

Chapter 13: Probability

NCERT Exemplar · Class 12 · Maths

Questions

Question 1

For a loaded die, the probabilities of outcomes are given as under:

$$P(1) = P(2) = 0.2, P(3) = P(5) = P(6) = 0.1 \text{ and } P(4) = 0.3.$$

The die is thrown two times. Let A and B be the events, 'same number each time', and 'a total score is 10 or more', respectively. Determine whether or not A and B are independent.

Solution

Given that a loaded die is thrown such that

$P(1) = P(2) = 0.2, P(3) = P(5) = P(6) = 0.1$ and $P(4) = 0.3$ and die is thrown two times. Also given that:

A = same number each time and

B = Total score is 10 or more.

$$\text{So, } P(A) = [P(1, 1) + P(2, 2) + P(3, 3) + P(4, 4) + P(5, 5) + P(6, 6)]$$

$$= P(1).P(1) + P(2).P(2) + P(3).P(3) + P(4).P(4) + P(5).P(5) + P(6).P(6)$$

$$= 0.2 \times 0.2 + 0.2 \times 0.2 + 0.1 \times 0.1 + 0.3 \times 0.3 + 0.1 \times 0.1 + 0.1 \times 0.1$$

$$= 0.04 + 0.04 + 0.01 + 0.09 + 0.01 + 0.01 = 0.20$$

$$\text{Now, } B = [(4, 6), (6, 4), (5, 5), (5, 6), (6, 5), (6, 6)]$$

$$P(B) = [P(4).P(6) + P(6).P(4) + P(5).P(5) + P(5).P(6) + P(6).P(5) + P(6).P(6)]$$

$$= 0.3 \times 0.1 + 0.1 \times 0.3 + 0.1 \times 0.1 + 0.1 \times 0.1 + 0.1 \times 0.1 + 0.1 \times 0.1$$

$$= 0.03 + 0.03 + 0.01 + 0.01 + 0.01 + 0.01 = 0.10$$

A and B both events will be independent if

$$P(A \cap B) = P(A).P(B) \dots (i)$$

And, here $(A \cap B) = \{(5, 5), (6, 6)\}$

So, $P(A \cap B) = P(5, 5) + P(6, 6) = P(5).P(5) + P(6).P(6)$

$$= 0.1 \times 0.1 + 0.1 \times 0.1 = 0.02$$

From equation (i) we get,

$$0.02 = 0.20 \times 0.10$$

$$0.02 = 0.02$$

Therefore, A and B are independent events.

Question 2

Refer to Exercise 1 above. If the die were fair, determine whether or not the events A and B are independent.

Solution

According to the solution of exercise 1, we have

$A = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$ and $n(A) = 6$ and $n(S) = 6 \times 6 = 36$

So, $P(A) = n(A)/n(S) = 6/36 = 1/6$

And, $B = \{(4, 6), (6, 4), (5, 5), (5, 6), (6, 5), (6, 6)\}$; $n(B) = 6$ and $n(S) = 36$

So, $P(B) = n(B)/n(S) = 6/36 = 1/6$

Now, $A \cap B = \{(5, 5), (6, 6)\}$

So, $P(A \cap B) = 2/36 = 1/18$

Hence, if A and B are not independent, then

$$P(A \cap B) \neq P(A).P(B)$$

$$1/18 \neq 1/6 \times 1/6 \Rightarrow 1/18 \neq 1/36$$

Therefore, A and B are not independent events.

Question 3

The probability that at least one of the two events A and B occurs is 0.6. If A and B occur

simultaneously with probability 0.3, evaluate $P(\bar{A}) + P(\bar{B})$.

Solution

W.k.t, $A \cup B$ denotes that atleast one of the events occurs and $A \cap B$ denotes that two events occur simultaneously.

$$\begin{aligned}\text{So, } P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ 0.6 &= P(A) + P(B) - 0.3 \\ 0.9 &= P(A) + P(B) \\ 0.9 &= 1 - P(\bar{A}) + 1 - P(\bar{B}) \\ \Rightarrow P(\bar{A}) + P(\bar{B}) &= 2 - 0.9 = 1.1\end{aligned}$$

Therefore, the required answer is 1.1.

Question 4

A bag contains 5 red marbles and 3 black marbles. Three marbles are drawn one by one without replacement. What is the probability that at least one of the three marbles drawn be black, if the first marble is red?

Solution

Let red marbles be presented with R and black marble with B. Also, let E be the event that at least one of the three marbles drawn be black when the first marble is red.

Now, the following three conditions are possible, if atleast one of the three marbles drawn be black and the first marble is red.

- (i) E_1 : 2nd ball is black and 3rd ball is red
- (ii) E_2 : 2nd ball is black and 3rd ball is also black
- (iii) E_3 : 2nd ball is red and 3rd ball is black

The probabilities of the above events are:

$$P(E_1) = P(R_1) \cdot P(B_1/R_1) \cdot P(R_2/R_1B_1) = \frac{5}{8} \cdot \frac{3}{7} \cdot \frac{4}{6} = \frac{60}{336} = \frac{5}{28}$$

$$P(E_2) = P(R_1) \cdot P(B_1/R_1) \cdot P(B_2/R_1B_1) = \frac{5}{8} \cdot \frac{3}{7} \cdot \frac{2}{6} = \frac{30}{336} = \frac{5}{56}$$

$$P(E_3) = P(R_1) \cdot P(R_2/R_1) \cdot P(B_1/R_1R_2) = \frac{5}{8} \cdot \frac{4}{7} \cdot \frac{3}{6} = \frac{60}{336} = \frac{5}{28}$$

$$\therefore P(E) = P(E_1) + P(E_2) + P(E_3) = \frac{5}{28} + \frac{5}{56} + \frac{5}{28} = \frac{25}{56}$$

Therefore, the required probability is 25/56.

Question 5

Two dice are thrown together and the total score is noted. The events E, F and G are 'a total of 4', 'a total of 9 or more', and 'a total divisible by 5', respectively. Calculate $P(E)$, $P(F)$ and $P(G)$ and decide which pairs of events, if any, are independent.

Solution

If two dice are thrown together, we have

$$n(S) = 36$$

Now, let's consider:

$$E = \text{A total of 4} = \{(2, 2), (1, 3), (3, 1)\} \Rightarrow n(E) = 3$$

$$F = \text{A total of 9 or more}$$

$$= \{(3, 6), (6, 3), (5, 4), (4, 5), (5, 5), (4, 6), (6, 4), (5, 6), (6, 5), (6, 6)\}$$

$$\text{So, } n(F) = 10$$

$$G = \text{A total divisible by 5}$$

$$= \{(1, 4), (4, 1), (2, 3), (3, 2), (4, 6), (6, 4), (5, 5)\}$$

$$\text{So, } n(G) = 7$$

It's seen that $(E \cap F) = \emptyset$ and $(E \cap G) = \emptyset$

And, $(F \cap G) = \{(4, 6), (6, 4), (5, 5)\}$

$\Rightarrow n(F \cap G) = 3$ and $(E \cap F \cap G) = \emptyset$

Hence, the probabilities of the events are:

$$P(E) = \frac{n(E)}{n(S)} = \frac{3}{36} = \frac{1}{12}$$

$$P(F) = \frac{n(F)}{n(S)} = \frac{10}{36} = \frac{5}{18}; P(G) = \frac{n(G)}{n(S)} = \frac{7}{36}$$

$$P(F \cap G) = \frac{3}{36} = \frac{1}{12} \text{ and } P(F) \cdot P(G) = \frac{5}{18} \cdot \frac{7}{36} = \frac{35}{648}$$

As, $P(F \cap G) \neq P(F) \cdot P(G)$

Therefore, there is no pair of independent events.

Question 6

Explain why the experiment of tossing a coin three times is said to have binomial distribution.

Solution

As the random variable X takes 0, 1, 2, 3, ..., n is said to be binomial distribution having parameters n and p , if the probability is given by

$P(X = r) = {}^nC_r p^r q^{n-r}$, where $q = 1 - p$ and $r = 0, 1, 2, 3, \dots$

Similarly, in case of tossing a coin 3 times, we have

$n = 3$ and X has the values 0, 1, 2, 3 with $p = \frac{1}{2}$, $q = \frac{1}{2}$.

Therefore, it is said that the experiment of tossing a coin three times have binomial distribution.

Question 7

A and B are two events such that $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$ and $P(A \cap B) = \frac{1}{4}$.

Find:

(i) $P(A|B)$ (ii) $P(B|A)$ (iii) $P(A'|B)$ (iv) $P(A'|B')$

Solution

Given, $P(A) = 1/2$, $P(B) = 1/3$ and $P(A \cap B) = 1/4$.

$$P(A') = 1 - 1/2 = 1/2, P(B') = 1 - 1/3 = 2/3$$

Now,

$$\begin{aligned} P(A' \cap B') &= 1 - P(A \cup B) = 1 - [P(A) + P(B) - P(A \cap B)] \\ &= 1 - \left[\frac{1}{2} + \frac{1}{3} - \frac{1}{4} \right] = 1 - \left[\frac{6 + 4 - 3}{12} \right] = 1 - \frac{7}{12} = \frac{5}{12} \end{aligned}$$

$$(i) P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{1/4}{1/3} = \frac{3}{4}$$

$$(ii) P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{1/4}{1/2} = \frac{1}{2}$$

$$\begin{aligned} (iii) P(A'/B) &= \frac{P(A' \cap B)}{P(B)} = \frac{P(B) - P(A \cap B)}{P(B)} \\ &= 1 - \frac{P(A \cap B)}{P(B)} = 1 - \frac{1/4}{1/3} = 1 - \frac{3}{4} = \frac{1}{4} \end{aligned}$$

$$(iv) P(A'/B') = \frac{P(A' \cap B')}{P(B')} = \frac{5/12}{2/3} = \frac{5}{12} \times \frac{3}{2} = \frac{5}{8}$$

$$\begin{aligned}
 P(A' \cap B') &= 1 - P(A \cup B) = 1 - [P(A) + P(B) - P(A \cap B)] \\
 &= 1 - \left[\frac{1}{2} + \frac{1}{3} - \frac{1}{4} \right] = 1 - \left[\frac{6+4-3}{12} \right] = 1 - \frac{7}{12} = \frac{5}{12}
 \end{aligned}$$

$$(i) \quad P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{1/4}{1/3} = \frac{3}{4}$$

$$(ii) \quad P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{1/4}{1/2} = \frac{1}{2}$$

$$\begin{aligned}
 (iii) \quad P(A'/B) &= \frac{P(A' \cap B)}{P(B)} = \frac{P(B) - P(A \cap B)}{P(B)} \\
 &= 1 - \frac{P(A \cap B)}{P(B)} = 1 - \frac{1/4}{1/3} = 1 - \frac{3}{4} = \frac{1}{4}
 \end{aligned}$$

$$(iv) \quad P(A'/B') = \frac{P(A' \cap B')}{P(B')} = \frac{5/12}{2/3} = \frac{5}{12} \times \frac{3}{2} = \frac{5}{8}$$

Question 8

Three events A, B and C have probabilities $2/5$, $1/3$ and $1/2$ respectively. Given that $P(A \cap C) = 1/5$ and $P(B \cap C) = 1/4$, find the values of $P(C | B)$ and $P(A' \cap C')$.

Solution

Given, $P(A) = 2/5$, $P(B) = 1/3$ and $P(C) = 1/2$

$P(A \cap C) = 1/5$ and $P(B \cap C) = 1/4$

So, $P(C/B) = P(B \cap C) / P(B) = (1/4) / (1/3) = 3/4$

$P(A' \cap C') = 1 - P(A \cup C)$

$= 1 - [P(A) + P(C) - P(A \cap C)]$

$= 1 - [2/5 + 1/2 - 1/5] = 1 - 7/10 = 3/10$

Therefore, the required probabilities are $3/4$ and $3/10$.

Question 9

Let E_1 and E_2 be two independent events such that $P(E_1) = p_1$ and $P(E_2) = p_2$. Describe in words of the events whose probabilities are:

- (i) $p_1 p_2$ (ii) $(1-p_1) p_2$ (iii) $1 - (1-p_1)(1-p_2)$ (iv) $p_1 + p_2 - 2p_1 p_2$

Solution

Here, $P(E_1) = p_1$ and $P(E_2) = p_2$

$$(i) \quad p_1 p_2 = P(E_1) \cdot P(E_2) = P(E_1 \cap E_2)$$

So, E_1 and E_2 occur.

$$(ii) \quad (1-p_1) \cdot p_2 = P(E_1)' \cdot P(E_2) = P(E_1' \cap E_2)$$

So, E_1 does not occur but E_2 occurs.

$$(iii) \quad 1 - (1-p_1)(1-p_2) = 1 - P(E_1)'P(E_2)' = 1 - P(E_1' \cap E_2') \\ = 1 - [1 - P(E_1 \cup E_2)] = P(E_1 \cup E_2)$$

So, either E_1 or E_2 or both E_1 and E_2 occur.

$$(iv) \quad p_1 + p_2 - 2p_1 p_2 = P(E_1) + P(E_2) - 2P(E_1) \cdot P(E_2) \\ = P(E_1) + P(E_2) - 2P(E_1 \cap E_2) \\ = P(E_1 \cup E_2) - 2P(E_1 \cap E_2)$$

Now, its clearly seen that either E_1 or E_2 occurs but not both.

Question 10

A discrete random variable X has the probability distribution given as below:

X	0.5	1	1.5	2
$P(X)$	k	k^2	$2k^2$	k

(i) Find the value of k

(ii) Determine the mean of the distribution.

Solution

For a probability distribution, we know that if $P_i \geq 0$

$$\begin{aligned}
 (i) \quad \sum_{i=1}^n P_i &= 1 \Rightarrow k + k^2 + 2k^2 + k = 1 \\
 &3k^2 + 2k - 1 = 0 \\
 &3k^2 + 3k - k - 1 = 0 \\
 &3k(k+1) - 1(k+1) = 0 \Rightarrow (3k-1)(k+1) = 0
 \end{aligned}$$

$$\text{So, } k = \frac{1}{3} \text{ and } k = -1$$

$$\text{But } k \geq 0 \quad \therefore k = \frac{1}{3}$$

(ii) Mean of the distribution

$$\begin{aligned}
 E(X) &= \sum_{i=1}^n X_i P_i = 0.5k + 1.k^2 + 1.5(2k^2) + 2k \\
 &= \frac{k}{2} + k^2 + 3k^2 + 2k = 4k^2 + \frac{5}{2}k \\
 &= 4\left(\frac{1}{3}\right)^2 + \frac{5}{2}\left(\frac{1}{3}\right) = \frac{4}{9} + \frac{5}{6} = \frac{23}{18}
 \end{aligned}$$

Question 11

Prove that

$$(i) P(A) = P(A \cap B) + P(A \cap B^c)$$

$$(ii) P(A \cap B) = P(A \cap B) + P(A \cap B^c) + P(A \cap B)$$

Solution

(i) Taking R.H.S.

$$\begin{aligned}
 \text{R.H.S.} &= P(A \cap B) + P(A \cap \bar{B}) \\
 &= P(A).P(B) + P(A).P(\bar{B}) = P(A)[P(B) + P(\bar{B})] \\
 &= P(A) \cdot 1 \\
 &= P(A) = \text{L.H.S. Hence proved.}
 \end{aligned}$$

(ii) Taking R.H.S.

$$\begin{aligned}
 \text{R.H.S.} &= P(A).P(B) + P(A).P(\bar{B}) + P(\bar{A}).P(B) \\
 &= P(A).P(B) + P(A)[1 - P(B)] + [1 - P(A)].P(B) \\
 &= P(A).P(B) + P(A) - P(A).P(B) + P(B) - P(A).P(B) \\
 &= P(A) + P(B) - P(A \cap B) \\
 &= P(A \cup B) = \text{L.H.S. Hence proved.}
 \end{aligned}$$

Question 12

If X is the number of tails in three tosses of a coin, determine the standard deviation of X .

Solution

Given, $X = 0, 1, 2, 3$

$$P(X = r) = {}^nC_r p^r q^{n-r}$$

Where $n = 3$, $p = \frac{1}{2}$, $q = \frac{1}{2}$ and $r = 0, 1, 2, 3$

$$P(X = 0) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$$

$$P(X = 1) = 3 \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{3}{8}$$

$$P(X = 2) = 3 \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{3}{8}$$

$$P(X = 3) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$$

The probability distribution table is:

X	0	1	2	3
---	---	---	---	---

P(X)	1/8	3/8	3/8	1/8
------	-----	-----	-----	-----

Now,

$$E(X) = 0 + 1 \times \frac{3}{8} + 2 \times \frac{3}{8} + 3 \times \frac{1}{8} = \frac{3}{8} + \frac{6}{8} + \frac{3}{8} = \frac{12}{8} = \frac{3}{2}$$

$$E(X^2) = 0 + 1 \times \frac{3}{8} + 4 \times \frac{3}{8} + 9 \times \frac{1}{8} = \frac{3}{8} + \frac{12}{8} + \frac{9}{8} = \frac{24}{8} = 3$$

$$\text{W.k.t, } \text{Var}(X) = E(X^2) - [E(X)]^2 = 3 - \left(\frac{3}{2}\right)^2 = 3 - \frac{9}{4} = \frac{3}{4}$$

$$\text{Thus, the standard deviation} = \text{Var}(X) = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

Question 13

In a dice game, a player pays a stake of Re1 for each throw of a die. She receives Rs 5 if the die shows a 3, Rs 2 if the die shows a 1 or 6, and nothing otherwise. What is the player's expected profit per throw over a long series of throws?

Solution

Let's take X to be the random variable of profit per throw.

X	-1	1	4
P(X)	1/2	1/3	1/6

As, she loses Rs 1 for giving any of 2, 4, 5.

$$\text{So, } P(X = -1) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2}$$

$$P(X = 1) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3} \text{ [Since, die showing 1 or 6]}$$

$$P(X = 4) = \frac{1}{6} \text{ [Since, die shows only a 3]}$$

Thus, the player's expected profit = $\sum p_i x_i$

$$= -1 \times \frac{1}{2} + 1 \times \frac{1}{3} + 4 \times \frac{1}{6} = -\frac{1}{2} + \frac{1}{3} + \frac{2}{3} = \frac{1}{2} = \text{Rs } 0.50$$

Question 14

Three dice are thrown at the same time. Find the probability of getting three two's, if it is known that the sum of the numbers on the dice was six.

Solution

Given that the dice is thrown three times

So, the sample space $n(S) = 6^3 = 216$

Let E_1 be the event when the sum of number on the dice was 6 and E_2 be the event when three 2's occur.

$$E_1 = \{(1, 1, 4), (1, 2, 3), (1, 4, 1), (2, 1, 3), (2, 2, 2), (2, 3, 1), (3, 1, 2), (3, 2, 1), (4, 1, 1)\}$$

$$n(E_1) = 10 \text{ and } n(E_2) = 1 \text{ [Since, } E_2 = (2, 2, 2)\text{]}$$

$$\text{Thus, } P(E_2/E_1) = P(E_1 \cap E_2)/P(E_1) = (1/216)/(10/216) = 1/10.$$

Question 15

Suppose 10,000 tickets are sold in a lottery each for Re 1. First prize is of Rs 3000 and the second prize is of Rs. 2000. There are three third prizes of Rs. 500 each. If you buy one ticket, what is your expectation.

Solution

Let's take X to be the random variable where $X = 0, 500, 2000$ and 3000

X	0	500	2000	3000
$P(X)$	9995/10000	3/10000	1/10000	1/10000

$$\begin{aligned} E(X) &= 0 \times \frac{9995}{10000} + 500 \times \frac{3}{10000} + 2000 \times \frac{1}{10000} + 3000 \times \frac{1}{10000} \\ &= 0 + \frac{1500}{10000} + \frac{2000}{10000} + \frac{3000}{10000} = \frac{6500}{10000} = \frac{13}{20} = ₹ 0.65 \end{aligned}$$

Question 16

A bag contains 4 white and 5 black balls. Another bag contains 9 white and 7 black balls. A ball is transferred from the first bag to the second and then a ball is drawn at random from the second bag. Find the probability that the ball drawn is white.

Solution

Let's take W_1 and W_2 to be two bags containing (4W, 5B) and (9W, 7B) balls respectively.

Let E_1 be the event that the transferred ball from the bag W_1 to W_2 is white and E_2 the event that the transferred ball is black.

And, E be the event that the ball drawn from the second bag is white.

So, the probabilities are:

$$P(E/E_1) = 10/17, P(E/E_2) = 9/17$$

$$P(E_1) = 4/9 \text{ and } P(E_2) = 5/9$$

$$\text{Now, } P(E) = P(E_1) \cdot P(E/E_1) + P(E_2) \cdot P(E/E_2)$$

$$= 4/9 \times 10/17 + 5/9 \times 9/17 = 40/153 + 45/153 = 85/153 = 5/9$$

Therefore, the required probability is $5/9$.

Question 17

Bag I contains 3 black and 2 white balls, Bag II contains 2 black and 4 white balls. A bag and a ball is selected at random. Determine the probability of selecting a black ball.

Solution

Given that:

Bag 1 has 3B, 2W balls and Bag 2 has 2B, 4W balls.

Let E_1 = The event that bag 1 is selected

E_2 = The event that bag 2 is selected

And, E = The event that a black ball is selected

Now, the probabilities are:

$$P(E_1) = \frac{1}{2}, P(E_2) = \frac{1}{2}, P(E/E_1) = \frac{3}{5} \text{ and } P(E/E_2) = \frac{1}{3}$$

$$\begin{aligned} P(E) &= P(E_1) \cdot P(E/E_1) + P(E_2) \cdot P(E/E_2) \\ &= \frac{1}{2} \times \frac{3}{5} + \frac{1}{2} \times \frac{1}{3} = \frac{3}{10} + \frac{1}{6} = \frac{9+5}{30} = \frac{14}{30} = \frac{7}{15} \end{aligned}$$

Therefore, the required probability is 7/15.

Question 18

A box has 5 blue and 4 red balls. One ball is drawn at random and not replaced. Its colour is also not noted. Then another ball is drawn at random. What is the probability of second ball being blue?

Solution

Given that the box has 5 blue and 4 red balls.

Let's consider E_1 be the event that first ball drawn is blue and E_2 be the event that first ball drawn is red.

And, E is the event that second ball drawn is blue.

Now, the probability of E is:

$$\begin{aligned} P(E) &= P(E_1) \cdot P(E/E_1) + P(E_2) \cdot P(E/E_2) \\ &= \frac{5}{9} \times \frac{4}{8} + \frac{4}{9} \times \frac{5}{8} = \frac{20}{72} + \frac{20}{72} = \frac{40}{72} = \frac{5}{9} \end{aligned}$$

Therefore, the required probability is 5/9.

Question 19

Four cards are successively drawn without replacement from a deck of 52 playing cards. What is the probability that all the four cards are kings?

Solution

Let E_1 , E_2 , E_3 and E_4 be the events that first, second, third and fourth card is King respectively.

$$\begin{aligned}
 &\therefore P(E_1 \cap E_2 \cap E_3 \cap E_4) \\
 &= P(E_1) \cdot P(E_2/E_1) \cdot P\left[\frac{E_3}{(E_1 \cap E_2)}\right] \cdot P\left[\frac{E_4}{(E_1 \cap E_2 \cap E_3 \cap E_4)}\right] \\
 &= \frac{4}{52} \times \frac{3}{51} \times \frac{2}{50} \times \frac{1}{49} = \frac{24}{52 \cdot 51 \cdot 50 \cdot 49} = \frac{1}{13 \cdot 17 \cdot 25 \cdot 49} = \frac{1}{27075}
 \end{aligned}$$

Therefore, the required probability is $1/27075$.

Question 20

A die is thrown 5 times. Find the probability that an odd number will come up exactly three times.

Solution

Here, $p = 1/6 + 1/6 + 1/6 = 1/2 \Rightarrow q = 1 - 1/2 = 1/2$ and $n = 5$

Now,

$$\begin{aligned}
 P(x = r) &= {}^nC_r p^r q^{n-r} = {}^5C_3 (1/2)^3 (1/2)^{5-3} \\
 &= [5!/(3!2!)] \cdot (1/2)^3 (1/2)^2 = 10 \cdot 1/8 \cdot 1/4 = 5/16
 \end{aligned}$$

Therefore, the required probability is $5/16$.

Question 21

Ten coins are tossed. What is the probability of getting at least 8 heads?

Solution

Here we have, $n = 10$, $p = 1/2$ and $q = 1 - 1/2 = 1/2$

$$P(X \geq 8) = P(x = 8) + P(x = 9) + P(x = 10)$$

$$\begin{aligned}
&= {}^{10}C_8 \left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^{10-8} + {}^{10}C_9 \left(\frac{1}{2}\right)^9 \left(\frac{1}{2}\right)^{10-9} + {}^{10}C_{10} \left(\frac{1}{2}\right)^{10} \left(\frac{1}{2}\right)^0 \\
&= \frac{10!}{8!2!} \cdot \left(\frac{1}{2}\right)^8 \cdot \left(\frac{1}{2}\right)^2 + \frac{10!}{9!1!} \left(\frac{1}{2}\right)^9 \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^{10} \\
&= 45 \cdot \left(\frac{1}{2}\right)^{10} + 10 \cdot \left(\frac{1}{2}\right)^{10} + \left(\frac{1}{2}\right)^{10} = \left(\frac{1}{2}\right)^{10} \cdot (45 + 10 + 1) \\
&= 56 \left(\frac{1}{2}\right)^{10} = 56 \times \frac{1}{1024} = \frac{7}{128}
\end{aligned}$$

Therefore, the required probability is 7/128.

Question 22

The probability of a man hitting a target is 0.25. He shoots 7 times. What is the probability of his hitting at least twice?

Solution

Here, we have $n = 7$, $p = 0.25 = 25/100 = \frac{1}{4}$ and $q = 1 - \frac{1}{4} = \frac{3}{4}$

$$P(X \geq 2) = 1 - [P(X = 0) + P(X = 1)]$$

$$\begin{aligned}
&= 1 - \left[{}^7C_0 \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^7 + {}^7C_1 \left(\frac{1}{4}\right)^1 \left(\frac{3}{4}\right)^6 \right] \\
&= 1 - \left[\left(\frac{3}{4}\right)^7 + \frac{7}{4} \left(\frac{3}{4}\right)^6 \right] = 1 - \left(\frac{3}{4}\right)^6 \left(\frac{3}{4} + \frac{7}{4}\right) \\
&= 1 - \left(\frac{3}{4}\right)^6 \left(\frac{10}{4}\right) = 1 - \frac{729}{4096} \times \frac{10}{4} = 1 - \frac{7290}{16384} \\
&= \frac{16384 - 7290}{16384} = \frac{9094}{16384} = \frac{4547}{8192}
\end{aligned}$$

Therefore, the required probability is 4547/8192.

Question 23

A lot of 100 watches is known to have 10 defective watches. If 8 watches are selected (one by one with replacement) at random, what is the probability that there will be at least one defective watch?

Solution

Given: Total number of watches = 100 and number of defective watches = 10

So, the probability of selecting a defective watch = $10/100 = 1/10$

Now,

$n = 8$, $p = 1/10$ and $q = 1 - 1/10 = 9/10$, $r \geq 1$

$$P(X \geq 1) = 1 - P(x = 0) = 1 - {}^8C_0 (1/10)^0 (9/10)^{8-0} = 1 - (9/10)^8$$

Therefore, the required probability is $1 - (9/10)^8$.

Question 24

Consider the probability distribution of a random variable X:

X	0	1	2	3	4
P(X)	0.1	0.25	0.3	0.2	0.15

Calculate: (i) $V(X/2)$ (ii) Variance of X.

Solution

Given:

X	0	1	2	3	4
P(X)	0.1	0.25	0.3	0.2	0.15

We know that: $\text{Var}(X) = E(X^2) - [E(X)]^2$

$$\text{where } E(X) = \sum_{i=1}^n x_i p_i \text{ and } E(X^2) = \sum_{i=1}^n p_i x_i^2$$

$$\therefore E(X) = 0 \times 0.1 + 1 \times 0.25 + 2 \times 0.3 + 3 \times 0.2 + 4 \times 0.15$$

$$= 0 + 0.25 + 0.6 + 0.6 + 0.6 = 2.05$$

$$E(X^2) = 0 \times 0.1 + 1 \times 0.25 + 4 \times 0.3 + 9 \times 0.2 + 16 \times 0.15$$

$$= 0 + 0.25 + 1.2 + 1.8 + 2.40 = 5.65$$

$$(i) \quad V\left(\frac{X}{2}\right) = \frac{1}{4} V(X) = \frac{1}{4} [5.65 - (2.05)^2] = \frac{1}{4} [5.65 - 4.2025]$$

$$= \frac{1}{4} \times 1.4475 = 0.361875$$

$$\left[\because V\left(\frac{X}{2}\right) = \frac{1}{4} V(X) \right]$$

$$(ii) \quad \text{Var}(X) = 1.4475$$

Question 25

The probability distribution of a random variable X is given below:

X	0	1	2	3
P(X)	k	k/2	k/4	k/8

(i) Determine the value of k.

(ii) Determine $P(X \leq 2)$ and $P(X > 2)$

(iii) Find $P(X \leq 2) + P(X > 2)$.

Solution

$$(i) \text{ W.k.t } P(0) + P(1) + P(2) + P(3) = 1$$

$$\Rightarrow k + k/2 + k/4 + k/8 = 1$$

$$(8k + 4k + 2k + k)/8 = 1$$

$$15k = 8$$

$$\text{Hence, } k = 8/15$$

$$(ii) P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2)$$

$$= k + k/2 + k/4 = 7k/4 = 7/4 \times 8/15 = 14/15$$

$$\text{And } P(X > 2) = P(X = 3) = k/8 = 1/8 \times 8/15 = 1/15$$

$$(iii) P(X \leq 2) + P(X \geq 2) = 14/15 + 1/15 = (14 + 1)/15 = 15/15 = 1$$

Question 26

For the following probability distribution determine standard deviation of the random variable X.

X	2	3	4
P(X)	0.2	0.5	0.3

Solution

We know that: Standard deviation (S.D.) = $\sqrt{\text{Variance}}$

$$\text{So, } \text{Var}(X) = E(X^2) - [E(X)]^2$$

$$E(X) = 2 \times 0.2 + 3 \times 0.5 + 4 \times 0.3 = 0.4 + 1.5 + 1.2 = 3.1$$

$$E(X^2) = 4 \times 0.2 + 9 \times 0.5 + 16 \times 0.3 = 0.8 + 4.5 + 4.8 = 10.1$$

$$\Rightarrow V(X) = 10.1 - (3.1)^2 = 10.1 - 9.61 = 0.49$$

$$\text{Therefore, S.D.} = \sqrt{0.49} = 0.7$$

Question 27

A biased die is such that $P(4) = 1/10$ and other scores being equally likely. The die

is tossed twice. If X is the 'number of fours seen', find the variance of the random variable X.

Solution

Here, random variable $X = 0, 1, 2$

$$P(4) = \frac{1}{10}, \quad P(\bar{4}) = 1 - \frac{1}{10} = \frac{9}{10}$$

$$P(X = 0) = P(\bar{4}) \cdot P(\bar{4}) = \frac{9}{10} \times \frac{9}{10} = \frac{81}{100}$$

$$P(X = 1) = P(\bar{4}) \cdot P(4) + P(4) \cdot P(\bar{4}) = \frac{9}{10} \times \frac{1}{10} + \frac{1}{10} \times \frac{9}{10} = \frac{18}{100}$$

$$P(X = 2) = P(4) \cdot P(4) = \frac{1}{10} \times \frac{1}{10} = \frac{1}{100}$$

X	0	1	2
P(X)	81/100	18/100	1/100

Now, we know $V(X) = E(X^2) - [E(X)]^2$

$$E(X) = 0 \times \frac{81}{100} + 1 \times \frac{18}{100} + 2 \times \frac{1}{100} = \frac{20}{100} = \frac{1}{5}$$

$$E(X^2) = 0 \times \frac{81}{100} + 1 \times \frac{18}{100} + 4 \times \frac{1}{100} = \frac{22}{100} = \frac{11}{50}$$

$$\therefore \text{Var}(X) = \frac{11}{50} - \left(\frac{1}{5}\right)^2 = \frac{11}{50} - \frac{1}{25} = \frac{9}{50} = 0.18$$

Therefore, the required variance = 0.18.

Question 28

A die is thrown three times. Let X be 'the number of twos seen'. Find the expectation of X.

Solution

Here, we have $X = 0, 1, 2, 3$ [Since, die is thrown 3 times]

And $p = 1/6$, $q = 5/6$

$$\therefore P(X=0) = P(\text{not } 2) \cdot P(\text{not } 2) \cdot P(\text{not } 2) = \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} = \frac{125}{216}$$

$$\begin{aligned} P(X=1) &= P(2) \cdot P(\text{not } 2) \cdot P(\text{not } 2) + P(\text{not } 2) \cdot P(2) \cdot P(\text{not } 2) + P(\text{not } 2) \cdot P(\text{not } 2) \cdot P(2) \\ &= \frac{1}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} + \frac{5}{6} \cdot \frac{1}{6} \cdot \frac{5}{6} + \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{1}{6} = \frac{25}{216} + \frac{25}{216} + \frac{25}{216} = \frac{75}{216} \end{aligned}$$

$$\begin{aligned} P(X=2) &= P(2) \cdot P(2) \cdot P(\text{not } 2) + P(2) \cdot P(\text{not } 2) \cdot P(2) + P(\text{not } 2) \cdot P(2) \cdot P(2) \\ &= \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{5}{6} + \frac{1}{6} \cdot \frac{5}{6} \cdot \frac{1}{6} + \frac{5}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} = \frac{5}{216} + \frac{5}{216} + \frac{5}{216} = \frac{15}{216} \end{aligned}$$

$$P(X=3) = P(2) \cdot P(2) \cdot P(2) = \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{216}$$

$$\begin{aligned} \text{Now, } E(X) &= \sum_{i=1}^n p_i x_i \\ &= 0 \times \frac{125}{216} + 1 \times \frac{75}{216} + 2 \times \frac{15}{216} + 3 \times \frac{1}{216} \\ &= 0 + \frac{75}{216} + \frac{30}{216} + \frac{3}{216} = \frac{75+30+3}{216} = \frac{108}{216} = \frac{1}{2} \end{aligned}$$

$$\therefore P(X=0) = P(\text{not } 2) \cdot P(\text{not } 2) \cdot P(\text{not } 2) = \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} = \frac{125}{216}$$

$$\begin{aligned} P(X=1) &= P(2) \cdot P(\text{not } 2) \cdot P(\text{not } 2) + P(\text{not } 2) \cdot P(2) \cdot P(\text{not } 2) + P(\text{not } 2) \cdot P(\text{not } 2) \cdot P(2) \\ &= \frac{1}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} + \frac{5}{6} \cdot \frac{1}{6} \cdot \frac{5}{6} + \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{1}{6} = \frac{25}{216} + \frac{25}{216} + \frac{25}{216} = \frac{75}{216} \end{aligned}$$

$$\begin{aligned} P(X=2) &= P(2) \cdot P(2) \cdot P(\text{not } 2) + P(2) \cdot P(\text{not } 2) \cdot P(2) + P(\text{not } 2) \cdot P(2) \cdot P(2) \\ &= \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{5}{6} + \frac{1}{6} \cdot \frac{5}{6} \cdot \frac{1}{6} + \frac{5}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} = \frac{5}{216} + \frac{5}{216} + \frac{5}{216} = \frac{15}{216} \end{aligned}$$

$$P(X=3) = P(2) \cdot P(2) \cdot P(2) = \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{216}$$

$$\begin{aligned} \text{Now, } E(X) &= \sum_{i=1}^n p_i x_i \\ &= 0 \times \frac{125}{216} + 1 \times \frac{75}{216} + 2 \times \frac{15}{216} + 3 \times \frac{1}{216} \\ &= 0 + \frac{75}{216} + \frac{30}{216} + \frac{3}{216} = \frac{75+30+3}{216} = \frac{108}{216} = \frac{1}{2} \end{aligned}$$

Therefore, the required expression is $\frac{1}{2}$.

Question 29

Two biased dice are thrown together. For the first die $P(6) = 1/2$, the other scores being equally likely while for the second die, $P(1) = 2/5$ and the other scores are equally likely. Find the probability

distribution of 'the number of ones seen'.

Solution

Given that: for the first die, $P(6) = \frac{1}{2}$ and $P(\bar{6}) = 1 - \frac{1}{2} = \frac{1}{2}$

$$\Rightarrow P(1) + P(2) + P(3) + P(4) + P(5) = \frac{1}{2}$$

But, $P(1) = P(2) = P(3) = P(4) = P(5)$

So,

$$5.P(1) = \frac{1}{2} \Rightarrow P(1) = \frac{1}{10} \text{ and } P(\bar{1}) = 1 - \frac{1}{10} = \frac{9}{10}$$

For the second die, $P(1) = \frac{2}{5}$ and $P(\bar{1}) = 1 - \frac{2}{5} = \frac{3}{5}$

Let X be the number of one's seen

$\therefore X = 0, 1, 2$

$$\Rightarrow P(X=0) = \underset{\text{I}}{P(\bar{1})} \cdot \underset{\text{II}}{P(\bar{1})} = \frac{9}{10} \cdot \frac{3}{5} = \frac{27}{50} = 0.54$$

$$\begin{aligned} P(X=1) &= \underset{\text{I}}{P(\bar{1})} \cdot \underset{\text{II}}{P(1)} + \underset{\text{I}}{P(1)} \cdot \underset{\text{II}}{P(\bar{1})} \\ &= \frac{9}{10} \cdot \frac{2}{5} + \frac{1}{10} \cdot \frac{3}{5} = \frac{18+3}{50} = \frac{21}{50} = 0.42 \end{aligned}$$

$$P(X=2) = \underset{\text{I}}{P(1)} \cdot \underset{\text{II}}{P(1)} = \frac{1}{10} \cdot \frac{2}{5} = \frac{2}{50} = 0.04$$

Therefore, the required probability distribution is

X	0	1	2
P(X)	0.54	0.42	0.04

Question 30

Two probability distributions of the discrete random variable X and Y are given below.

X	0	1	2	3
P(X)	1/5	2/5	1/5	1/5

Y	0	1	2	3
P(Y)	1/5	3/10	2/5	1/10

Prove that $E(Y^2) = 2 E(X)$.

Solution

The probability distribution of random variable X is

X	0	1	2	3
P(X)	1/5	2/5	1/5	1/5

We know that,

$$E(X) = \sum_{i=1}^n P_i X_i$$

$$E(X) = 0 \cdot \frac{1}{5} + 1 \cdot \frac{2}{5} + 2 \cdot \frac{1}{5} + 3 \cdot \frac{1}{5} = 0 + \frac{2}{5} + \frac{2}{5} + \frac{3}{5} = \frac{7}{5}$$

For the second probability distribution of random variable Y,

Y	0	1	2	3
P(Y)	1/5	3/10	2/5	1/10

$$E(Y^2) = 0 \cdot \frac{1}{5} + 1 \cdot \frac{3}{10} + 4 \cdot \frac{2}{5} + 9 \cdot \frac{1}{10}$$

$$= 0 + \frac{3}{10} + \frac{8}{5} + \frac{9}{10} = \frac{28}{10} = \frac{14}{5}$$

$$\text{Now, } E(Y^2) = \frac{14}{5} \text{ and } 2 E(X) = 2 \cdot \frac{7}{5} = \frac{14}{5}$$

$$\text{Therefore, } E(Y^2) = 2E(X)$$

Question 31

A factory produces bulbs. The probability that any one bulb is defective is $\frac{1}{50}$ and they are packed in boxes of 10. From a single box, find the probability that

- (i) none of the bulbs is defective
- (ii) exactly two bulbs are defective
- (iii) more than 8 bulbs work properly

Solution

Let's assume X to be the random variable denoting a bulb to be defective.

Here, $n = 10$, $p = 1/50$, $q = 1 - 1/50 = 49/50$

We know that, $P(X = r) = {}^nC_r p^r q^{n-r}$

(i) None of the bulbs is defective, i.e., $r = 0$

$$P(x = 0) = {}^{10}C_0 (1/50)^0 (49/50)^{10-0} = (49/50)^{10}$$

(ii) Exactly two bulbs are defective

$$\text{So, } P(x = 2) = {}^{10}C_2 (1/50)^2 (49/50)^{10-2}$$

$$= 45 \cdot 49^8 / 50^{10} = 45 \times (1/50)^{10} \times 49^8$$

(iii) More than 8 bulbs work properly

We can say that less than 2 bulbs are defective

$$P(x < 2) = P(x = 0) + P(x = 1)$$

$$\begin{aligned} &= {}^{10}C_0 \left(\frac{1}{50}\right)^0 \left(\frac{49}{50}\right)^{10} + {}^{10}C_1 \left(\frac{1}{50}\right)^1 \left(\frac{49}{50}\right)^9 = \left(\frac{49}{50}\right)^{10} + \frac{1}{5} \left(\frac{49}{50}\right)^9 \\ &= \left(\frac{49}{50}\right)^9 \left(\frac{49}{50} + \frac{1}{5}\right) = \left(\frac{49}{50}\right)^9 \left(\frac{59}{50}\right) = \frac{59(49)^9}{(50)^{10}}. \end{aligned}$$

Question 32

Suppose you have two coins which appear identical in your pocket. You know that one is fair and one is 2-headed. If you take one out, toss it and get a head, what is the probability that it was a fair coin?

Solution

Let's consider

E_1 = Event that the coin is fair

E_2 = Event that the coin is 2 headed

And H = Event that the tossed coin gets head.

Now,

$$P(E_1) = \frac{1}{2}, P(E_2) = \frac{1}{2}, P(H/E_1) = \frac{1}{2}, P(H/E_2) = 1$$

Using Baye's Theorem, we get

$$\begin{aligned} P(E_1/H) &= \frac{P(E_1) \cdot P(H/E_1)}{P(E_1) \cdot P(H/E_1) + P(E_2) \cdot P(H/E_2)} \\ &= \frac{\frac{1}{2} \cdot \frac{1}{2}}{\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot 1} = \frac{\frac{1}{4}}{\frac{1}{4} + \frac{1}{2}} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3} \end{aligned}$$

Therefore, the required probability is $1/3$.

Question 33

Suppose that 6% of the people with blood group O are left-handed and 10% of those with other blood groups are left-handed 30% of the people have blood group O. If a left-handed person is selected at random, what is the probability that he/she will have blood group O?

Solution

Let's assume

E_1 = The event that a person selected is of blood group O

E_2 = The event that the people selected is of other group

And H = The event that selected person is left handed

Now,

$$P(E_1) = 0.30 \text{ and } P(E_2) = 0.70$$

$$\text{Further, } P(H/E_1) = 0.06 \text{ and } P(H/E_2) = 0.10$$

Using Baye's Theorem, we have

$$\begin{aligned}
 P\left(\frac{E_1}{H}\right) &= \frac{P(E_1) \cdot P(H/E_1)}{P(E_1) \cdot P(H/E_1) + P(E_2) \cdot P(H/E_2)} \\
 &= \frac{0.30 \times 0.06}{0.30 \times 0.06 + 0.70 \times 0.10} = \frac{0.018}{0.018 + 0.070} = \frac{0.018}{0.088} = \frac{9}{44}
 \end{aligned}$$

Therefore, the required probability is 9/44.

Question 34

Two natural numbers r, s are drawn one at a time, without replacement from the set $S = \{1, 2, 3, \dots, n\}$. Find $P[r \leq p | s \leq p]$, where $p \in S$.

Solution

Given, $S = \{1, 2, 3, \dots, n\}$

So, $P(r \leq p | s \leq p) = P(P \cap S) / P(S)$

$$= p - 1/n \times n/(n - 1)$$

$$= (p - 1)/(n - 1)$$

Therefore, the required probability is $(p - 1)/(n - 1)$.

Question 35

Find the probability distribution of the maximum of the two scores obtained when a die is thrown twice. Determine also the mean of the distribution.

Solution

Let X be the random variable scores when a die is thrown twice.

$$X = 1, 2, 3, 4, 5, 6$$

$$\text{And } S = \{(1, 1), (1, 2), (2, 1), (2, 2), (1, 3), (2, 3), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), \dots, (6, 6)\}$$

So,

$$P(X = 1) = 1/6 \cdot 1/6 = 1/36$$

$$P(X = 2) = 1/6 \cdot 1/6 + 1/6 \cdot 1/6 + 1/6 \cdot 1/6 = 3/36$$

$$P(X = 3) = 1/6 \cdot 1/6 + 1/6 \cdot 1/6 + 1/6 \cdot 1/6 + 1/6 \cdot 1/6 + 1/6 \cdot 1/6 = 5/36$$

Similarly,

$$P(X = 4) = 7/36; P(X = 5) = 9/36 \text{ and } P(X = 6) = 11/36$$

Hence, the required distribution is

X	1	2	3	4	5	6
P(X)	1/36	3/36	5/36	7/36	9/36	11/36

$$\begin{aligned}
 \text{Now, the mean } E(X) &= \sum_{i=1}^n x_i p_i \\
 &= 1 \times \frac{1}{36} + 2 \times \frac{3}{36} + 3 \times \frac{5}{36} + 4 \times \frac{7}{36} + 5 \times \frac{9}{36} + 6 \times \frac{11}{36} \\
 &= \frac{1}{36} + \frac{6}{36} + \frac{15}{36} + \frac{28}{36} + \frac{45}{36} + \frac{66}{36} = \frac{161}{36}
 \end{aligned}$$

Therefore, the required mean = 161/36.

Question 36

The random variable X can take only the values 0, 1, 2. Given that $P(X = 0) = P(X = 1) = p$ and that $E(X^2) = E[X]$, find the value of p .

Solution

Given, $X = 0, 1, 2$ and $P(X = 0) = P(X = 1) = p$

Let $P(X)$ at $X = 2$ is x

$$\Rightarrow p + p + x = 1 \Rightarrow x = 1 - 2p$$

Now, we have the following distribution:

X	0	1	2
P(X)	p	p	1 - 2p

So,

$$E(X) = 0 \cdot p + 1 \cdot p + 2(1 - 2p) = p + 2 - 4p = 2 - 3p$$

$$\text{And, } E(X^2) = 0 \cdot p + 1 \cdot p + 4(1 - 2p) = p + 4 - 8p = 4 - 7p$$

$$\text{Also, given } E(X) = E(X^2)$$

$$2 - 3p = 4 - 7p$$

$$4p = 2$$

$$p = \frac{1}{2}$$

Therefore, the required value of p is $\frac{1}{2}$.

Question 37

Find the variance of the distribution:

x	0	1	2	3	4	5
P(x)	1/6	5/18	2/9	1/6	1/9	1/18

Solution

We know that,

$$\text{Variance}(X) = E(X^2) - [E(X)]^2$$

Now,

$$\begin{aligned} E(X) &= \sum_{i=1}^n p_i x_i \\ &= 0 \times \frac{1}{6} + 1 \times \frac{5}{18} + 2 \times \frac{2}{9} + 3 \times \frac{1}{6} + 4 \times \frac{1}{9} + 5 \times \frac{1}{18} \\ &= 0 + \frac{5}{18} + \frac{4}{9} + \frac{3}{6} + \frac{4}{9} + \frac{5}{18} = \frac{5+8+9+8+5}{18} = \frac{35}{18} \end{aligned}$$

$$\begin{aligned} E(X^2) &= 0 \times \frac{1}{6} + 1 \times \frac{5}{18} + 4 \times \frac{2}{9} + 9 \times \frac{1}{6} + 16 \times \frac{1}{9} + 25 \times \frac{1}{18} \\ &= \frac{5}{18} + \frac{8}{9} + \frac{9}{6} + \frac{16}{9} + \frac{25}{18} = \frac{5+16+27+32+25}{18} = \frac{105}{18} \end{aligned}$$

$$\therefore \text{Var}(X) = \frac{105}{18} - \frac{35}{18} \times \frac{35}{18} = \frac{1890 - 1225}{324} = \frac{665}{324}$$

Therefore, the required variance is $665/324$.

Question 38

A and B throw a pair of dice alternately. A wins the game if he gets a total of 6 and B wins if she gets a total of 7. If A starts the game, find the probability of winning the game by A in third throw of the pair of dice.

Solution

Let's take A_1 to be the event of getting a total of 6

$$A_1 = \{(2, 4), (4, 2), (1, 5), (5, 1), (3, 3)\}$$

And, B_1 be the event of getting a total of 7

$$B_1 = \{(2, 5), (5, 2), (1, 6), (6, 1), (3, 4), (4, 3)\}$$

Let $P(A_1)$ is the probability, if A wins in a throw = $5/36$

And $P(B_1)$ is the probability, if B wins in a throw = $1/6$

Therefore, the required probability of winning A in his third throw

$$= P(\bar{A}_1) \cdot P(\bar{B}_1) \cdot P(A_1) = \frac{31}{36} \cdot \frac{5}{6} \cdot \frac{5}{36} = \frac{775}{7776}$$

Question 39

Two dice are tossed. Find whether the following two events A and B are independent:

$$A = \{(x, y) : x + y = 11\} \quad B = \{(x, y) : x \neq 5\}$$

where (x, y) denotes a typical sample point.

Solution

Given, two events A and B are independent such that $A = \{(x, y) : x + y = 11\}$ $B = \{(x, y) : x \neq 5\}$

Now,

$$A = \{(5, 6), (6, 5)\}$$

$B = \{(1, 1), (1, 2), (1, 3), (1, 5), (1, 6), (2, 1), (2, 2), (2, 4), (2, 5), (2, 6), (3, 1), (3, 3), (3, 4), (3, 5), (3, 6), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}$

$n(A) = 2$, $n(B) = 30$ and $n(A \cap B) = 1$

So, $P(A) = 2/36 = 1/18$ and $P(B) = 30/36 = 5/6$

Now,

$P(A) \cdot P(B) = 1/18 \cdot 5/6 = 5/108$ and $P(A \cap B) = 1/36$

As $P(A) \cdot P(B) \neq P(A \cap B)$

Therefore, events A and B are not independent.

Question 40

An urn contains m white and n black balls. A ball is drawn at random and is put back into the urn along with k additional balls of the same colour as that of the ball drawn. A ball is again drawn at random. Show that the probability of drawing a white ball now does not depend on k .

Solution

Let's consider A to be the event of having m white and n black balls

E_1 = First ball drawn of white colour

E_2 = First ball drawn of black colour

E_3 = Second ball drawn of white colour

$$\therefore P(E_1) = \frac{m}{m+n} \text{ and } P(E_2) = \frac{n}{m+n}$$

$$P(E_3/E_1) = \frac{m+k}{m+n+k} \text{ and } P(E_3/E_2) = \frac{m}{m+n+k}$$

$$\begin{aligned} \text{Now, } P(E_3) &= P(E_1) \cdot P(E_3/E_1) + P(E_2) \cdot P(E_3/E_2) \\ &= \frac{m}{m+n} \times \frac{m+k}{m+n+k} + \frac{n}{m+n} \times \frac{m}{m+n+k} \\ &= \frac{m}{m+n+k} \left[\frac{m+k}{m+n} + \frac{n}{m+n} \right] \\ &= \frac{m}{m+n+k} \left[\frac{m+n+k}{m+n} \right] = \frac{m}{m+n} \end{aligned}$$

Therefore, the probability of drawing a white ball does not depend upon k .

Long Answer (L.A.)

Question 41

Three bags contain a number of red and white balls as follows:

Bag 1 : 3 red balls, Bag 2 : 2 red balls and 1 white ball

Bag 3 : 3 white balls.

The probability that bag i will be chosen and a ball is selected from it is $i/6$, $i = 1, 2, 3$. What is the probability that

(i) a red ball will be selected? (ii) a white ball is selected?

Solution

Given:

Bag 1 : 3 red balls,

Bag 2 : 2 red balls and 1 white ball

Bag 3 : 3 white balls

Now, let E_1 , E_2 and E_3 be the events of choosing Bag 1, Bag 2 and Bag 3 respectively and a ball is drawn from it.

And, we have

$$P(E_1) = 1/6, P(E_2) = 2/6 \text{ and } P(E_3) = 3/6$$

(i) Let E be the event that red ball is selected

$$\begin{aligned}\therefore P(E) &= P(E_1) \cdot P(E/E_1) + P(E_2) \cdot P(E/E_2) + P(E_3) \cdot P(E/E_3) \\ &= \frac{1}{6} \cdot \frac{3}{3} + \frac{2}{6} \cdot \frac{2}{3} + \frac{3}{6} \cdot 0 = \frac{3}{18} + \frac{4}{18} = \frac{7}{18}\end{aligned}$$

(ii) Let F be the event that a white ball is selected

$$\begin{aligned}\therefore P(F) &= 1 - P(E) & [P(E) + P(F) = 1] \\ &= 1 - \frac{7}{18} = \frac{11}{18}\end{aligned}$$

Therefore, the required probabilities are $7/18$ and $11/18$.

Question 42

Refer to Question 41 above. If a white ball is selected, what is the probability that it came from (i) Bag 2 (ii) Bag 3

Solution

Referring the Question 41, here we will use Baye's Theorem

$$\begin{aligned}
 (i) P(E_2/F) &= \frac{P(E_2) \cdot P(F/E_2)}{P(E_1) \cdot P(F/E_1) + P(E_2) \cdot P(F/E_2) + P(E_3) \cdot P(F/E_3)} \\
 &= \frac{\frac{2}{6} \cdot \frac{1}{3}}{\frac{1}{6} \cdot 0 + \frac{2}{6} \cdot \frac{1}{3} + \frac{3}{6} \cdot 1} = \frac{\frac{2}{18}}{\frac{2}{18} + \frac{3}{6}} = \frac{2}{11} \\
 (ii) P(E_3/F) &= \frac{P(E_3) \cdot P(F/E_3)}{P(E_1) \cdot P(F/E_1) + P(E_2) \cdot P(F/E_2) + P(E_3) \cdot P(F/E_3)} \\
 &= \frac{\frac{3}{6} \cdot 1}{\frac{1}{6} \cdot 0 + \frac{2}{6} \cdot \frac{1}{3} + \frac{3}{6} \cdot 1} = \frac{\frac{3}{6}}{\frac{2}{18} + \frac{3}{6}} = \frac{3}{6} \times \frac{18}{11} = \frac{9}{11}
 \end{aligned}$$

Therefore, the required probabilities are 2/11 and 9/11.

Question 43

A shopkeeper sells three types of flower seeds A_1 , A_2 and A_3 . They are sold as a mixture where the proportions are 4:4:2 respectively. The germination rates of the three types of seeds are 45%, 60% and 35%. Calculate the probability

- (i) of a randomly chosen seed to germinate
- (ii) that it will not germinate given that the seed is of type A_3 ,
- (iii) that it is of the type A_2 given that a randomly chosen seed does not germinate.

Solution

Given that: $A_1 : A_2 : A_3 = 4 : 4 : 2$

So, the probabilities will be

$P(A_1) = 4/10$, $P(A_2) = 4/10$ and $P(A_3) = 2/10$,

Where A_1 , A_2 and A_3 are the three types of seeds.

Let E be the event that a seed germinates and $\bar{E}$ be the event that a seed does not germinate

$$\therefore P\left(\frac{E}{A_1}\right) = \frac{45}{100}, P\left(\frac{E}{A_2}\right) = \frac{60}{100} \text{ and } P\left(\frac{E}{A_3}\right) = \frac{35}{100}$$

$$\text{and } P\left(\frac{\bar{E}}{A_1}\right) = \frac{55}{100}, P\left(\frac{\bar{E}}{A_2}\right) = \frac{40}{100} \text{ and } P\left(\frac{\bar{E}}{A_3}\right) = \frac{65}{100}$$

$$\begin{aligned} (i) \quad P(E) &= P(A_1) \cdot P\left(\frac{E}{A_1}\right) + P(A_2) \cdot P\left(\frac{E}{A_2}\right) + P(A_3) \cdot P\left(\frac{E}{A_3}\right) \\ &= \frac{4}{10} \cdot \frac{45}{100} + \frac{4}{10} \cdot \frac{60}{100} + \frac{2}{10} \cdot \frac{35}{100} \\ &= \frac{180}{1000} + \frac{240}{1000} + \frac{70}{1000} = \frac{490}{1000} = 0.49 \end{aligned}$$

$$(ii) \quad P(\bar{E}/A_3) = 1 - P(E/A_3) = 1 - \frac{35}{100} = \frac{65}{100} = 0.65$$

(iii) Using Bayes' Theorem, we get

$$\begin{aligned} P(A_2/\bar{E}) &= \frac{P(A_2) \cdot P(\bar{E}/A_2)}{P(A_1) \cdot P(\bar{E}/A_1) + P(A_2) \cdot P(\bar{E}/A_2) + P(A_3) \cdot P(\bar{E}/A_3)} \\ &= \frac{\frac{4}{10} \cdot \frac{40}{100}}{\frac{4}{10} \cdot \frac{55}{100} + \frac{4}{10} \cdot \frac{40}{100} + \frac{2}{10} \cdot \frac{65}{100}} \\ &= \frac{\frac{160}{1000}}{\frac{220}{1000} + \frac{160}{1000} + \frac{130}{1000}} = \frac{160}{220 + 160 + 130} = \frac{160}{510} = \frac{16}{51} = 0.314 \end{aligned}$$

Therefore, the required probability is $16/51$ or 0.314 .

Question 44

A letter is known to have come either from TATA NAGAR or from CALCUTTA. On the envelope, just two consecutive letter TA are visible. What is the probability that the letter came from TATA NAGAR.

Solution

Let E_1 be the event that the letter comes from TATA NAGAR,

E_2 be the event that the letter comes from CALCUTTA

And, E_3 be the event that on the letter, two consecutive letters TA are visible

Now,

$$P(E_1) = \frac{1}{2}, P(E_2) = \frac{1}{2}, P(E_3/E_1) = \frac{2}{8} \text{ and } P(E_3/E_2) = \frac{1}{7}$$

For TATA NAGAR, the two consecutive letters visible are TA, AT, TA, AN, NA, AG, GA and AR

$$\text{So, } P(E_3/E_1) = 2/8$$

And, for CALCULTTA the two consecutive letters visible are CA, AL, LC, CU, UT, TT and TA

$$\text{So, } P(E_3/E_2) = 1/7$$

Now, using Baye's Theorem we have

$$\begin{aligned} P(E_1/E_3) &= \frac{P(E_1) \cdot P(E_3/E_1)}{P(E_1) \cdot P(E_3/E_1) + P(E_2) \cdot P(E_3/E_2)} \\ &= \frac{\frac{1}{2} \cdot \frac{2}{8}}{\frac{1}{2} \cdot \frac{2}{8} + \frac{1}{2} \cdot \frac{1}{7}} = \frac{\frac{1}{8}}{\frac{1}{8} + \frac{1}{14}} = \frac{\frac{1}{8}}{\frac{7+4}{56}} = \frac{7}{11} \end{aligned}$$

Therefore, the required probability is 7/11.

Question 45

There are two bags, one of which contains 3 black and 4 white balls while the other contains 4 black and 3 white balls. A die is thrown. If it shows up 1 or 3, a ball is taken from the 1st bag; but it shows up any other number, a ball is chosen from the second bag. Find the probability of choosing a black ball.

Solution

Let E_1 be the event of selecting Bag 1 and E_2 be the event of selecting Bag 2.

Also, let E_3 be the event that black ball is selected

Now,

$$P(E_1) = 2/6 = 1/3 \text{ and } P(E_2) = 1 - 1/3 = 2/3$$

$$P(E_3/E_1) = 3/7 \text{ and } P(E_3/E_2) = 4/7$$

So,

$$P(E_3) = P(E_1) \cdot P(E_3/E_1) + P(E_2) \cdot P(E_3/E_2)$$

$$= 1/3 \cdot 3/7 + 2/3 \cdot 4/7 = (3 + 8)/21 = 11/21$$

Therefore, the required probability is $11/21$.

Question 46

There are three urns containing 2 white and 3 black balls, 3 white and 2 black balls, and 4 white and 1 black balls, respectively. There is an equal probability of each urn being chosen. A ball is drawn at random from the chosen urn and it is found to be white. Find the probability that the ball drawn was from the second urn.

Solution

Given, we have 3 urns:

Urn 1 = 2 white and 3 black balls

Urn 2 = 3 white and 2 black balls

Urn 3 = 4 white and 1 black balls

Now, the probabilities of choosing either of the urns are

$$P(U_1) = P(U_2) = P(U_3) = 1/3$$

Let H be the event of drawing white ball from the chosen urn.

So,

$$P(H/U_1) = 2/5, P(H/U_2) = 3/5 \text{ and } P(H/U_3) = 4/5$$

By using Baye's Theorem, we have

$$\begin{aligned} P(U_2/H) &= \frac{P(U_2) \cdot P(H/U_2)}{P(U_1) \cdot P(H/U_1) + P(U_2) \cdot P(H/U_2) + P(U_3) \cdot P(H/U_3)} \\ &= \frac{\frac{1}{3} \cdot \frac{3}{5}}{\frac{1}{3} \cdot \frac{2}{5} + \frac{1}{3} \cdot \frac{3}{5} + \frac{1}{3} \cdot \frac{4}{5}} = \frac{\frac{3}{5}}{\frac{2}{5} + \frac{3}{5} + \frac{4}{5}} = \frac{3}{9} = \frac{1}{3} \end{aligned}$$

Therefore, the required probability is $1/3$.

Question 47

By examining the chest X ray, the probability that TB is detected when a person is actually suffering is 0.99. The probability of an healthy person diagnosed to have TB is 0.001. In a certain city, 1 in 1000 people suffers from TB. A person is selected at random and is diagnosed to have TB. What is the probability that he actually has TB?

Solution

Let E_1 = Event that a person has TB

E_2 = Event that a person does not have TB

And H = Event that the person is diagnosed to have TB.

So,

$$P(E_1) = 1/1000 = 0.001, P(E_2) = 1 - 1/1000 = 999/1000 = 0.999$$

$$P(H/E_1) = 0.99, P(H/E_2) = 0.001$$

Now, using Baye's theorem we have

$$\begin{aligned} \therefore P(E_1/H) &= \frac{P(E_1) \cdot P(H/E_1)}{P(E_1) \cdot P(H/E_1) + P(E_2) \cdot P(H/E_2)} \\ &= \frac{0.001 \times 0.99}{0.001 \times 0.99 + 0.999 \times 0.001} = \frac{0.99}{0.99 + 0.999} \\ &= \frac{0.990}{0.990 + 0.999} = \frac{990}{1989} = \frac{110}{221} \end{aligned}$$

Therefore, the required probability is 110/221.

Question 48

An item is manufactured by three machines A, B and C. Out of the total number of items manufactured during a specified period, 50% are manufactured on A, 30% on B and 20% on C. 2% of the items produced on A and 2% of items produced on B are defective, and 3% of these produced on C are defective. All the items are stored at one godown. One item is drawn at random and is found to be defective. What is the probability that it was manufactured on machine A?

Solution

Let's consider:

E_1 = The event that the item is manufactured on machine A

E_2 = The event that the item is manufactured on machine B

E_3 = The event that the item is manufactured on machine C

Let H be the event that the selected item is defective.

Thus, by using Baye's theorem we have

$$P(E_1) = 50/100, P(E_2) = 30/100, P(E_3) = 20/100$$

$$P(H/E_1) = 2/100, P(H/E_2) = 2/100 \text{ and } P(H/E_3) = 3/100$$

$$\begin{aligned} \therefore P(E_1/H) &= \frac{P(E_1).P(H/E_1)}{P(E_1).P(H/E_1) + P(E_2).P(H/E_2) + P(E_3).P(H/E_3)} \\ &= \frac{\frac{50}{100} \times \frac{2}{100}}{\frac{50}{100} \times \frac{2}{100} + \frac{30}{100} \times \frac{2}{100} + \frac{20}{100} \times \frac{3}{100}} \\ &= \frac{100}{100 + 60 + 60} = \frac{100}{220} = \frac{10}{22} = \frac{5}{11} \end{aligned}$$

Therefore, the required probability is 5/11.

Question 49

Let X be a discrete random variable whose probability distribution is defined as follows:

$$P(X = x) = \begin{cases} k(x+1) & \text{for } x = 1, 2, 3, 4 \\ 2kx & \text{for } x = 5, 6, 7 \\ 0 & \text{otherwise} \end{cases}$$

where k is a constant. Calculate

(i) the value of k (ii) $E(X)$ (iii) Standard deviation of X .

Solution

(i) Given, $P(X = x) = k(x + 1)$ for $x = 1, 2, 3, 4$

$$\text{So, } P(X = 1) = k(1 + 1) = 2k$$

$$P(X = 2) = k(2 + 1) = 3k$$

$$P(X = 3) = k(3 + 1) = 4k$$

$$P(X = 4) = k(4 + 1) = 5k$$

Also, $P(X = x) = 2kx$ for $x = 5, 6, 7$

$$P(X = 5) = 2k(5) = 10k$$

$$P(X = 6) = 2k(6) = 12k$$

$$P(X = 7) = 2k(7) = 14k$$

And, for otherwise it is 0.

Thus, the probability distribution is given by

X	1	2	3	4	5	6	7	Otherwise
P(X)	2k	3k	4k	5k	10k	12k	14k	0

We know that,

$$\sum_{i=1}^n P(X_i) = 1$$

$$\text{So, } 2k + 3k + 4k + 5k + 10k + 12k + 14k = 1$$

$$\Rightarrow 50k = 1$$

$$\Rightarrow k = 1/50$$

Therefore, the value of k is $1/50$.

(ii) Now, the probability distribution is

X	1	2	3	4	5	6	7
---	---	---	---	---	---	---	---

P(X)	2/50	3/50	4/50	5/50	10/50	12/50	14/50
------	------	------	------	------	-------	-------	-------

$$E(X) = 1 \times \frac{2}{50} + 2 \times \frac{3}{50} + 3 \times \frac{4}{50} + 4 \times \frac{5}{50} + 5 \times \frac{10}{50} + 6 \times \frac{12}{50} + 7 \times \frac{14}{50}$$

$$= \frac{2}{50} + \frac{6}{50} + \frac{12}{50} + \frac{20}{50} + \frac{50}{50} + \frac{72}{50} + \frac{98}{50}$$

$$= \frac{260}{50}$$

$$= \frac{26}{5} = 5.2$$

(ii) We know that the standard deviation (SD) = $\sqrt{\text{Variance}}$

$$\text{Variance} = E(X^2) - [E(X)]^2$$

$$\begin{aligned} E(X^2) &= 1 \times \frac{2}{50} + 4 \times \frac{3}{50} + 9 \times \frac{4}{50} + 16 \times \frac{5}{50} + 25 \times \frac{10}{50} + 36 \times \frac{12}{50} + 49 \times \frac{14}{50} \\ &= \frac{2}{50} + \frac{12}{50} + \frac{36}{50} + \frac{80}{50} + \frac{250}{50} + \frac{432}{50} + \frac{686}{50} = \frac{1498}{50} \end{aligned}$$

$$\begin{aligned} \text{Now,} \\ \text{Variance (X)} &= \frac{1498}{50} - \left(\frac{26}{5}\right)^2 \\ &= \frac{1498}{50} - \frac{676}{25} = \frac{1498 - 1352}{50} = \frac{146}{50} = 2.92 \end{aligned}$$

$$\therefore \text{S.D} = \sqrt{2.92} = 1.7 \text{ (approx.)}$$

Question 50

The probability distribution of a discrete random variable X is given as under:

X	1	2	4	2A	3A	5A
P(X)	1/2	1/5	3/25	1/10	1/25	1/25

Calculate:

(i) The value of A if $E(X) = 2.94$

(ii) Variance of X.

Solution

(i) We know that:

$$E(X) = \sum_{i=1}^n P_i X_i$$

$$E(X) = 1 \times \frac{1}{2} + 2 \times \frac{1}{5} + 4 \times \frac{3}{25} + 2A \times \frac{1}{10} + 3A \times \frac{1}{25} + 5A \times \frac{1}{25}$$

$$2.94 = \frac{1}{2} + \frac{2}{5} + \frac{12}{25} + \frac{A}{5} + \frac{3A}{25} + \frac{A}{5}$$

$$2.94 = 0.5 + 0.4 + 0.48 + \frac{13A}{25} = 1.38 + \frac{13A}{25}$$

$$2.94 - 1.38 = \frac{13A}{25} \Rightarrow 1.56 = \frac{13A}{25}$$

$$A = \frac{1.56 \times 25}{13} = 0.12 \times 25$$

$$\therefore A = 3$$

(ii) Now, the distribution becomes

X	1	2	4	6	9	15
P(X)	1/2	1/5	3/25	1/10	1/25	1/25

$$E(X^2) = 1 \times \frac{1}{2} + 4 \times \frac{1}{5} + 16 \times \frac{3}{25} + 36 \times \frac{1}{10} + 81 \times \frac{1}{25} + 225 \times \frac{1}{25}$$

$$= \frac{1}{2} + \frac{4}{5} + \frac{48}{25} + \frac{36}{10} + \frac{81}{25} + \frac{225}{25}$$

$$= 0.5 + 0.8 + 1.92 + 3.6 + 3.24 + 9 = 19.06$$

$$\text{Variance (X)} = E(X^2) - [E(X)]^2$$

$$= 19.06 - (2.94)^2 = 19.06 - 8.64 = 10.42$$

Question 51

The probability distribution of a random variable x is given as under:

$$P(X = x) = \begin{cases} kx^2 & \text{for } x = 1, 2, 3 \\ 2kx & \text{for } x = 4, 5, 6 \\ 0 & \text{otherwise} \end{cases}$$

where k is a constant. Calculate

(i) $E(X)$ (ii) $E(3X^2)$ (iii) $P(X \geq 4)$

Solution

The probability distribution of random variable X is given by:

X	1	2	3	4	5	6	Otherwise
$P(X)$	k	$4k$	$9k$	$8k$	$10k$	$12k$	0

We know that

$$\sum_{i=1}^n P(X_i) = 1$$

$$\text{So, } k + 4k + 9k + 8k + 10k + 12k = 1$$

$$44k = 1$$

$$\sum_{i=1}^n P(X_i) = 1$$

$$k = 1/44$$

$$(i) E(X) =$$

$$= 1 \times k + 2 \times 4k + 3 \times 9k + 4 \times 8k + 5 \times 10k + 6 \times 12k$$

$$= k + 8k + 27k + 32k + 50k + 72k = 190k$$

$$= 190 \times 1/44 = 95/22 = 4.32 \text{ (approx.)}$$

$$(ii) E(X^2) = 3(k + 4 \times 4k + 9 \times 9k + 16 \times 8k + 25 \times 10k + 36 \times 12k)$$

$$= 3(k + 16k + 81k + 128k + 250k + 432k) = 3(908k)$$

$$= 3 \times 908 \times 1/44 = 2724/44 = 61.9 \text{ (approx.)}$$

$$(iii) P(X \geq 4) = P(X = 4) + P(X = 5) + P(X = 6)$$

$$= 8k + 10k + 12k = 30k$$

$$= 30 \times \frac{1}{44} = \frac{15}{22}$$

Question 52

A bag contains $(2n + 1)$ coins. It is known that n of these coins has a head on both sides whereas the rest of the coins are fair. A coin is picked up at random from the bag and is tossed. If the probability that the toss results in a head is $\frac{31}{42}$, determine the value of n .

Solution

Given, n coins are two headed coins and the remaining $(n + 1)$ coins are fair.

Let E_1 : the event that unfair coin is selected

E_2 : the event that the fair coin is selected

E : the event that the toss results in a head

So,

$$P(E_1) = \frac{n}{(2n + 1)} \text{ and } P(E_2) = \frac{(n + 1)}{(2n + 1)}$$

$$P(E/E_1) = 1 \text{ (As it's a sure event)}$$

$$P(E/E_2) = \frac{1}{2}$$

$$\begin{aligned}
 \therefore P(E) &= P(E_1) \cdot P(E/E_1) + P(E_2) \cdot P(E/E_2) \\
 &= \frac{n}{2n+1} \cdot 1 + \frac{n+1}{2n+1} \cdot \frac{1}{2} = \frac{1}{2n+1} \left(n + \frac{n+1}{2} \right) \\
 &= \frac{1}{2n+1} \left(\frac{2n+n+1}{2} \right) = \frac{3n+1}{2(2n+1)}
 \end{aligned}$$

$$\text{But, } P(E) = \frac{31}{42} \text{ (given)}$$

$$\Rightarrow \frac{3n+1}{2(2n+1)} = \frac{31}{42}$$

$$\frac{3n+1}{2n+1} = \frac{31}{21}$$

$$63n + 21 = 62n + 31$$

$$\therefore n = 10$$

Therefore, the required value of n is 10.

Question 53

Two cards are drawn successively without replacement from a well shuffled deck of cards. Find the mean and standard variation of the random variable X where X is the number of aces.

Solution

Let's consider X to be the random variable such that $X = 0, 1, 2$

Now, let E = the event of drawing an ace

And, F = the event of drawing non – ace

So,

$$\therefore P(E) = \frac{4}{52} \text{ and } P(\bar{E}) = \frac{48}{52}$$

$$\text{Now, } P(X=0) = P(\bar{E}) \cdot P(\bar{E}) = \frac{48}{52} \cdot \frac{47}{51} = \frac{188}{221}$$

$$P(X=1) = P(E) \cdot P(\bar{E}) + P(\bar{E}) \cdot P(E) = \frac{4}{52} \times \frac{48}{51} + \frac{48}{52} \times \frac{4}{51} = \frac{32}{221}$$

$$P(X=2) = P(E) \cdot P(E) = \frac{4}{52} \cdot \frac{3}{51} = \frac{1}{221}$$

We have Distribution Table:

X	0	1	2
P(X)	$\frac{188}{221}$	$\frac{32}{221}$	$\frac{1}{221}$

$$\text{Now, Mean } E(X) = 0 \times \frac{188}{221} + 1 \times \frac{32}{221} + 2 \times \frac{1}{221} = \frac{32}{221} + \frac{2}{221} = \frac{34}{221} = \frac{2}{13}$$

$$E(X^2) = 0 \times \frac{188}{221} + 1 \times \frac{32}{221} + 4 \times \frac{1}{221} = \frac{32}{221} + \frac{4}{221} = \frac{36}{221}$$

$$\therefore \text{ Variance} = E(X^2) - [E(X)]^2$$

$$= \frac{36}{221} - \left(\frac{2}{13}\right)^2 = \frac{36}{221} - \frac{4}{169} = \frac{468 - 68}{13 \times 221} = \frac{400}{2873}$$

$$\text{Standard deviation} = \sqrt{\frac{400}{2873}} = 0.377 \text{ (approx.)}$$

$$\therefore P(E) = \frac{4}{52} \text{ and } P(\bar{E}) = \frac{48}{52}$$

$$\text{Now, } P(X=0) = P(\bar{E}) \cdot P(\bar{E}) = \frac{48}{52} \cdot \frac{47}{51} = \frac{188}{221}$$

$$P(X=1) = P(E) \cdot P(\bar{E}) + P(\bar{E}) \cdot P(E) = \frac{4}{52} \times \frac{48}{51} + \frac{48}{52} \times \frac{4}{51} = \frac{32}{221}$$

$$P(X=2) = P(E) \cdot P(E) = \frac{4}{52} \cdot \frac{3}{51} = \frac{1}{221}$$

We have Distribution Table:

X	0	1	2
P(X)	$\frac{188}{221}$	$\frac{32}{221}$	$\frac{1}{221}$

$$\text{Now, Mean } E(X) = 0 \times \frac{188}{221} + 1 \times \frac{32}{221} + 2 \times \frac{1}{221} = \frac{32}{221} + \frac{2}{221} = \frac{34}{221} = \frac{2}{13}$$

$$E(X^2) = 0 \times \frac{188}{221} + 1 \times \frac{32}{221} + 4 \times \frac{1}{221} = \frac{32}{221} + \frac{4}{221} = \frac{36}{221}$$

$$\therefore \text{ Variance} = E(X^2) - [E(X)]^2$$

$$= \frac{36}{221} - \left(\frac{2}{13}\right)^2 = \frac{36}{221} - \frac{4}{169} = \frac{468 - 68}{13 \times 221} = \frac{400}{2873}$$

$$\text{Standard deviation} = \sqrt{\frac{400}{2873}} = 0.377 \text{ (approx.)}$$

Question 54

A die is tossed twice. A 'success' is getting an even number on a toss. Find the variance of the number of successes.

Solution

Let's consider E to be the event of getting even number on tossing a die.

$$\therefore P(E) = \frac{3}{6} = \frac{1}{2} \text{ and } P(\bar{E}) = 1 - \frac{1}{2} = \frac{1}{2}$$

Here $X = 0, 1, 2$

$$P(X = 0) = P(\bar{E}) \cdot P(\bar{E}) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

$$P(X = 1) = P(E) \cdot P(\bar{E}) + P(\bar{E}) \cdot P(E) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} + \frac{1}{4} = \frac{2}{4}$$

$$P(X = 2) = P(E) \cdot P(E) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

Hence, the probability distribution table is

X	0	1	2
P(X)	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$

$$E(X) = 0 \times \frac{1}{4} + 1 \times \frac{2}{4} + 2 \times \frac{1}{4} = \frac{2}{4} + \frac{2}{4} = 1$$

$$E(X^2) = 0 \times \frac{1}{4} + 1 \times \frac{2}{4} + 4 \times \frac{1}{4} = \frac{3}{2}$$

$$\therefore \text{Variance (X)} = E(X^2) - [E(X)]^2 = \frac{3}{2} - 1 = \frac{1}{2} = 0.5$$