-6x^2}{\sqrt{K-6x^2}} + \sqrt{K-6x^2} \right] \\
&= 6x - \frac{3}{\sqrt{\pi}} \left[\frac{-6x^2 + K - 6x^2}{\sqrt{K-6x^2}} \right] = 6x + \frac{3}{\sqrt{\pi}} \left[\frac{12x^2 - K}{\sqrt{K-6x^2}} \right] \\
\text{Put } x &= \sqrt{\frac{K}{\pi+6}} = 6\sqrt{\frac{K}{\pi+6}} + \frac{3}{\sqrt{\pi}} \left[\frac{\frac{12K}{\pi+6} - K}{\sqrt{K - \frac{6K}{\pi+6}}} \right] \\
&= 6\sqrt{\frac{K}{\pi+6}} + \frac{3}{\sqrt{\pi}} \left[\frac{12K - \pi K - 6K}{\sqrt{\frac{\pi K + 6K - 6K}{\pi+6}}} \right] \\
&= 6\sqrt{\frac{K}{\pi+6}} + \frac{3}{\sqrt{\pi}} \left[\frac{6K - \pi K}{\sqrt{\frac{\pi K}{\pi+6}}} \right] \\
&= 6\sqrt{\frac{K}{\pi+6}} + \frac{3}{\pi\sqrt{K}} [(6K - \pi K) \sqrt{\pi+6}] > 0
\end{aligned}$$

So, it is the minima.

Therefore, the required ratio is 1: 1 when the combined volume is minimum.

Question 32

AB is a diameter of a circle and C is any point on the circle. Show that the area of ΔABC is maximum, when it is isosceles.

Solution

Let consider AB be the diameter and C is any point on the circle with radius r.

$\angle ACB = 90^\circ$ [angle in the semi-circle is 90°]

Let $AC = x$

$$BC = \sqrt{AB^2 - AC^2} \quad (\text{By Phythagorus Theorem})$$

$$BC = \sqrt{(2r)^2 - x^2} \Rightarrow BC = \sqrt{4r^2 - x^2} \quad \dots(i)$$

Now area of $\triangle ABC$, $A = \frac{1}{2} \times AC \times BC$

$$\Rightarrow A = \frac{1}{2} x \cdot \sqrt{4r^2 - x^2}$$

Squaring on both the sides, we get

$$A^2 = \frac{1}{4}x^2(4r^2 - x^2)$$

Let $A^2 = Z$

$$\text{So, } Z = \frac{1}{4}x^2(4r^2 - x^2) \Rightarrow Z = \frac{1}{4}(4x^2r^2 - x^4)$$

Differentiating both sides w.r.t. x , we get

$$\frac{dZ}{dx} = \frac{1}{4}[8xr^2 - 4x^3] \quad \dots(ii)$$

For local maxima and local minima $\frac{dZ}{dx} = 0$

$$\text{So, } \frac{1}{4}[8xr^2 - 4x^3] = 0 \Rightarrow x[2r^2 - x^2] = 0$$

$$x \neq 0 \quad \text{and} \quad 2r^2 - x^2 = 0$$

$$\Rightarrow x^2 = 2r^2 \Rightarrow x = \sqrt{2}r = AC$$

Now from eq. (i) we have

$$BC = \sqrt{4r^2 - 2r^2} \Rightarrow BC = \sqrt{2r^2} \Rightarrow BC = \sqrt{2}r$$

Thus, $AC = BC$

Hence, ΔABC is an isosceles triangle.

Differentiating eq. (ii) w.r.t. x , we get $\frac{d^2Z}{dx^2} = \frac{1}{4}[8r^2 - 12x^2]$

Put $x = \sqrt{2}r$

$$\begin{aligned} \therefore \frac{d^2Z}{dx^2} &= \frac{1}{4}[8r^2 - 12 \times 2r^2] = \frac{1}{4}[8r^2 - 24r^2] \\ &= \frac{1}{4} \times (-16r^2) = -4r^2 < 0 \quad \text{maxima} \end{aligned}$$

Therefore, the area of ΔABC is minimum when it is an isosceles triangle.