

Chapter 2: Inverse Trigonometric Functions

NCERT Exemplar · Class 12 · Maths

Questions

Question 1

Find the value of $\tan^{-1} [\tan (5\pi/6)] + \cos^{-1} [\cos (13\pi/6)]$

Solution

We know that,

$$\tan^{-1} \tan x = x, x \in (-\pi/2, \pi/2)$$

And, here

$$\tan^{-1} \tan (5\pi/6) \neq 5\pi/6 \text{ as } 5\pi/6 \notin (-\pi/2, \pi/2)$$

Also,

$$\cos^{-1} \cos x = x; x \in [0, \pi]$$

So,

$$\cos^{-1} \cos (13\pi/6) \neq 13\pi/6 \text{ as } 13\pi/6 \notin [0, \pi]$$

Now,

$$\begin{aligned} & \tan^{-1} [\tan (5\pi/6)] + \cos^{-1} [\cos (13\pi/6)] \\ &= \tan^{-1} [\tan (\pi - \pi/6)] + \cos^{-1} [\cos (2\pi + \pi/6)] \\ &= \tan^{-1} [-\tan \pi/6] + \cos^{-1} [-\cos (7\pi/6)] \\ &= -\tan^{-1} [\tan \pi/6] + \cos^{-1} [\cos (\pi/6)] \\ &= -\pi/6 + \pi/6 \\ &= 0 \end{aligned}$$

Question 2

Evaluate $\cos[\cos^{-1}(-\sqrt{3}/2) + \pi/6]$

Solution

$$\begin{aligned} & \cos\left[\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right) + \frac{\pi}{6}\right] \\ &= \cos\left[\cos^{-1}\left(\cos\frac{5\pi}{6}\right) + \frac{\pi}{6}\right] \quad \left(\because \cos\frac{5\pi}{6} = \frac{-\sqrt{3}}{2}\right) \\ &= \cos\left(\frac{5\pi}{6} + \frac{\pi}{6}\right) \quad (\because \cos^{-1} \cos x = x; x \in [0, \pi]) \\ &= \cos(\pi) = -1 \end{aligned}$$

Question 3

Prove that $\cot(\pi/4 - 2 \cot^{-1} 3) = 7$

Solution

Re-writing the given,

$$\frac{\pi}{4} - 2 \cot^{-1} 3 = \cot^{-1} 7$$

$$2 \tan^{-1} \frac{1}{3} = \frac{\pi}{4} - \tan^{-1} \frac{1}{7}$$

$$2 \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{7} = \frac{\pi}{4}$$

Now, $2 \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{7}$

$$= \tan^{-1} \frac{2/3}{1 - (1/3)^2} + \tan^{-1} \frac{1}{7} \quad \left(\because 2 \tan^{-1} x = \tan^{-1} \frac{2x}{1 - x^2} \right)$$

$$= \tan^{-1} \frac{3}{4} + \tan^{-1} \frac{1}{7} = \tan^{-1} \frac{\frac{3}{4} + \frac{1}{7}}{1 - \frac{3}{4} \cdot \frac{1}{7}}$$

$$= \tan^{-1} \frac{(21 + 4)/28}{(28 - 3)/28} = \tan^{-1} \frac{25}{25} = \tan^{-1} 1 = \frac{\pi}{4}$$

$$\frac{\pi}{4} - 2 \cot^{-1} 3 = \cot^{-1} 7$$

$$2 \tan^{-1} \frac{1}{3} = \frac{\pi}{4} - \tan^{-1} \frac{1}{7}$$

$$2 \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{7} = \frac{\pi}{4}$$

Now, $2 \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{7}$

$$= \tan^{-1} \frac{2/3}{1 - (1/3)^2} + \tan^{-1} \frac{1}{7} \quad \left(\because 2 \tan^{-1} x = \tan^{-1} \frac{2x}{1 - x^2} \right)$$

$$= \tan^{-1} \frac{3}{4} + \tan^{-1} \frac{1}{7} = \tan^{-1} \frac{\frac{3}{4} + \frac{1}{7}}{1 - \frac{3}{4} \cdot \frac{1}{7}}$$

$$= \tan^{-1} \frac{(21 + 4)/28}{(28 - 3)/28} = \tan^{-1} \frac{25}{25} = \tan^{-1} 1 = \frac{\pi}{4}$$

L.H.S = R.H.S

– Hence Proved

Question 4

Find the value of $\tan^{-1}(-1/\sqrt{3}) + \cot^{-1}(1/\sqrt{3}) + \tan^{-1}(\sin(-\pi/2))$

Solution

Given,

$$\begin{aligned} & \tan^{-1}(-1/\sqrt{3}) + \cot^{-1}(1/\sqrt{3}) + \tan^{-1}(\sin(-\pi/2)) \\ &= \tan^{-1}\left(\tan\left(-\frac{\pi}{6}\right)\right) + \cot^{-1}\left(\cot\frac{\pi}{3}\right) + \tan^{-1}(-1) \\ &= -\frac{\pi}{6} + \frac{\pi}{3} + \left(-\frac{\pi}{4}\right) = -\frac{\pi}{12} \end{aligned}$$

Question 5

Find the value of $\tan^{-1}(\tan 2\pi/3)$.

Solution

We know that,

$$\tan^{-1} \tan x = x, x \in (-\pi/2, \pi/2)$$

$$\tan^{-1}\left(\tan\frac{2\pi}{3}\right) = \tan^{-1}\tan\left(\pi - \frac{\pi}{3}\right) = \tan^{-1}\left(\tan\left(-\frac{\pi}{3}\right)\right) = -\frac{\pi}{3}$$

Question 6

Show that $2 \tan^{-1}(-3) = -\pi/2 + \tan^{-1}(-4/3)$

Solution

Taking L.H.S = $2 \tan^{-1}(-3) = -2 \tan^{-1} 3$ ($\because \tan^{-1}(-x) = -\tan^{-1} x, x \in \mathbb{R}$)

$$\begin{aligned}
&= -2 \left[\frac{\pi}{2} - \cot^{-1} 3 \right] && \left(\because \tan^{-1} x + \cot^{-1} x = \frac{\pi}{2} \right) \\
&= -2 \left[\frac{\pi}{2} - \tan^{-1} \frac{1}{3} \right] && \left(\because \tan^{-1} x = \cot^{-1} \frac{1}{x}, x > 0 \right) \\
&= -\pi + 2 \tan^{-1} \frac{1}{3} \\
&= -\pi + \tan^{-1} \frac{2 \cdot \frac{1}{3}}{1 - \left(\frac{1}{3} \right)^2} && \left(\because 2 \tan^{-1} x = \tan^{-1} \frac{2x}{1-x^2} \right) \\
&= -\pi + \tan^{-1} \frac{2/3}{8/9} = -\pi + \tan^{-1} \frac{3}{4} \\
&= -\pi + \frac{\pi}{2} - \cot^{-1} \frac{3}{4} && \left(\because \tan^{-1} x + \cot^{-1} x = \frac{\pi}{2} \right) \\
&= -\frac{\pi}{2} - \tan^{-1} \frac{4}{3} && \left(\because \tan^{-1} x = \cot^{-1} \frac{1}{x}, x > 0 \right) \\
&= -\frac{\pi}{2} + \tan^{-1} \left(-\frac{4}{3} \right) && (\because \tan^{-1}(-x) = -\tan^{-1} x, x \in \mathbb{R}) \\
&= \text{R.H.S}
\end{aligned}$$

– Hence Proved.

$$\tan^{-1} \sqrt{x(x+1)} + \sin^{-1} \sqrt{x^2 + x + 1} = \frac{\pi}{2}$$

Question 7

Find the real solution of the equation

Solution

Given equation,

$$\tan^{-1} \sqrt{x(x+1)} + \sin^{-1} \sqrt{x^2 + x + 1} = \frac{\pi}{2}$$

$$\tan^{-1} \sqrt{x(x+1)} = \frac{\pi}{2} - \sin^{-1} \sqrt{x^2 + x + 1}$$

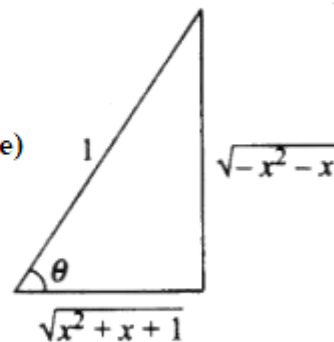
$$= \cos^{-1} \sqrt{x^2 + x + 1}$$

$$= \tan^{-1} \frac{\sqrt{-x^2 - x}}{\sqrt{x^2 + x + 1}} \quad (\text{From figure})$$

$$\sqrt{x(x+1)} = \frac{\sqrt{-x^2 - x}}{\sqrt{x^2 + x + 1}}$$

$$x^2 + x = 0$$

$$x = 0, -1$$



Hence, the real solutions of the given trigonometric equation are 0 and -1.

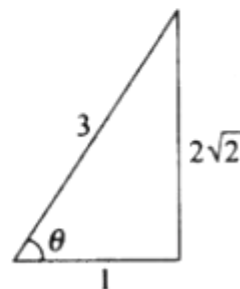
Question 8

Find the value of the expression $\sin(2 \tan^{-1} 1/3) + \cos(\tan^{-1} 2\sqrt{2})$.

Solution

Given expression, $\sin(2 \tan^{-1} 1/3) + \cos(\tan^{-1} 2\sqrt{2})$

$$\begin{aligned}
 \sin\left(2 \tan^{-1} \frac{1}{3}\right) &= \sin\left(\sin^{-1} \frac{2 \times \frac{1}{3}}{1 + \left(\frac{1}{3}\right)^2}\right) \quad \left(\because 2 \tan^{-1} x = \sin^{-1} \frac{2x}{1+x^2}\right) \\
 &= \sin\left(\sin^{-1} \frac{2/3}{10/9}\right) \\
 &= \sin\left(\sin^{-1} \frac{3}{5}\right) = \frac{3}{5} \quad (\because \sin(\sin^{-1} x) = x, x \in [-1, 1]) \\
 \cos(\tan^{-1} 2\sqrt{2}) &= \cos\left(\cos^{-1} \frac{1}{3}\right) = \frac{1}{3} \\
 &\quad (\because \cos(\cos^{-1} x) = x, x \in [-1, 1])
 \end{aligned}$$



Hence,

$$\sin\left(2 \tan^{-1} \frac{1}{3}\right) + \cos(\tan^{-1} 2\sqrt{2}) = \frac{3}{5} + \frac{1}{3} = \frac{9+5}{15} = \frac{14}{15}$$

Question 9

If $2 \tan^{-1}(\cos \theta) = \tan^{-1}(2 \operatorname{cosec} \theta)$, then show that $\theta = \pi/4$.

Solution

Given, $2 \tan^{-1}(\cos \theta) = \tan^{-1}(2 \operatorname{cosec} \theta)$

So,

$$\begin{aligned}
 \tan^{-1}\left(\frac{2 \cos \theta}{1 - \cos^2 \theta}\right) &= \tan^{-1}(2 \operatorname{cosec} \theta) \quad \left(\because 2 \tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)\right) \\
 \frac{2 \cos \theta}{\sin^2 \theta} &= 2 \operatorname{cosec} \theta \\
 \frac{2 \cos \theta}{\sin^2 \theta} &= \frac{2}{\sin \theta} \\
 \frac{\cos \theta}{\sin \theta} &= 1 \Rightarrow \cot \theta = 1 \Rightarrow \theta = \frac{\pi}{4}
 \end{aligned}$$

Question 10

Show that $\cos(2 \tan^{-1} 1/7) = \sin(4 \tan^{-1} 1/3)$.

Solution

Taking L.H.S, we have

$$\begin{aligned}
 \text{L.H.S.} &= \cos\left(2 \tan^{-1} \frac{1}{7}\right) \\
 &= \cos\left(\cos^{-1} \frac{1 - \left(\frac{1}{7}\right)^2}{1 + \left(\frac{1}{7}\right)^2}\right) \quad \left(\because 2 \tan^{-1} x = \cos^{-1} \frac{1 - x^2}{1 + x^2}\right) \\
 &= \cos\left(\cos^{-1} \frac{48/49}{50/49}\right) \\
 &= \cos\left(\cos^{-1} \frac{24}{25}\right) = \frac{24}{25} \quad (\because \cos(\cos^{-1} x) = x, x \in [-1, 1])
 \end{aligned}$$

$$\begin{aligned}
 \text{R.H.S.} &= \sin\left(4 \tan^{-1} \frac{1}{3}\right) \\
 &= \sin\left(2\left(2 \tan^{-1} \frac{1}{3}\right)\right) \\
 &= \sin\left(2\left(\tan^{-1} \frac{2 \cdot \frac{1}{3}}{1 - \left(\frac{1}{3}\right)^2}\right)\right) \quad \left(\because 2 \tan^{-1} x = \tan^{-1} \left(\frac{2x}{1 - x^2}\right)\right) \\
 &= \sin\left(2 \tan^{-1} \frac{2/3}{8/9}\right) \\
 &= \sin\left(2 \tan^{-1} \frac{3}{4}\right)
 \end{aligned}$$

$$\begin{aligned}
 &= \sin \left(\sin^{-1} \frac{2 \times \frac{3}{4}}{1 + \left(\frac{3}{4}\right)^2} \right) \quad \left(\because 2 \tan^{-1} x = \sin^{-1} \frac{2x}{1+x^2} \right) \\
 &= \sin \left(\sin^{-1} \frac{3/2}{25/16} \right) \\
 &= \sin \left(\sin^{-1} \frac{24}{25} \right) = \frac{24}{25} \quad (\because \sin(\sin^{-1} x) = x, x \in [-1, 1])
 \end{aligned}$$

L.H.S. = R.H.S.

Thus, L.H.S = R.H.S

– Hence proved

Question 11

Solve the equation $\cos(\tan^{-1} x) = \sin(\cot^{-1} 3/4)$.

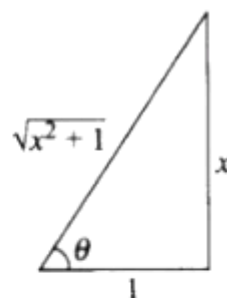
Solution

Given equation, $\cos(\tan^{-1} x) = \sin(\cot^{-1} 3/4)$

Taking L.H.S,

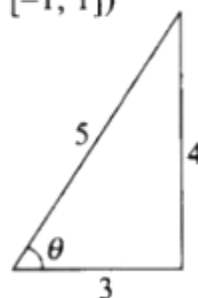
We have, $\cos(\tan^{-1} x) = \sin\left(\cot^{-1} \frac{3}{4}\right)$

$$\begin{aligned}\text{L.H.S.} &= \cos(\tan^{-1} x) \\ &= \cos\left(\cos^{-1} \frac{1}{\sqrt{x^2+1}}\right) \\ &= \frac{1}{\sqrt{x^2+1}}\end{aligned}$$



$$(\because \cos(\cos^{-1} x) = x, x \in [-1, 1])$$

$$\begin{aligned}\text{R.H.S.} &= \sin\left(\cot^{-1} \frac{3}{4}\right) \\ &= \sin\left(\sin^{-1} \frac{4}{5}\right) \\ &= \frac{4}{5}\end{aligned}$$



$$(\because \sin(\sin^{-1} x) = x, x \in [-1, 1])$$

Hence, on equating the L.H.S and R.H.S, we get $\frac{1}{\sqrt{x^2+1}} = \frac{4}{5}$

On squaring on both sides,

$$16(x^2 + 1) = 25$$

$$16x^2 + 16 = 25$$

$$16x^2 = 9$$

$$x^2 = 9/16$$

Therefore, $x = \pm 3/4$

$$\tan^{-1} \frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1+x^2} - \sqrt{1-x^2}} = \frac{\pi}{4} + \frac{1}{2} \cos^{-1} x^2.$$

Question 12

Prove that

Solution

$$\tan^{-1} \frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1+x^2} - \sqrt{1-x^2}} = \frac{\pi}{4} + \frac{1}{2} \cos^{-1} x^2.$$

Taking L.H.S,

$$\text{Let } x^2 = \cos 2\theta \Rightarrow \theta = \frac{1}{2} \cos^{-1} x^2$$

$$\begin{aligned} \text{So, L.H.S.} &= \tan^{-1} \left[\frac{\sqrt{1+\cos 2\theta} + \sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta} - \sqrt{1-\cos 2\theta}} \right] \\ &= \tan^{-1} \left[\frac{\sqrt{2\cos^2 \theta} + \sqrt{2\sin^2 \theta}}{\sqrt{2\cos^2 \theta} - \sqrt{2\sin^2 \theta}} \right] \\ &= \tan^{-1} \left[\frac{\sqrt{2}\cos \theta + \sqrt{2}\sin \theta}{\sqrt{2}\cos \theta - \sqrt{2}\sin \theta} \right] \\ &= \tan^{-1} \left[\frac{1 + \tan \theta}{1 - \tan \theta} \right] \\ &= \tan^{-1} \left[\frac{\tan \frac{\pi}{4} + \tan \theta}{1 - \tan \frac{\pi}{4} \tan \theta} \right] \\ &= \tan^{-1} \left(\tan \left(\frac{\pi}{4} + \theta \right) \right) = \frac{\pi}{4} + \theta = \frac{\pi}{4} + \frac{1}{2} \cos^{-1} x^2 \end{aligned}$$

= R.H.S

– Hence Proved

Question 13

Find the simplified form of $\cos^{-1} [3/5 \cos x + 4/5 \sin x]$, $x \in [-3\pi/4, \pi/4]$.

Solution

We have,

$$\cos^{-1} [3/5 \cos x + 4/5 \sin x], x \in [-3\pi/4, \pi/4]$$

Now, let $\cos \alpha = 3/5$

So, $\sin \alpha = 4/5$ and $\tan \alpha = 4/3$

$$\cos^{-1} [3/5 \cos x + 4/5 \sin x]$$

$$\Rightarrow \cos^{-1} [3/5 \cos x + 4/5 \sin x] = \cos^{-1} [\cos \alpha \cos x + \sin \alpha \sin x]$$

$$= \cos^{-1} [\cos (\alpha - x)]$$

$$= \alpha - x$$

$$= \tan^{-1} 4/3 - x$$

Question 14

Prove that $\sin^{-1} 8/17 + \sin^{-1} 3/5 = \sin^{-1} 77/85$

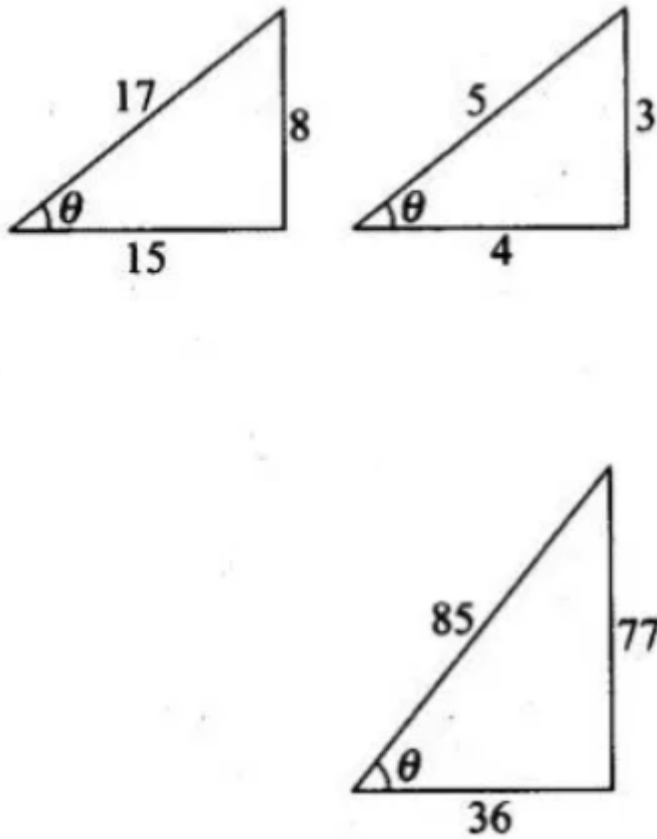
Solution

Taking the L.H.S,

$$= \sin^{-1} 8/17 + \sin^{-1} 3/5$$

$$\begin{aligned}
&= \tan^{-1} \frac{\frac{8}{15} + \frac{3}{4}}{1 - \frac{8}{15} \times \frac{3}{4}} \\
&= \tan^{-1} \frac{\frac{32 + 45}{60}}{\frac{60 - 24}{60}} \\
&= \tan^{-1} \frac{77}{36} \\
&= \sin^{-1} \frac{77}{\sqrt{5929 + 1296}} \\
&= \sin^{-1} \frac{77}{85} \\
&= \text{R.H.S.}
\end{aligned}$$

$$= \tan^{-1} 8/15 + \tan^{-1} 3/4$$



– Hence proved

Question 15

Show that $\sin^{-1} \frac{5}{13} + \cos^{-1} \frac{3}{5} = \tan^{-1} \frac{63}{16}$

Solution

Here, $\sin^{-1} \frac{5}{13} = \tan^{-1} \frac{5}{12}$

And,

$$\cos^{-1} \frac{3}{5} = \tan^{-1} \frac{4}{3}$$

Taking the L.H.S, we have

$$\begin{aligned}
 \text{L.H.S.} &= \sin^{-1} \frac{5}{13} + \cos^{-1} \frac{3}{5} \\
 &= \tan^{-1} \frac{5}{12} + \tan^{-1} \frac{4}{3} = \tan^{-1} \frac{\frac{5}{12} + \frac{4}{3}}{1 - \frac{5}{12} \cdot \frac{4}{3}} \\
 &= \tan^{-1} \frac{\frac{15 + 48}{36}}{\frac{36 - 20}{36}} = \tan^{-1} \frac{63}{16}
 \end{aligned}$$

Thus, L.H.S = R.H.S

– Hence Proved

Question 16

Prove that $\tan^{-1} \frac{1}{4} + \tan^{-1} \frac{2}{9} = \sin^{-1} \frac{1}{\sqrt{5}}$

Solution

Taking the LHS,

$$\tan^{-1} \frac{1}{4} + \tan^{-1} \frac{2}{9}$$

$$= \tan^{-1} \frac{\frac{1}{4} + \frac{2}{9}}{1 - \frac{1}{4} \cdot \frac{2}{9}} = \tan^{-1} \frac{\frac{9 + 8}{36}}{1 - \frac{2}{36}} = \tan^{-1} \frac{1}{2} = \sin^{-1} \frac{1}{\sqrt{5}}$$

= RHS

– Hence Proved

Question 17

Find the value of $4 \tan^{-1} \frac{1}{5} - \tan^{-1} \frac{1}{239}$

Solution

$$4 \tan^{-1} \frac{1}{5} - \tan^{-1} \frac{1}{239}$$

$$= 2 (\tan^{-1} \frac{1}{5}) - \tan^{-1} \frac{1}{239}$$

$$= 2 \tan^{-1} \frac{\frac{2}{5}}{1 - \left(\frac{1}{5}\right)^2} - \tan^{-1} \frac{1}{239} \quad \left(\because 2 \tan^{-1} x = \tan^{-1} \frac{2x}{1 - x^2} \right)$$

$$= 2 \tan^{-1} \frac{2/5}{24/25} - \tan^{-1} \frac{1}{239}$$

$$= 2 \tan^{-1} \frac{5}{12} - \tan^{-1} \frac{1}{239}$$

$$= 2 \tan^{-1} \frac{\frac{2}{5}}{1 - \left(\frac{1}{5}\right)^2} - \tan^{-1} \frac{1}{239} \quad \left(\because 2 \tan^{-1} x = \tan^{-1} \frac{2x}{1 - x^2} \right)$$

$$= 2 \tan^{-1} \frac{2/5}{24/25} - \tan^{-1} \frac{1}{239}$$

$$= 2 \tan^{-1} \frac{5}{12} - \tan^{-1} \frac{1}{239}$$

$$\begin{aligned}
&= \tan^{-1} \frac{2 \cdot \frac{5}{12}}{1 - \left(\frac{5}{12}\right)^2} - \tan^{-1} \frac{1}{239} \quad \left(\because 2 \tan^{-1} x = \tan^{-1} \frac{2x}{1 - x^2} \right) \\
&= \tan^{-1} \frac{144 \times 5}{119 \times 6} - \tan^{-1} \frac{1}{239} \\
&= \tan^{-1} \frac{120}{119} - \tan^{-1} \frac{1}{239} \\
&= \tan^{-1} \frac{\frac{120}{119} - \frac{1}{239}}{1 + \frac{120}{119} \cdot \frac{1}{239}} \quad \left(\because \tan^{-1} x - \tan^{-1} y = \tan^{-1} \frac{x - y}{1 + xy} \right) \\
&= \tan^{-1} \frac{120 \times 239 - 119}{119 \times 239 + 120} = \tan^{-1} \frac{28680 - 119}{28441 + 120} \\
&= \tan^{-1} \frac{28561}{28561} = \tan^{-1} 1 = \frac{\pi}{4}
\end{aligned}$$

Thus,

$$4 \tan^{-1} \frac{1}{5} - \tan^{-1} \frac{1}{239} = \pi/4$$

Question 18

Show that $\tan^{-1} (1/2 \sin^{-1} 3/4) = (4 - \sqrt{7})/3$ and justify why the other value $(4 + \sqrt{7})/3$ is ignored.

Solution

We have, $\tan^{-1} (1/2 \sin^{-1} 3/4)$

$$\text{Let } \frac{1}{2} \sin^{-1} \frac{3}{4} = \theta \Rightarrow \sin^{-1} \frac{3}{4} = 2\theta \Rightarrow \sin 2\theta = \frac{3}{4}$$

$$2 \tan \theta / 1 + \tan^2 \theta = \frac{3}{4}$$

$$3 \tan^2 \theta - 8 \tan \theta + 3 = 0$$

$$\tan \theta = \frac{8 \pm \sqrt{64 - 36}}{6}$$

$$\tan \theta = \frac{8 \pm \sqrt{28}}{6} = \frac{8 \pm 2\sqrt{7}}{6} = \frac{4 \pm \sqrt{7}}{3}$$

Now,

$$-\frac{\pi}{2} \leq \sin^{-1} \frac{3}{4} \leq \frac{\pi}{2}$$

$$-\frac{\pi}{4} \leq \frac{1}{2} \sin^{-1} \frac{3}{4} \leq \frac{\pi}{4}$$

$$\tan\left(-\frac{\pi}{4}\right) \leq \tan\left(\frac{1}{2}\left(\sin^{-1} \frac{3}{4}\right)\right) \leq \tan \frac{\pi}{4}$$

$$-1 \leq \tan\left(\frac{1}{2}\sin^{-1} \frac{3}{4}\right) \leq 1$$

Hence,

$$\tan \theta = \frac{4 - \sqrt{7}}{3} \left(\tan \theta = \frac{4 + \sqrt{7}}{3} > 1, \text{ which is not possible} \right)$$

Question 19

If $a_1, a_2, a_3, \dots, a_n$ is an arithmetic progression with common difference d , then evaluate the following expression.

$$\tan\left[\tan^{-1}\left(\frac{d}{1+a_1a_2}\right) + \tan^{-1}\left(\frac{d}{1+a_2a_3}\right) + \tan^{-1}\left(\frac{d}{1+a_3a_4}\right) + \dots + \tan^{-1}\left(\frac{d}{1+a_{n-1}a_n}\right)\right]$$

Solution

As $a_1, a_2, a_3, \dots, a_n$ is an arithmetic progression with common difference d .

$$d = a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = \dots = a_n - a_{n-1}$$

So,

$$\tan^{-1} \frac{d}{1 + a_1 a_2} = \tan^{-1} \frac{a_2 - a_1}{1 + a_1 a_2} = \tan^{-1} a_2 - \tan^{-1} a_1$$

$$\text{Similarly } \tan^{-1} \frac{d}{1 + a_2 a_3} = \tan^{-1} \frac{a_3 - a_2}{1 + a_2 a_3} = \tan^{-1} a_3 - \tan^{-1} a_2$$

...

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$$\tan^{-1} \frac{d}{1 + a_{n-1} a_n} = \tan^{-1} \frac{a_n - a_{n-1}}{1 + a_{n-1} a_n} = \tan^{-1} a_n - \tan^{-1} a_{n-1}$$

$$\begin{aligned} \therefore \tan & \left[\tan^{-1} \left(\frac{d}{1 + a_1 a_2} \right) + \tan^{-1} \left(\frac{d}{1 + a_2 a_3} \right) \right. \\ & \left. + \tan^{-1} \left(\frac{d}{1 + a_3 a_4} \right) + \dots + \tan^{-1} \left(\frac{d}{1 + a_{n-1} a_n} \right) \right] \\ & = \tan [(\tan^{-1} a_2 - \tan^{-1} a_1) + (\tan^{-1} a_3 - \tan^{-1} a_2) \\ & \quad + \dots + (\tan^{-1} a_n - \tan^{-1} a_{n-1})] \\ & = \tan [\tan^{-1} a_n - \tan^{-1} a_1] \\ & = \tan \left[\tan^{-1} \frac{a_n - a_1}{1 + a_n a_1} \right] \left[\because \tan^{-1} x - \tan^{-1} y = \tan^{-1} \left(\frac{x - y}{1 + xy} \right) \right] \\ & = \frac{a_n - a_1}{1 + a_n a_1} \quad [\because \tan (\tan^{-1} x) = x] \end{aligned}$$

Question 20

Which of the following is the principal value branch of $\cos^{-1} x$?

- (a) $[-\pi/2, \pi/2]$ (b) $(0, \pi)$ (c) $[0, \pi]$ (d) $[0, \pi] - \{\pi/2\}$

Solution

- (c) $[0, \pi]$

As we know that the principal value branch $\cos^{-1} x$ is $[0, \pi]$.

Question 21

Which of the following is the principal value branch of $\operatorname{cosec}^{-1} x$?

- (a) $(-\pi/2, \pi/2)$ (b) $[0, \pi] - \{\pi/2\}$ (c) $[-\pi/2, \pi/2]$ (d) $[-\pi/2, \pi/2] - \{0\}$

Solution

- (d) $[-\pi/2, \pi/2] - \{0\}$

As the principal branch of $\operatorname{cosec}^{-1} x$ is $[-\pi/2, \pi/2] - \{0\}$.

Question 22

If $3 \tan^{-1} x + \cot^{-1} x = \pi$, then x equals

- (a) 0 (b) 1 (c) -1 (d) $\frac{1}{2}$

Solution

- (b) 1

$$\text{Given, } 3 \tan^{-1} x + \cot^{-1} x = \pi$$

$$2 \tan^{-1} x + \tan^{-1} x + \cot^{-1} x = \pi$$

$$2 \tan^{-1} x + \pi/2 = \pi \text{ (As } \tan^{-1} + \cot^{-1} = \pi/2)$$

$$2 \tan^{-1} x = \pi/2$$

$$\tan^{-1} x = \pi/4$$

$$x = 1$$

Question 23

The value of $\sin^{-1} \cos 33\pi/5$ is

- (a) $3\pi/5$ (b) $-7\pi/5$ (c) $\pi/10$ (d) $-\pi/10$

Solution

- (d) $-\pi/10$

$$\begin{aligned}\sin^{-1}\left(\cos\frac{33\pi}{5}\right) &= \sin^{-1}\left(\cos\left(6\pi + \frac{3\pi}{5}\right)\right) = \sin^{-1}\left(\cos\frac{3\pi}{5}\right) \\ &= \sin^{-1}\left[\sin\left(\frac{\pi}{2} - \frac{3\pi}{5}\right)\right] \\ &= \sin^{-1}\left(\sin\left(-\frac{\pi}{10}\right)\right) \\ &= -\frac{\pi}{10} \quad \left(\because \sin^{-1}(\sin x) = x, \text{ for } x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right)\end{aligned}$$

Question 24

The domain of the function $\cos^{-1}(2x - 1)$ is

- (a) $[0, 1]$ (b) $[-1, 1]$ (c) $[-1, 1]$ (d) $[0, \pi]$

Solution

- (a) $[0, 1]$

Since, $\cos^{-1} x$ is defined for $x \in [-1, 1]$

So, $f(x) = \cos^{-1}(2x - 1)$ is defined if

$$-1 \leq 2x - 1 \leq 1$$

$$0 \leq 2x \leq 2$$

Hence,

$$0 \leq x \leq 1$$

Question 25

The domain of the function by $f(x) = \sin^{-1} \sqrt{x-1}$ is

(a) $[1, 2]$ (b) $[-1, 1]$ (c) $[0, 1]$ (d) none of these

Solution

(a) $[1, 2]$

We know that, $\sin^{-1} x$ is defined for $x \in [-1, 1]$

So, $f(x) = \sin^{-1} \sqrt{x-1}$ is defined if

$$0 \leq \sqrt{x-1} \leq 1$$

$$0 \leq x-1 \leq 1$$

$$1 \leq x \leq 2$$

Hence,

$$x \in [1, 2]$$

Question 26

If $\cos (\sin^{-1} 2/5 + \cos^{-1} x) = 0$, then x is equal to

(a) $1/5$ (b) $2/5$ (c) 0 (d) 1

Solution

(b) $2/5$

Given,

$$\cos (\sin^{-1} 2/5 + \cos^{-1} x) = 0$$

So, this can be rewritten as

$$\sin^{-1} 2/5 + \cos^{-1} x = \cos^{-1} 0$$

$$\sin^{-1} 2/5 + \cos^{-1} x = \pi/2$$

$$\cos^{-1} x = \pi/2 - \sin^{-1} 2/5$$

$$\cos^{-1} x = \cos^{-1} 2/5 \text{ [Since, } \cos^{-1} x + \sin^{-1} x = \pi/2]$$

Hence,

$$x = 2/5$$

Question 27

The value of $\sin(2 \tan^{-1}(0.75))$ is equal to

(a) 0.75 (b) 1.5 (c) 0.96 (d) sin 1.5

Solution

(c) 0.96

We have, $\sin(2 \tan^{-1}(0.75))$

$$\begin{aligned} &= \sin\left(2 \tan^{-1} \frac{3}{4}\right) = \sin\left(\sin^{-1} \frac{2 \cdot \frac{3}{4}}{1 + \frac{9}{16}}\right) \left(\because 2 \tan^{-1} x = \sin^{-1} \frac{2x}{1+x^2}\right) \\ &= \sin\left(\sin^{-1} \frac{3/2}{25/16}\right) = \sin\left(\sin^{-1} \frac{24}{25}\right) = \frac{24}{25} = 0.96 \end{aligned}$$
