

Chapter 11: Three Dimensional Geometry

NCERT Exemplar · Class 12 · Maths

Questions

Question 1

Find the position vector of a point A in space such that $\overrightarrow{OA}$ is inclined at 60° to OX and at 45° to OY

and $|\overrightarrow{OA}| = 10$ units.

Solution

We know that, $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$

$$\cos^2 60^\circ + \cos^2 45^\circ + \cos^2 \gamma = 1$$

$$(1/2)^2 + (1/\sqrt{2})^2 + \cos^2 \gamma = 1$$

$$1/4 + 1/2 + \cos^2 \gamma = 1$$

$$\cos^2 \gamma = 1 - 3/4 = 1/4$$

So, $\cos \gamma = \pm 1/2 \Rightarrow \cos \gamma = 1/2$ [Rejecting $\cos \gamma = -1/2$, as $\gamma < 90^\circ$]

Now,

$$\begin{aligned}\overrightarrow{OA} &= |\overrightarrow{OA}| \left(\frac{1}{2} \hat{i} + \frac{1}{\sqrt{2}} \hat{j} + \frac{1}{2} \hat{k} \right) = 10 \left(\frac{1}{2} \hat{i} + \frac{1}{\sqrt{2}} \hat{j} + \frac{1}{2} \hat{k} \right) \\ &= 5\hat{i} + 5\sqrt{2}\hat{j} + 5\hat{k}\end{aligned}$$

Thus, the position vector of A is $(5\hat{i} + 5\sqrt{2}\hat{j} + 5\hat{k})$.

Question 2

Find the vector equation of the line which is parallel to the vector $3\hat{i} - 2\hat{j} + 6\hat{k}$ and which passes through the point (1, -2, 3).

Solution

We know that the equation of line is

$$\vec{r} = \vec{a} + \vec{b}\lambda$$

$$\text{Here, } \vec{a} = \hat{i} - 2\hat{j} + 3\hat{k} \text{ and } \vec{b} = 3\hat{i} - 2\hat{j} + 6\hat{k}$$

$$\therefore \text{Equation of line is } \vec{r} = (\hat{i} - 2\hat{j} + 3\hat{k}) + \lambda(3\hat{i} - 2\hat{j} + 6\hat{k})$$

$$\Rightarrow (x\hat{i} + y\hat{j} + z\hat{k}) = (\hat{i} - 2\hat{j} + 3\hat{k}) + \lambda(3\hat{i} - 2\hat{j} + 6\hat{k})$$

$$\Rightarrow (x\hat{i} + y\hat{j} + z\hat{k}) - (\hat{i} - 2\hat{j} + 3\hat{k}) = \lambda(3\hat{i} - 2\hat{j} + 6\hat{k})$$

$$\Rightarrow (x-1)\hat{i} + (y+2)\hat{j} + (z-3)\hat{k} = \lambda(3\hat{i} - 2\hat{j} + 6\hat{k})$$

Thus, the required equation is

$$(x-1)\hat{i} + (y+2)\hat{j} + (z-3)\hat{k} = \lambda(3\hat{i} - 2\hat{j} + 6\hat{k})$$

Question 3

Show that the lines

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$$

$$\text{and } \frac{x-4}{5} = \frac{y-1}{2} = z \text{ intersect.}$$

Also, find their point of intersection.

Solution

Given equation are,

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} \text{ and } \frac{x-4}{5} = \frac{y-1}{2} = z$$

$$\text{Let, } \frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} = \lambda$$

$$\Rightarrow x = 2\lambda + 1, y = 3\lambda + 2 \text{ and } z = 4\lambda + 3$$

$$\text{And, } \frac{x-4}{5} = \frac{y-1}{2} = \frac{z}{1} = \mu$$

$$\Rightarrow x = 5\mu + 4, y = 2\mu + 1 \text{ and } z = \mu$$

If the two lines intersect each other at one point,

$$\text{then } 2\lambda + 1 = 5\mu + 4 \Rightarrow 2\lambda - 5\mu = 3 \quad \dots(i)$$

$$3\lambda + 2 = 2\mu + 1 \Rightarrow 3\lambda - 2\mu = -1 \quad \dots(ii)$$

$$\text{and } 4\lambda + 3 = \mu \Rightarrow 4\lambda - \mu = -3 \quad \dots(iii)$$

Solving eqns. (i) and (ii) we get

$$2\lambda - 5\mu = 3$$

$$3\lambda - 2\mu = -1$$

[multiply by 3]

[multiply by 2]

$$\Rightarrow 6\lambda - 15\mu = 9$$

$$6\lambda - 4\mu = -2$$

$$\begin{array}{r} (-) \quad (+) \quad (+) \\ \hline \end{array}$$

$$-11\mu = 11 \quad \therefore \mu = -1$$

Putting the value of μ in eq. (i) we get,

$$2\lambda - 5(-1) = 3$$

$$2\lambda + 5 = 3$$

$$2\lambda = -2 \Rightarrow \lambda = -1$$

$$4(-1) - (-1) = -3$$

$$-4 + 1 = -3$$

$$-3 = -3 \text{ (satisfied)}$$

So, Coordinates of the point of intersection are

$$x = 5(-1) + 4 = -5 + 4 = -1$$

$$y = 2(-1) + 1 = -2 + 1 = -1$$

$$z = -1$$

Thus, the given lines intersect each other at $(-1, -1, -1)$.

Question 4

Find the angle between the lines

$$\vec{r} = 3\hat{i} - 2\hat{j} + 6\hat{k} + \lambda(2\hat{i} + \hat{j} + 2\hat{k}) \text{ and } \vec{r} = (2\hat{j} - 5\hat{k}) + \mu(6\hat{i} + 3\hat{j} + 2\hat{k})$$

Solution

$$\text{Here, } \vec{b}_1 = 2\hat{i} + \hat{j} + 2\hat{k} \text{ and } \vec{b}_2 = 6\hat{i} + 3\hat{j} + 2\hat{k}$$

$$\begin{aligned} \text{So, } \cos \theta &= \frac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1| |\vec{b}_2|} = \frac{(2\hat{i} + \hat{j} + 2\hat{k}) \cdot (6\hat{i} + 3\hat{j} + 2\hat{k})}{\sqrt{(2)^2 + (1)^2 + (2)^2} \cdot \sqrt{(6)^2 + (3)^2 + (2)^2}} \\ &= \frac{12 + 3 + 4}{\sqrt{4 + 1 + 4} \cdot \sqrt{36 + 9 + 4}} = \frac{19}{\sqrt{9} \cdot \sqrt{49}} = \frac{19}{3 \cdot 7} = \frac{19}{21} \\ \therefore \theta &= \cos^{-1}\left(\frac{19}{21}\right) \end{aligned}$$

$$\text{Thus, the required angle is } \cos^{-1}\left(\frac{19}{21}\right).$$

$$\text{Here, } \vec{b}_1 = 2\hat{i} + \hat{j} + 2\hat{k} \text{ and } \vec{b}_2 = 6\hat{i} + 3\hat{j} + 2\hat{k}$$

$$\begin{aligned} \text{So, } \cos \theta &= \frac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1| |\vec{b}_2|} = \frac{(2\hat{i} + \hat{j} + 2\hat{k}) \cdot (6\hat{i} + 3\hat{j} + 2\hat{k})}{\sqrt{(2)^2 + (1)^2 + (2)^2} \cdot \sqrt{(6)^2 + (3)^2 + (2)^2}} \\ &= \frac{12 + 3 + 4}{\sqrt{4 + 1 + 4} \cdot \sqrt{36 + 9 + 4}} = \frac{19}{\sqrt{9} \cdot \sqrt{49}} = \frac{19}{3 \cdot 7} = \frac{19}{21} \\ \therefore \theta &= \cos^{-1}\left(\frac{19}{21}\right) \end{aligned}$$

$$\text{Thus, the required angle is } \cos^{-1}\left(\frac{19}{21}\right).$$

Question 5

Prove that the line through A (0, -1, -1) and B (4, 5, 1) intersects the line through C (3, 9, 4) and D (-4, 4, 4).

Solution

Given points, A $(0, -1, -1)$ and B $(4, 5, 1)$

C $(3, 9, 4)$ and D $(-4, 4, 4)$.

Cartesian form of equation AB is

$$\frac{x-0}{4-0} = \frac{y+1}{5+1} = \frac{z+1}{1+1} \Rightarrow \frac{x}{4} = \frac{y+1}{6} = \frac{z+1}{2}$$

and its vector form is $\vec{r} = (-\hat{j} - \hat{k}) + \lambda(4\hat{i} + 6\hat{j} + 2\hat{k})$

Similarly, equation of CD is

$$\frac{x-3}{-4-3} = \frac{y-9}{4-9} = \frac{z-4}{4-4} \Rightarrow \frac{x-3}{-7} = \frac{y-9}{-5} = \frac{z-4}{0}$$

and its vector form is $\vec{r} = (3\hat{i} + 9\hat{j} + 4\hat{k}) + \mu(-7\hat{i} - 5\hat{j})$

Now, here $\vec{a}_1 = -\hat{j} - \hat{k}$, $\vec{b}_1 = 4\hat{i} + 6\hat{j} + 2\hat{k}$
 $\vec{a}_2 = 3\hat{i} + 9\hat{j} + 4\hat{k}$, $\vec{b}_2 = -7\hat{i} - 5\hat{j}$

Shortest distance between AB and CD

$$\text{S.D.} = \left| \frac{(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|} \right|$$

$$\vec{a}_2 - \vec{a}_1 = (3\hat{i} + 9\hat{j} + 4\hat{k}) - (-\hat{j} - \hat{k}) = 3\hat{i} + 10\hat{j} + 5\hat{k}.$$

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 6 & 2 \\ -7 & -5 & 0 \end{vmatrix}$$

$$= \hat{i}(0 + 10) - \hat{j}(0 + 14) + \hat{k}(-20 + 42)$$

$$= 10\hat{i} - 14\hat{j} + 22\hat{k}$$

$$|\vec{b}_1 \times \vec{b}_2| = \sqrt{(10)^2 + (-14)^2 + (22)^2}$$

$$= \sqrt{100 + 196 + 484} = \sqrt{780}$$

$$\therefore \text{S.D} = \frac{(3\hat{i} + 10\hat{j} + 5\hat{k}) \cdot (10\hat{i} - 14\hat{j} + 22\hat{k})}{\sqrt{780}}$$

$$= \frac{30 - 140 + 110}{\sqrt{780}} = 0$$

Thus, the two lines intersect each other.

Question 6

Prove that the lines $x = py + q$, $z = ry + s$ and $x = p'y + q'$, $z = r'y + s'$ are perpendicular if $pp' + rr' + 1 = 0$.

Solution

Given , $x = py + q \Rightarrow y = \frac{x - q}{p}$

and $z = ry + s \Rightarrow y = \frac{z - s}{r}$

So, the equation becomes

$$\frac{x - q}{p} = \frac{y}{1} = \frac{z - s}{r} \text{ in which direction ratios are } a_1 = p, b_1 = 1, c_1 = r$$

Similarly $x = p'y + q' \Rightarrow y = \frac{x - q'}{p'}$

and $z = r'y + s' \Rightarrow y = \frac{z - s'}{r'}$

Hence, the equation becomes

$$\frac{x - q'}{p'} = \frac{y}{1} = \frac{z - s'}{r'} \text{ in which } a_2 = p', b_2 = 1, c_2 = r'$$

If the lines are perpendicular to each other, then

$$a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$$

$$pp' + 1 \cdot 1 + rr' = 0$$

Thus, $pp' + rr' + 1 = 0$ is the required condition.

Question 7

Find the equation of a plane which bisects perpendicularly the line joining the points A (2, 3, 4) and B (4, 5, 8) at right angles.

Solution

Given coordinates are A (2, 3, 4) and B (4, 5, 8)

Now, the coordinates of the mid-point C are $(2+4/2, 3+5/2, 4+8/2) = (3, 4, 6)$

And, the direction ratios of the normal to the plane = direction ratios of AB

$$= 4 - 2, 5 - 3, 8 - 4 = (2, 2, 4)$$

Equation of the plane is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

$$2(x - 3) + 2(y - 4) + 4(z - 6) = 0$$

$$2x - 6 + 2y - 8 + 4z - 24 = 0$$

$$2x + 2y + 4z = 38$$

$$x + y + 2z = 19$$

Thus, the required equation of plane is $x + y + 2z = 19$ or

$$\vec{r}(\hat{i} + \hat{j} + 2\hat{k}) = 19.$$

Question 8

Find the equation of a plane which is at a distance $3\sqrt{3}$ units from origin and the normal to which is equally inclined to coordinate axis.

Solution

As the normal to the plane is equally inclined to the axes we have,

$$\cos \alpha = \cos \beta = \cos \gamma$$

$$\text{So, } \cos^2 \alpha + \cos^2 \alpha + \cos^2 \alpha = 1$$

$$3 \cos^2 \alpha = 1 \Rightarrow \cos \alpha = \pm 1/\sqrt{3}$$

$$\text{And, } \cos \alpha = \cos \beta = \cos \gamma = \pm 1/\sqrt{3}$$

Now, the normal is $\vec{N} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}$

$\therefore$ Equation of the plane is $\vec{r} \cdot \vec{N} = d$

$$\vec{r} \cdot \frac{\vec{N}}{|\vec{N}|} = d$$

$$\frac{\vec{r} \cdot \left(\frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k} \right)}{1} = 3\sqrt{3}$$

$$\vec{r} \cdot \left(\frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k} \right) = 3\sqrt{3}$$

$$(x\hat{i} + y\hat{j} + z\hat{k}) \cdot \frac{1}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k}) = 3\sqrt{3}$$

$$x + y + z = 3\sqrt{3} \cdot \sqrt{3}$$

$$\Rightarrow x + y + z = 9$$

Thus, the equation of the plane is $x + y + z = 9$

Question 9

If the line drawn from the point $(-2, -1, -3)$ meets a plane at right angle at the point $(1, -3, 3)$, find the equation of the plane.

Solution

Given, points $(-2, -1, -3)$ and $(1, -3, 3)$

Direction ratios of the normal to the plane are $(1 + 2, -3 + 1, 3 + 3) = (3, -2, 6)$

Now, the equation of plane passing through one point (x_1, y_1, z_1) is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

$$3(x - 1) - 2(y + 3) + 6(z - 3) = 0$$

$$3x - 3 - 2y - 6 + 6z - 18 = 0$$

$$3x - 2y + 6z - 27 = 0 \Rightarrow 3x - 2y + 6z = 27$$

Thus, the required equation of plane is $3x - 2y + 6z = 27$.

Question 10

Find the equation of the plane through the points $(2, 1, 0)$, $(3, -2, -2)$ and $(3, 1, 7)$.

Solution

Given points are $(2, 1, 0)$, $(3, -2, -2)$ and $(3, 1, 7)$

As the equation of the plane passing through the points (x_1, y_1, z_1) , (x_2, y_2, z_2) and (x_3, y_3, z_3) is

$$(x - 2)(-21) - (y - 1)(7 + 2) + z(3) = 0$$

$$-21(x - 2) - 9(y - 1) + 3z = 0$$

$$-21x + 42 - 9y + 9 + 3z = 0$$

$$-21x - 9y + 3z + 51 = 0 \Rightarrow 7x + 3y - z - 17 = 0$$

Thus, the required equation of plane is $7x + 3y - z - 17 = 0$.

Question 11

Find the equations of the two lines through the origin which intersect the line

$$\frac{x-3}{2} = \frac{y-3}{1} = \frac{z}{1} \text{ at angles of } \pi/3 \text{ each.}$$

Solution

Any point in the given line is

$$x - 3/2 = y - 3/1 = z/1 = \lambda$$

$$x = 2\lambda + 3, y = \lambda + 3 \text{ and } z = \lambda$$

Let it be the coordinates of P

So, the direction ratios of OP are $(2\lambda + 3 - 0)$, $(\lambda + 3 - 0)$ and $(\lambda - 0) \Rightarrow 2\lambda + 3, \lambda + 3, \lambda$

But the direction ratios of the line PQ are 2, 1, 1

Now, we know that

$$\cos \theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \cdot \sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$$\cos \frac{\pi}{3} = \frac{2(2\lambda + 3) + 1(\lambda + 3) + 1 \cdot \lambda}{\sqrt{(2)^2 + (1)^2 + (1)^2} \cdot \sqrt{(2\lambda + 3)^2 + (\lambda + 3)^2 + \lambda^2}}$$

$$\frac{1}{2} = \frac{4\lambda + 6 + \lambda + 3 + \lambda}{\sqrt{6} \cdot \sqrt{4\lambda^2 + 9 + 12\lambda + \lambda^2 + 9 + 6\lambda + \lambda^2}}$$

$$\frac{\sqrt{6}}{2} = \frac{6\lambda + 9}{\sqrt{6\lambda^2 + 18\lambda + 18}} = \frac{6\lambda + 9}{\sqrt{6}\sqrt{\lambda^2 + 3\lambda + 3}}$$

$$\frac{6}{2} = \frac{3(2\lambda + 3)}{\sqrt{\lambda^2 + 3\lambda + 3}} \Rightarrow 3 = \frac{3(2\lambda + 3)}{\sqrt{\lambda^2 + 3\lambda + 3}}$$

$$1 = \frac{2\lambda + 3}{\sqrt{\lambda^2 + 3\lambda + 3}} \Rightarrow \sqrt{\lambda^2 + 3\lambda + 3} = 2\lambda + 3$$

$$\lambda^2 + 3\lambda + 3 = 4\lambda^2 + 9 + 12\lambda \text{ (On squaring on both sides)}$$

$$3\lambda^2 + 9\lambda + 6 = 0$$

$$\lambda^2 + 3\lambda + 2 = 0$$

$$(\lambda + 1)(\lambda + 2) = 0$$

$$\lambda = -1, -2$$

So, the direction are:

Thus, the required equation of planes are

$$x/1 = y/2 = z/-1 \text{ and } x/-1 = y/1 = z/-2$$

Question 12

Find the angle between the lines whose direction cosines are given by the equations $l + m + n = 0$, $l^2 + m^2 - n^2 = 0$.

Solution

Given equations are,

$$l + m + n = 0 \dots (i)$$

$$l^2 + m^2 - n^2 = 0 \dots (ii)$$

From equation (i), we have $n = -(l + m)$

Putting the value of n in equation (ii), we get

$$l^2 + m^2 + [-(l + m)]^2 = 0$$

$$l^2 + m^2 - l^2 - m^2 - 2lm = 0$$

$$-2lm = 0$$

$$lm = 0 \Rightarrow (-m - n)m = 0 \text{ [Since, } l = -m - n]$$

$$(m + n)m = 0 \Rightarrow m = 0 \text{ or } m = -n$$

$$\Rightarrow l = 0 \text{ or } l = -n$$

Now, the direction cosines of the two lines are

$$0, -n, n \text{ and } -n, 0, n \Rightarrow 0, -1, 1 \text{ and } -1, 0, 1$$

$$\cos \theta = \frac{(0\hat{i} - \hat{j} + \hat{k}) \cdot (-\hat{i} + 0\hat{j} + \hat{k})}{\sqrt{(-1)^2 + (1)^2} \sqrt{(-1)^2 + (1)^2}} = \frac{1}{\sqrt{2} \cdot \sqrt{2}} = \frac{1}{2}$$

$$\theta = \frac{\pi}{3}$$

Thus, the required angle $\pi/3$.

Question 13

If a variable line in two adjacent positions has direction cosines l, m, n and $l + dl, m + dm, n + dn$, show that the small angle dq between the two positions is given by

$$dq^2 = dl^2 + dm^2 + dn^2$$

Solution

Given that l, m, n and $l + dl, m + dm, n + dn$ are the direction cosines of a variable line in two positions

$$l^2 + m^2 + n^2 = 1 \dots (i) \text{ and}$$

$$(l + dl)^2 + (m + dm)^2 + (n + dn)^2 = 1 \dots (ii)$$

$$\begin{aligned} l^2 + \delta l^2 + 2l.\delta l + m^2 + \delta m^2 + 2m.\delta m + n^2 + \delta n^2 + 2n.\delta n &= 1 \\ (l^2 + m^2 + n^2) + (\delta l^2 + \delta m^2 + \delta n^2) + 2(l.\delta l + m.\delta m + n.\delta n) &= 1 \\ 1 + (\delta l^2 + \delta m^2 + \delta n^2) + 2(l.\delta l + m.\delta m + n.\delta n) &= 1 \end{aligned}$$

$$\Rightarrow l.\delta l + m.\delta m + n.\delta n = -\frac{1}{2}(\delta l^2 + \delta m^2 + \delta n^2)$$

Let $\vec{a}$ and $\vec{b}$ be the unit vectors along a line with d'cosines l, m, n and $(l + \delta l), (m + \delta m), (n + \delta n)$.

$$\therefore \vec{a} = l\hat{i} + m\hat{j} + n\hat{k} \text{ and } \vec{b} = (l + \delta l)\hat{i} + (m + \delta m)\hat{j} + (n + \delta n)\hat{k}$$

$$\cos \delta\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$$

$$\cos \delta\theta = \frac{(l\hat{i} + m\hat{j} + n\hat{k}) \cdot [(l + \delta l)\hat{i} + (m + \delta m)\hat{j} + (n + \delta n)\hat{k}]}{1.1} \quad \left[\because |\vec{a}| = |\vec{b}| = 1 \right]$$

$$\cos \delta\theta = l(l + \delta l) + m(m + \delta m) + n(n + \delta n)$$

$$\cos \delta\theta = l^2 + l.\delta l + m^2 + m.\delta m + n^2 + n.\delta n$$

$$\cos \delta\theta = (l^2 + m^2 + n^2) + (l.\delta l + m.\delta m + n.\delta n)$$

$$\cos \delta\theta = 1 - \frac{1}{2}(\delta l^2 + \delta m^2 + \delta n^2)$$

$$1 - \cos \delta\theta = \frac{1}{2}(\delta l^2 + \delta m^2 + \delta n^2)$$

$$2 \sin^2 \frac{\delta\theta}{2} = \frac{1}{2}(\delta l^2 + \delta m^2 + \delta n^2)$$

$$4 \sin^2 \frac{\delta\theta}{2} = \delta l^2 + \delta m^2 + \delta n^2$$

$$4 \left(\frac{\delta\theta}{2} \right)^2 = \delta l^2 + \delta m^2 + \delta n^2$$

$$(\delta\theta)^2 = \delta l^2 + \delta m^2 + \delta n^2$$

Hence proved.

$$\left[\begin{array}{l} \because \frac{\delta\theta}{2} \text{ is very small so,} \\ \sin \frac{\delta\theta}{2} = \frac{\delta\theta}{2} \end{array} \right]$$

$$\begin{aligned}
 l^2 + \delta l^2 + 2l.\delta l + m^2 + \delta m^2 + 2m.\delta m + n^2 + \delta n^2 + 2n.\delta n &= 1 \\
 (l^2 + m^2 + n^2) + (\delta l^2 + \delta m^2 + \delta n^2) + 2(l.\delta l + m.\delta m + n.\delta n) &= 1 \\
 1 + (\delta l^2 + \delta m^2 + \delta n^2) + 2(l.\delta l + m.\delta m + n.\delta n) &= 1 \\
 \Rightarrow l.\delta l + m.\delta m + n.\delta n &= -\frac{1}{2}(\delta l^2 + \delta m^2 + \delta n^2)
 \end{aligned}$$

Let $\vec{a}$ and $\vec{b}$ be the unit vectors along a line with d'cosines l, m, n and $(l + \delta l), (m + \delta m), (n + \delta n)$.

$$\therefore \vec{a} = l\hat{i} + m\hat{j} + n\hat{k} \text{ and } \vec{b} = (l + \delta l)\hat{i} + (m + \delta m)\hat{j} + (n + \delta n)\hat{k}$$

$$\cos \delta\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|}$$

$$\cos \delta\theta = \frac{(l\hat{i} + m\hat{j} + n\hat{k}) \cdot [(l + \delta l)\hat{i} + (m + \delta m)\hat{j} + (n + \delta n)\hat{k}]}{1.1} \quad \left[\because |\vec{a}| = |\vec{b}| = 1 \right]$$

$$\cos \delta\theta = l(l + \delta l) + m(m + \delta m) + n(n + \delta n)$$

$$\cos \delta\theta = l^2 + l.\delta l + m^2 + m.\delta m + n^2 + n.\delta n$$

$$\cos \delta\theta = (l^2 + m^2 + n^2) + (l.\delta l + m.\delta m + n.\delta n)$$

$$\cos \delta\theta = 1 - \frac{1}{2}(\delta l^2 + \delta m^2 + \delta n^2)$$

$$1 - \cos \delta\theta = \frac{1}{2}(\delta l^2 + \delta m^2 + \delta n^2)$$

$$2 \sin^2 \frac{\delta\theta}{2} = \frac{1}{2}(\delta l^2 + \delta m^2 + \delta n^2)$$

$$4 \sin^2 \frac{\delta\theta}{2} = \delta l^2 + \delta m^2 + \delta n^2$$

$$4 \left(\frac{\delta\theta}{2} \right)^2 = \delta l^2 + \delta m^2 + \delta n^2$$

$$(\delta\theta)^2 = \delta l^2 + \delta m^2 + \delta n^2$$

Hence proved.

$$\left[\begin{array}{l} \because \frac{\delta\theta}{2} \text{ is very small so,} \\ \sin \frac{\delta\theta}{2} = \frac{\delta\theta}{2} \end{array} \right]$$

Question 14

O is the origin and A is (a, b, c). Find the direction cosines of the line OA and the equation of plane through A at right angle to OA.

Solution

Given, O (0, 0, 0) and A(a, b, c)

So, the direction ratios of OA = a - 0, b - 0, c - 0 = a, b, c

And, the direction cosines of line OA

$$= \frac{a}{\sqrt{a^2 + b^2 + c^2}}, \frac{b}{\sqrt{a^2 + b^2 + c^2}}, \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

Now, the direction ratios of the normal to the plane are (a, b, c).

We know that, the equation of the plane passing through the point A(a, b, c) is

$$a(x - a) + b(y - b) + c(z - c) = 0$$

$$ax - a^2 + by - b^2 + cz - c^2 = 0$$

$$ax + by + cz = a^2 + b^2 + c^2$$

Thus, the required equation of the plane is $ax + by + cz = a^2 + b^2 + c^2$

Question 15

Two systems of rectangular axis have the same origin. If a plane cuts them at distances a, b, c and a', b', c' , respectively, from the origin, prove that

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{a'^2} + \frac{1}{b'^2} + \frac{1}{c'^2}.$$

Solution

Let's take OX, OY, OZ and ox, oy, oz to be two rectangular systems.

And, the equations of two planes are

$$\frac{X}{a} + \frac{Y}{b} + \frac{Z}{c} = 1 \dots(i) \quad \text{and} \quad \frac{x}{a'} + \frac{y}{b'} + \frac{z}{c'} = 1 \dots(ii)$$

Length of perpendicular from origin to plane (i) is

$$= \frac{\left| \frac{0}{a} + \frac{0}{b} + \frac{0}{c} - 1 \right|}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}} = \frac{1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}}$$

Length of perpendicular from origin to plane (ii)

$$= \frac{\left| \frac{0}{a'} + \frac{0}{b'} + \frac{0}{c'} - 1 \right|}{\sqrt{\frac{1}{a'^2} + \frac{1}{b'^2} + \frac{1}{c'^2}}} = \frac{1}{\sqrt{\frac{1}{a'^2} + \frac{1}{b'^2} + \frac{1}{c'^2}}}$$

As per the condition of the question

$$\frac{1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}} = \frac{1}{\sqrt{\frac{1}{a'^2} + \frac{1}{b'^2} + \frac{1}{c'^2}}}$$

$$\text{Thus,} \quad \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{a'^2} + \frac{1}{b'^2} + \frac{1}{c'^2}$$

$$\frac{X}{a} + \frac{Y}{b} + \frac{Z}{c} = 1 \dots(i) \quad \text{and} \quad \frac{x}{a'} + \frac{y}{b'} + \frac{z}{c'} = 1 \dots(ii)$$

Length of perpendicular from origin to plane (i) is

$$= \frac{\left| \frac{0}{a} + \frac{0}{b} + \frac{0}{c} - 1 \right|}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}} = \frac{1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}}$$

Length of perpendicular from origin to plane (ii)

$$= \frac{\left| \frac{0}{a'} + \frac{0}{b'} + \frac{0}{c'} - 1 \right|}{\sqrt{\frac{1}{a'^2} + \frac{1}{b'^2} + \frac{1}{c'^2}}} = \frac{1}{\sqrt{\frac{1}{a'^2} + \frac{1}{b'^2} + \frac{1}{c'^2}}}$$

As per the condition of the question

$$\frac{1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}} = \frac{1}{\sqrt{\frac{1}{a'^2} + \frac{1}{b'^2} + \frac{1}{c'^2}}}$$

$$\text{Thus,} \quad \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{a'^2} + \frac{1}{b'^2} + \frac{1}{c'^2}$$

Long Answer (L.A.)

Question 16

$$\frac{4-x}{2} = \frac{y}{6} = \frac{1-z}{3}.$$

Find the foot of perpendicular from the point (2, 3, -8) to the line

Also, find the perpendicular distance from the given point to the line.

Solution

Given,

$$\frac{4-x}{2} = \frac{y}{6} = \frac{1-z}{3}$$

$$\text{So, } \frac{x-4}{-2} = \frac{y}{6} = \frac{z-1}{-3} = \lambda \quad \text{is the equation of the line.}$$

Now, the coordinates of any point Q on the line are

$$x = -2\lambda + 4, y = 6\lambda \text{ and } z = -3\lambda + 1 \text{ and the given point is } P(2, 3, -8)$$

The direction ratios of PQ are $-2\lambda + 4 - 2, 6\lambda - 3, -3\lambda + 1 + 8$ i.e. $-2\lambda + 2, 6\lambda - 3, -3\lambda + 9$

And the direction ratios of the given line are $-2, 6, -3$.

If $PQ \perp$ line, then

$$-2(-2\lambda + 2) + 6(6\lambda - 3) - 3(-3\lambda + 9) = 0$$

$$4\lambda - 4 + 36\lambda - 18 + 9\lambda - 27 = 0$$

$$49\lambda - 49 = 0$$

$$\lambda = 1$$

Now, the foot of the perpendicular is $-2(1) + 4, 6(1), -3(1) + 1$ i.e. $2, 6, -2$

Hence, the distance PQ is

$$\begin{aligned} &= \sqrt{(2-2)^2 + (3-6)^2 + (-8+2)^2} \\ &= \sqrt{9+36} = \sqrt{45} = 3\sqrt{5} \end{aligned}$$

Thus, the required coordinates of the foot of perpendicular are $2, 6, -2$ and the required distance is $3\sqrt{5}$ units.

Question 17

Find the distance of a point $(2, 4, -1)$ from the line

$$\frac{x+5}{1} = \frac{y+3}{4} = \frac{z-6}{-9}$$

Solution

Given equation of line,

$$\frac{x+5}{1} = \frac{y+3}{4} = \frac{z-6}{-9} = \lambda \text{ and any point } P(2, 4, -1)$$

Let Q be any point on the given line

So, Coordinates of Q are $x = \lambda - 5, y = 4\lambda - 3, z = -9\lambda + 6$

D'r ratios of PQ are $\lambda - 5 - 2, 4\lambda - 3 - 4, -9\lambda + 6 + 1$ i.e., $\lambda - 7, 4\lambda - 7, -9\lambda + 7$

and the d'r ratios of the line are 1, 4, -9

If $PQ \perp$ line then

$$1(\lambda - 7) + 4(4\lambda - 7) - 9(-9\lambda + 7) = 0$$

$$\lambda - 7 + 16\lambda - 28 + 81\lambda - 63 = 0$$

$$98\lambda - 98 = 0 \Rightarrow \lambda = 1$$

So, the coordinates of Q are $1 - 5, 4 \times 1 - 3, -9 \times 1 + 6$ i.e., $-4, 1, -3$

$$\begin{aligned} \therefore PQ &= \sqrt{(-4 - 2)^2 + (1 - 4)^2 + (-3 + 1)^2} \\ &= \sqrt{(-6)^2 + (-3)^2 + (-2)^2} = \sqrt{36 + 9 + 4} = \sqrt{49} = 7 \end{aligned}$$

Thus, the required distance is 7 units.

Question 18

Find the length and the foot of perpendicular from the point $(1, 3/2, 2)$ to the plane $2x - 2y + 4z + 5 = 0$.

Solution

Given plane is $2x - 2y + 4z + 5 = 0$ and point $(1, 3/2, 2)$

The direction ratios of the normal to the plane are 2, -2, 4

So, the equation of the line passing through $(1, 3/2, 2)$ and direction ratios are equal to the direction ratios of the normal to the plane i.e. 2, -2, 4 is

$$\frac{x-1}{2} = \frac{y-\frac{3}{2}}{-2} = \frac{z-2}{4} = \lambda$$

Now, any point in the plane is $2\lambda + 1, -2\lambda + 3/2, 4\lambda + 2$

Since, the point lies in the plane, then

$$2(2\lambda + 1) - 2(-2\lambda + 3/2) + 4(4\lambda + 2) + 5 = 0$$

$$4\lambda + 2 + 4\lambda - 3 + 16\lambda + 8 + 5 = 0$$

$$24\lambda + 12 = 0 \quad \lambda = \frac{1}{2}$$

So, the coordinates of the point in the plane are

$$2(-1/2) + 1, -2(-1/2) + 3/2, 4(-1/2) + 2 = 0, 5/2, 0$$

Thus, the foot of the perpendicular is $(0, 5/2, 0)$ and the required length

$$\begin{aligned} &= \sqrt{(1-0)^2 + \left(\frac{3}{2} - \frac{5}{2}\right)^2 + (2-0)^2} \\ &= \sqrt{1+1+4} = \sqrt{6} \text{ units} \end{aligned}$$

Question 19

Find the equations of the line passing through the point $(3,0,1)$ and parallel to the planes $x + 2y = 0$ and $3y - z = 0$.

Solution

Given point is $(3, 0, 1)$ and the equation of planes are

$$x + 2y = 0 \dots (i) \text{ and } 3y - z = 0 \dots (ii)$$

Equation of any line l passing through $(3, 0, 1)$ is

$$l: (x - 3)/a = (y - 0)/b = (z - 1)/c$$

Now, the direction ratios of the normal to plane (i) and plane (ii) are $(1, 2, 0)$ and $(0, 3, 1)$.

As the line is parallel to both the planes, we have

$$1.a + 2.b + 0.c = 0 \Rightarrow a + 2b + 0c = 0 \text{ and}$$

$$0.a + 3.b - 1.c = 0 \Rightarrow 0a + 3b - c = 0$$

$$\text{So, } \frac{a}{-2-0} = \frac{-b}{-1-0} = \frac{c}{3-0} = \lambda$$

$$\therefore a = -2\lambda, b = \lambda, c = 3\lambda$$

So, the equation of line is

$$\frac{x-3}{-2\lambda} = \frac{y}{\lambda} = \frac{z-1}{3\lambda}$$

Thus, the required equation is

$$\frac{x-3}{-2} = \frac{y}{1} = \frac{z-1}{3}$$

or in vector form is

$$(x-3)\hat{i} + y\hat{j} + (z-1)\hat{k} = \lambda(-2\hat{i} + \hat{j} + 3\hat{k})$$

Question 20

Find the equation of the plane through the points (2, 1, -1) and (-1, 3, 4) and perpendicular to the plane $x - 2y + 4z = 10$.

Solution

We know that, equation of the plane passing through two points (x_1, y_1, z_1) and (x_2, y_2, z_2) with its normal's direction ratios is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0 \dots (i)$$

Now, if the plane is passing through two points (2, 1, -1) and (-1, 3, 4) then

$$a(x_2 - x_1) + b(y_2 - y_1) + c(z_2 - z_1) = 0$$

$$a(-1 - 2) + b(3 - 1) + c(4 + 1) = 0$$

$$-3a + 2b + 5c = 0 \dots (ii)$$

As the required plane is perpendicular to the given plane $x - 2y + 4z = 10$, then

$$1.a - 2.b + 4.c = 10 \dots (iii)$$

On solving (ii) and (iii) we get,

$$\frac{a}{8+10} = \frac{-b}{-12-5} = \frac{c}{6-2} = \lambda$$

So, $a = 18\lambda$, $b = 17\lambda$ and $c = 4\lambda$

Thus, the required plane is

$$18\lambda (x - 2) + 17\lambda (y - 1) + 4\lambda (z + 1) = 0$$

$$18x - 36 + 17y - 17 + 4z + 4 = 0$$

$$\Rightarrow 18x + 17y + 4z - 49 = 0$$

Question 21

Find the shortest distance between the lines given by

$$\vec{r} = (8 + 3\lambda)\hat{i} - (9 + 16\lambda)\hat{j} + (10 + 7\lambda)\hat{k} \quad \text{and}$$
$$\vec{r} = 15\hat{i} + 29\hat{j} + 5\hat{k} + \mu(3\hat{i} + 8\hat{j} - 5\hat{k}) .$$

Solution

Given equations of lines are

$$\vec{r} = (8 + 3\lambda)\hat{i} - (9 + 16\lambda)\hat{j} + (10 + 7\lambda)\hat{k} \quad \dots(i)$$

and $\vec{r} = 15\hat{i} + 29\hat{j} + 5\hat{k} + \mu(3\hat{i} + 8\hat{j} - 5\hat{k}) \quad \dots(ii)$

Equation (i) can be re-written as

$$\vec{r} = 8\hat{i} - 9\hat{j} + 10\hat{k} + \lambda(3\hat{i} - 16\hat{j} + 7\hat{k}) \quad \dots(iii)$$

Here, $\vec{a}_1 = 8\hat{i} - 9\hat{j} + 10\hat{k}$ and $\vec{a}_2 = 15\hat{i} + 29\hat{j} + 5\hat{k}$

$$\vec{b}_1 = 3\hat{i} - 16\hat{j} + 7\hat{k} \text{ and } \vec{b}_2 = 3\hat{i} + 8\hat{j} - 5\hat{k}$$

$$\vec{a}_2 - \vec{a}_1 = 7\hat{i} + 38\hat{j} - 5\hat{k}$$

$$\begin{aligned} \vec{b}_1 \times \vec{b}_2 &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -16 & 7 \\ 3 & 8 & -5 \end{vmatrix} \\ &= \hat{i}(80 - 56) - \hat{j}(-15 - 21) + \hat{k}(24 + 48) \\ &= 24\hat{i} + 36\hat{j} + 72\hat{k} \end{aligned}$$

Thus,

$$\begin{aligned} \text{Shortest distance, SD} &= \frac{|(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2)|}{|\vec{b}_1 \times \vec{b}_2|} \\ &= \frac{|(7\hat{i} + 38\hat{j} - 5\hat{k}) \cdot (24\hat{i} + 36\hat{j} + 72\hat{k})|}{\sqrt{(24)^2 + (36)^2 + (72)^2}} \\ &= \frac{|168 + 1368 - 360|}{\sqrt{576 + 1296 + 5184}} = \frac{|168 + 1008|}{\sqrt{7056}} = \frac{1176}{84} = 14 \text{ units} \end{aligned}$$

Question 22

Find the equation of the plane which is perpendicular to the plane $5x + 3y + 6z + 8 = 0$ and which contains the line of intersection of the planes $x + 2y + 3z - 4 = 0$ and $2x + y - z + 5 = 0$.

Solution

The given planes are

$$P_1: 5x + 3y + 6z + 8 = 0$$

$$P_2: x + 2y + 3z - 4 = 0$$

$$P_3: 2x + y - z + 5 = 0$$

Now, the equation of the plane passing through the line of intersection of P_1 and P_3 is

$$(x + 2y + 3z - 4) + \lambda(2x + y - z + 5) = 0$$

$$(1 + 2\lambda)x + (2 + \lambda)y + (3 - \lambda)z - 4 + 5\lambda = 0 \dots (i)$$

From the question its understood that plane (i) is perpendicular to P_1 , then

$$5(1 + 2\lambda) + 3(2 + \lambda) + 6(3 - \lambda) = 0$$

$$5 + 10\lambda + 6 + 3\lambda + 18 - 6\lambda = 0$$

$$7\lambda + 29 = 0$$

$$\lambda = -29/7$$

Putting the value of λ in equation (i), we get

$$\left[1 + 2\left(-\frac{29}{7}\right)\right]x + \left[2 - \frac{29}{7}\right]y + \left[3 + \frac{29}{7}\right]z - 4 + 5\left(-\frac{29}{7}\right) = 0$$

$$\Rightarrow \frac{-15}{7}x - \frac{15}{7}y + \frac{50}{7}z - 4 - \frac{145}{7} = 0$$

$$-15x - 15y + 50z - 28 - 145 = 0$$

$$-15x - 15y + 50z - 173 = 0 \Rightarrow 15x + 15y - 50z + 173 = 0$$

Thus, the required equation of plane is $15x + 15y - 50z + 173 = 0$.

Question 23

The plane $ax + by = 0$ is rotated about its line of intersection with the plane $z = 0$ through an angle α .

Prove that the equation of the plane in its new position is

$$ax + by \pm (\sqrt{a^2 + b^2} \tan \alpha) z = 0.$$

Solution

Given planes are: $ax + by = 0 \dots (i)$ and $z = 0 \dots (ii)$

Now, the equation of any plane passing through the line of intersection of plane (i) and (ii) is

$$(ax + by) + kz = 0 \Rightarrow ax + by + kz = 0 \dots (iii)$$

Dividing both sides by $\sqrt{a^2 + b^2 + k^2}$, we get

$$\frac{a}{\sqrt{a^2 + b^2 + k^2}}x + \frac{b}{\sqrt{a^2 + b^2 + k^2}}y + \frac{k}{\sqrt{a^2 + b^2 + k^2}}z = 0$$

So, direction cosines of the normal to the plane are

$$\frac{a}{\sqrt{a^2 + b^2 + k^2}}, \frac{b}{\sqrt{a^2 + b^2 + k^2}}, \frac{k}{\sqrt{a^2 + b^2 + k^2}}$$

and the direction cosines of the plane (i) are

$$\frac{a}{\sqrt{a^2 + b^2}}, \frac{b}{\sqrt{a^2 + b^2}}, 0$$

As, α is the angle between the planes (i) and (iii), we get

$$\Rightarrow \cos \alpha = \frac{a.a + b.b + k.0}{\sqrt{a^2 + b^2 + k^2} \cdot \sqrt{a^2 + b^2}}$$

$$\cos \alpha = \frac{a^2 + b^2}{\sqrt{a^2 + b^2 + k^2} \cdot \sqrt{a^2 + b^2}}$$

$$\cos \alpha = \frac{\sqrt{a^2 + b^2}}{\sqrt{a^2 + b^2 + k^2}} \Rightarrow \cos^2 \alpha = \frac{a^2 + b^2}{a^2 + b^2 + k^2}$$

$$(a^2 + b^2 + k^2) \cos^2 \alpha = a^2 + b^2$$

$$a^2 \cos^2 \alpha + b^2 \cos^2 \alpha + k^2 \cos^2 \alpha = a^2 + b^2$$

$$k^2 \cos^2 \alpha = a^2 - a^2 \cos^2 \alpha + b^2 - b^2 \cos^2 \alpha$$

$$k^2 \cos^2 \alpha = a^2(1 - \cos^2 \alpha) + b^2(1 - \cos^2 \alpha)$$

$$k^2 \cos^2 \alpha = a^2 \sin^2 \alpha + b^2 \sin^2 \alpha$$

$$k^2 \cos^2 \alpha = (a^2 + b^2) \sin^2 \alpha$$

$$\Rightarrow k^2 = (a^2 + b^2) \frac{\sin^2 \alpha}{\cos^2 \alpha} \Rightarrow k = \pm \sqrt{a^2 + b^2} \cdot \tan \alpha$$

Putting the value of k in eq. (iii) we get

$ax + by \pm (\sqrt{a^2 + b^2} \cdot \tan \alpha)z = 0$ which is the required equation of plane.

- Hence proved.

Dividing both sides by $\sqrt{a^2 + b^2 + k^2}$, we get

$$\frac{a}{\sqrt{a^2 + b^2 + k^2}}x + \frac{b}{\sqrt{a^2 + b^2 + k^2}}y + \frac{k}{\sqrt{a^2 + b^2 + k^2}}z = 0$$

So, direction cosines of the normal to the plane are

$$\frac{a}{\sqrt{a^2 + b^2 + k^2}}, \frac{b}{\sqrt{a^2 + b^2 + k^2}}, \frac{k}{\sqrt{a^2 + b^2 + k^2}}$$

and the direction cosines of the plane (i) are

$$\frac{a}{\sqrt{a^2 + b^2}}, \frac{b}{\sqrt{a^2 + b^2}}, 0$$

As, α is the angle between the planes (i) and (iii), we get

$$\Rightarrow \cos \alpha = \frac{a.a + b.b + k.0}{\sqrt{a^2 + b^2 + k^2} \cdot \sqrt{a^2 + b^2}}$$

$$\cos \alpha = \frac{a^2 + b^2}{\sqrt{a^2 + b^2 + k^2} \cdot \sqrt{a^2 + b^2}}$$

$$\cos \alpha = \frac{\sqrt{a^2 + b^2}}{\sqrt{a^2 + b^2 + k^2}} \Rightarrow \cos^2 \alpha = \frac{a^2 + b^2}{a^2 + b^2 + k^2}$$

$$(a^2 + b^2 + k^2) \cos^2 \alpha = a^2 + b^2$$

$$a^2 \cos^2 \alpha + b^2 \cos^2 \alpha + k^2 \cos^2 \alpha = a^2 + b^2$$

$$k^2 \cos^2 \alpha = a^2 - a^2 \cos^2 \alpha + b^2 - b^2 \cos^2 \alpha$$

$$k^2 \cos^2 \alpha = a^2(1 - \cos^2 \alpha) + b^2(1 - \cos^2 \alpha)$$

$$k^2 \cos^2 \alpha = a^2 \sin^2 \alpha + b^2 \sin^2 \alpha$$

$$k^2 \cos^2 \alpha = (a^2 + b^2) \sin^2 \alpha$$

$$\Rightarrow k^2 = (a^2 + b^2) \frac{\sin^2 \alpha}{\cos^2 \alpha} \Rightarrow k = \pm \sqrt{a^2 + b^2} \cdot \tan \alpha$$

Putting the value of k in eq. (iii) we get

$ax + by \pm (\sqrt{a^2 + b^2} \cdot \tan \alpha)z = 0$ which is the required equation of plane.

- Hence proved.

Question 24

Find the equation of the plane through the intersection of the planes

$\vec{r} \cdot (\hat{i} + 3\hat{j}) - 6 = 0$ and $\vec{r} \cdot (3\hat{i} - \hat{j} - 4\hat{k}) = 0$, whose perpendicular distance from origin is unity.

Solution

Given planes are

$$\vec{r} \cdot (\hat{i} + 3\hat{j}) - 6 = 0 \Rightarrow x + 3y - 6 = 0 \quad \dots(i)$$

$$\text{and } \vec{r} \cdot (3\hat{i} - \hat{j} - 4\hat{k}) = 0 \Rightarrow 3x - y - 4z = 0 \quad \dots(ii)$$

Equation of the plane passing through the line of intersection of plane (i) and (ii) is

$$\begin{aligned} (x + 3y - 6) + k(3x - y - 4z) &= 0 \\ (1 + 3k)x + (3 - k)y - 4kz - 6 &= 0 \end{aligned} \quad \dots(iii)$$

Perpendicular distance from origin

$$\Rightarrow \left| \frac{-6}{\sqrt{(1+3k)^2 + (3-k)^2 + (-4k)^2}} \right| = 1$$

$$\frac{36}{1 + 9k^2 + 6k + 9 + k^2 - 6k + 16k^2} = 1 \quad [\text{Squaring both sides}]$$

$$\frac{36}{26k^2 + 10} = 1 \Rightarrow 26k^2 + 10 = 36$$

$$26k^2 = 26 \Rightarrow k^2 = 1 \therefore k = \pm 1$$

Putting the value of k in eq. (iii) we get,

$$\begin{aligned} (x + 3y - 6) \pm (3x - y - 4z) &= 0 \\ \Rightarrow x + 3y - 6 + 3x - y - 4z &= 0 \text{ and } x + 3y - 6 - 3x + y + 4z = 0 \\ \Rightarrow 4x + 2y - 4z - 6 &= 0 \text{ and } -2x + 4y + 4z - 6 = 0 \end{aligned}$$

Thus, the required equations of planes are

$$4x + 2y - 4z - 6 = 0 \text{ and } -2x - 4y + 4z - 6 = 0.$$