

Chapter 10: Vector Algebra

NCERT Exemplar · Class 12 · Maths

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Question 1

Find the unit vector in the direction of sum of vectors

Solution

Given vectors are,

$$\vec{a} = 2\hat{i} - \hat{j} + \hat{k} \text{ and } \vec{b} = 2\hat{j} + \hat{k}$$

$$\text{So, } \vec{a} + \vec{b} = (2\hat{i} - \hat{j} + \hat{k}) + (2\hat{j} + \hat{k}) = 2\hat{i} + \hat{j} + 2\hat{k}$$

$$\begin{aligned} \text{Unit vector in the direction of } \vec{a} + \vec{b} &= \frac{\vec{a} + \vec{b}}{|\vec{a} + \vec{b}|} \\ &= \frac{2\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{(2)^2 + (1)^2 + (2)^2}} = \frac{2\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{4 + 1 + 4}} \\ &= \frac{2\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{9}} = \frac{2\hat{i} + \hat{j} + 2\hat{k}}{3} = \frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k} \end{aligned}$$

Thus, the required unit vector is $\frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}$.

$$\vec{a} = \hat{i} + \hat{j} + 2\hat{k} \text{ and } \vec{b} = 2\hat{i} + \hat{j} - 2\hat{k},$$

Question 2

If find the unit vector in the direction of

(i) $6\vec{b}$

(ii) $2\vec{a} - \vec{b}$

Solution

Given, $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} - 2\hat{k}$

$$(i) \quad 6\vec{b} = 6(2\hat{i} + \hat{j} - 2\hat{k}) = 12\hat{i} + 6\hat{j} - 12\hat{k}$$

$$\begin{aligned} \text{So, Unit vector in the direction of } 6\vec{b} &= \frac{6\vec{b}}{|6\vec{b}|} \\ &= \frac{12\hat{i} + 6\hat{j} - 12\hat{k}}{\sqrt{(12)^2 + (6)^2 + (-12)^2}} = \frac{12\hat{i} + 6\hat{j} - 12\hat{k}}{\sqrt{144 + 36 + 144}} \\ &= \frac{12\hat{i} + 6\hat{j} - 12\hat{k}}{\sqrt{324}} = \frac{12\hat{i} + 6\hat{j} - 12\hat{k}}{18} \\ &= \frac{6}{18} (2\hat{i} + \hat{j} - 2\hat{k}) = \frac{1}{3} (2\hat{i} + \hat{j} - 2\hat{k}) \end{aligned}$$

Thus, the required unit vector is $\frac{1}{3} (2\hat{i} + \hat{j} - 2\hat{k})$.

$$(ii) \quad 2\vec{a} - \vec{b} = 2(\hat{i} + \hat{j} + 2\hat{k}) - (2\hat{i} + \hat{j} - 2\hat{k})$$

$$\text{So,} \quad = 2\hat{i} + 2\hat{j} + 4\hat{k} - 2\hat{i} - \hat{j} + 2\hat{k} = \hat{j} + 6\hat{k}$$

Unit vector in the direction of $2\vec{a} - \vec{b}$

$$\begin{aligned} &= \frac{2\vec{a} - \vec{b}}{|2\vec{a} - \vec{b}|} = \frac{\hat{j} + 6\hat{k}}{\sqrt{(1)^2 + (6)^2}} = \frac{\hat{j} + 6\hat{k}}{\sqrt{1 + 36}} \\ &= \frac{\hat{j} + 6\hat{k}}{\sqrt{37}} = \frac{1}{\sqrt{37}} [\hat{j} + 6\hat{k}] \end{aligned}$$

Thus, the required unit vector is $\frac{1}{\sqrt{37}} [\hat{j} + 6\hat{k}]$.

Given, $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} - 2\hat{k}$

$$(i) \ 6\vec{b} = 6(2\hat{i} + \hat{j} - 2\hat{k}) = 12\hat{i} + 6\hat{j} - 12\hat{k}$$

$$\begin{aligned} \text{So, Unit vector in the direction of } 6\vec{b} &= \frac{6\vec{b}}{|6\vec{b}|} \\ &= \frac{12\hat{i} + 6\hat{j} - 12\hat{k}}{\sqrt{(12)^2 + (6)^2 + (-12)^2}} = \frac{12\hat{i} + 6\hat{j} - 12\hat{k}}{\sqrt{144 + 36 + 144}} \\ &= \frac{12\hat{i} + 6\hat{j} - 12\hat{k}}{\sqrt{324}} = \frac{12\hat{i} + 6\hat{j} - 12\hat{k}}{18} \\ &= \frac{6}{18} (2\hat{i} + \hat{j} - 2\hat{k}) = \frac{1}{3} (2\hat{i} + \hat{j} - 2\hat{k}) \end{aligned}$$

Thus, the required unit vector is $\frac{1}{3} (2\hat{i} + \hat{j} - 2\hat{k})$.

$$(ii) \ 2\vec{a} - \vec{b} = 2(\hat{i} + \hat{j} + 2\hat{k}) - (2\hat{i} + \hat{j} - 2\hat{k})$$

$$\text{So,} \quad = 2\hat{i} + 2\hat{j} + 4\hat{k} - 2\hat{i} - \hat{j} + 2\hat{k} = \hat{j} + 6\hat{k}$$

Unit vector in the direction of $2\vec{a} - \vec{b}$

$$\begin{aligned} &= \frac{2\vec{a} - \vec{b}}{|2\vec{a} - \vec{b}|} = \frac{\hat{j} + 6\hat{k}}{\sqrt{(1)^2 + (6)^2}} = \frac{\hat{j} + 6\hat{k}}{\sqrt{1 + 36}} \\ &= \frac{\hat{j} + 6\hat{k}}{\sqrt{37}} = \frac{1}{\sqrt{37}} [\hat{j} + 6\hat{k}] \end{aligned}$$

Thus, the required unit vector is $\frac{1}{\sqrt{37}} [\hat{j} + 6\hat{k}]$.

$\overrightarrow{PQ}$,

Question 3

Find a unit vector in the direction of where P and Q have co-ordinates (5, 0, 8) and (3, 3, 2), respectively.

Solution

Given coordinates are P(5, 0, 8) and Q(3, 3, 2).

$$\text{So, } \overrightarrow{PQ} = (3 - 5)\hat{i} + (3 - 0)\hat{j} + (2 - 8)\hat{k} = -2\hat{i} + 3\hat{j} - 6\hat{k}$$

And,

$$\begin{aligned}\text{Unit vector in the direction of } \overrightarrow{PQ} &= \frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} \\&= \frac{-2\hat{i} + 3\hat{j} - 6\hat{k}}{\sqrt{(-2)^2 + (3)^2 + (-6)^2}} = \frac{-2\hat{i} + 3\hat{j} - 6\hat{k}}{\sqrt{4 + 9 + 36}} = \frac{-2\hat{i} + 3\hat{j} - 6\hat{k}}{\sqrt{49}} \\&= \frac{-2\hat{i} + 3\hat{j} - 6\hat{k}}{7} = \frac{1}{7}(-2\hat{i} + 3\hat{j} - 6\hat{k})\end{aligned}$$

Thus, the required unit vector is $\frac{1}{7}(-2\hat{i} + 3\hat{j} - 6\hat{k})$.

$\vec{a}$ and $\vec{b}$

Question 4

If $\vec{a}$ and $\vec{b}$ are the position vectors of A and B, respectively, find the position vector of a point C in BA produced such that $BC = 1.5 BA$.

Solution

Given,

$$BC = 1.5 BA$$

$$\Rightarrow \frac{BC}{BA} = 1.5 = \frac{3}{2}$$

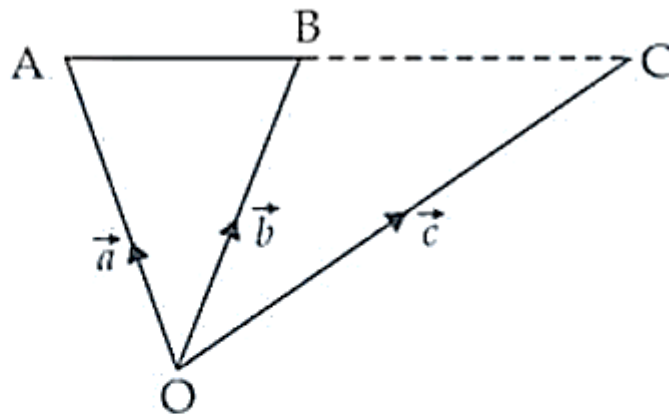
$$\frac{\vec{c} - \vec{b}}{\vec{a} - \vec{b}} = \frac{3}{2}$$

$$2\vec{c} - 2\vec{b} = 3\vec{a} - 3\vec{b}$$

$$2\vec{c} = 3\vec{a} - 3\vec{b} + 2\vec{b} \Rightarrow 2\vec{c} = 3\vec{a} - \vec{b}$$

$$\therefore \vec{c} = \frac{3\vec{a} - \vec{b}}{2}$$

Thus, the required vector is $\vec{c} = \frac{3\vec{a} - \vec{b}}{2}$.

**Question 5**

Using vectors, find the value of k such that the points $(k, -10, 3)$, $(1, -1, 3)$ and $(3, 5, 3)$ are collinear.

Solution

Let the given points be $A(k, -10, 3)$, $B(1, -1, 3)$ and $C(3, 5, 3)$.

$$\overline{AB} = (1-k)\hat{i} + (-1+10)\hat{j} + (3-3)\hat{k}$$

$$\overline{AB} = (1-k)\hat{i} + 9\hat{j} + 0\hat{k}$$

$$\text{So, } |\overline{AB}| = \sqrt{(1-k)^2 + (9)^2} = \sqrt{(1-k)^2 + 81}$$

$$\overline{BC} = (3-1)\hat{i} + (5+1)\hat{j} + (3-3)\hat{k} = 2\hat{i} + 6\hat{j} + 0\hat{k}$$

$$\text{So, } |\overline{BC}| = \sqrt{(2)^2 + (6)^2} = \sqrt{4+36} = \sqrt{40} = 2\sqrt{10}$$

$$\overline{AC} = (3-k)\hat{i} + (5+10)\hat{j} + (3-3)\hat{k} = (3-k)\hat{i} + 15\hat{j} + 0\hat{k}$$

$$\text{So, } |\overline{AC}| = \sqrt{(3-k)^2 + (15)^2} = \sqrt{(3-k)^2 + 225}$$

If A, B and C are collinear, then

$$|\overline{AB}| + |\overline{BC}| = |\overline{AC}|$$

$$\sqrt{(1-k)^2 + 81} + \sqrt{40} = \sqrt{(3-k)^2 + 225}$$

Squaring both sides, we have

$$\left[\sqrt{(1-k)^2 + 81} + \sqrt{40} \right]^2 = \left[\sqrt{(3-k)^2 + 225} \right]^2$$

$$(1-k)^2 + 81 + 40 + 2\sqrt{40} \sqrt{(1-k)^2 + 81} = (3-k)^2 + 225$$

$$1 + k^2 - 2k + 121 + 2\sqrt{40} \sqrt{1+k^2 - 2k + 81} = 9 + k^2 - 6k + 225$$

$$\Rightarrow 122 - 2k + 2\sqrt{40} \sqrt{k^2 - 2k + 82} = 234 - 6k$$

Now, on dividing by 2, we get

$$61 - k + \sqrt{40} \sqrt{k^2 - 2k + 82} = 117 - 3k$$

$$\sqrt{40} \sqrt{k^2 - 2k + 82} = 117 - 61 - 3k + k$$

$$\sqrt{40} \sqrt{k^2 - 2k + 82} = 56 - 2k \Rightarrow 2\sqrt{10} \sqrt{k^2 - 2k + 82} = 56 - 2k$$

$$\Rightarrow \sqrt{10} \sqrt{k^2 - 2k + 82} = 28 - k \quad (\text{Dividing by 2})$$

Squaring both sides, we get

$$10(k^2 - 2k + 82) = 784 + k^2 - 56k$$

$$10k^2 - 20k + 820 = 784 + k^2 - 56k$$

$$10k^2 - 20k + 820 = 784 + k^2 - 56k$$

$$10k^2 - k^2 - 20k + 56k + 820 - 784 = 0$$

$$9k^2 + 36k + 36 = 0 \Rightarrow k^2 + 4k + 4 = 0 \Rightarrow (k + 2)^2 = 0$$

$$k + 2 = 0 \Rightarrow k = -2$$

Thus, the required value is $k = -2$

Question 6

A vector $\vec{r}$ is inclined at equal angles to the three axes. If the magnitude of $\vec{r}$ is $2\sqrt{3}$ units, find $\vec{r}$.

Solution

As the vector $\vec{r}$

makes equal angles with the axes, their direction cosines should also be same

So, $l = m = n$

And we know that,

$$l^2 + m^2 + n^2 = 1 \Rightarrow l^2 + l^2 + l^2 = 1$$

$$3l^2 = 1$$

$$l = \pm 1/\sqrt{3}$$

$$\text{Now, } \hat{r} = \pm \frac{1}{\sqrt{3}} \hat{i} \pm \frac{1}{\sqrt{3}} \hat{j} \pm \frac{1}{\sqrt{3}} \hat{k} \Rightarrow \hat{r} = \pm \frac{1}{\sqrt{3}} (\hat{i} + \hat{j} + \hat{k})$$

$$\begin{aligned} \text{And, w.k.t } \vec{r} &= (\hat{r}) |\vec{r}| \\ &= \pm \frac{1}{\sqrt{3}} (\hat{i} + \hat{j} + \hat{k}) 2\sqrt{3} = \pm 2(\hat{i} + \hat{j} + \hat{k}) \end{aligned}$$

Thus, the required value of $\vec{r}$ is $\pm 2(\hat{i} + \hat{j} + \hat{k})$.

Question 7

A vector $\vec{r}$ has magnitude 14 and direction ratios 2, 3, -6. Find the direction cosines and components of $\vec{r}$, given that $\vec{r}$ makes an acute angle with x-axis.

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three vectors such that $\vec{a} = 2k$, $\vec{b} = 3k$ and $\vec{c} = -6k$

If l , m and n are the direction cosines of vector $\vec{r}$, then

$$l = \frac{\vec{a}}{|\vec{r}|} = \frac{2k}{14} = \frac{k}{7}$$

$$m = \frac{\vec{b}}{|\vec{r}|} = \frac{3k}{14} \quad \text{and} \quad n = \frac{\vec{c}}{|\vec{r}|} = \frac{-6k}{14} = \frac{-3k}{7}$$

We know that $l^2 + m^2 + n^2 = 1$

$$\text{So, } \frac{k^2}{49} + \frac{9k^2}{196} + \frac{9k^2}{49} = 1$$

$$\frac{4k^2 + 9k^2 + 36k^2}{196} = 1 \Rightarrow 49k^2 = 196 \Rightarrow k^2 = 4$$

$$\therefore k = \pm 2 \quad \text{and} \quad l = \frac{k}{7} = \frac{2}{7}$$

$$m = \frac{3k}{14} = \frac{3 \times 2}{14} = \frac{3}{7} \quad \text{and} \quad n = \frac{-3k}{7} = \frac{-3 \times 2}{7} = \frac{-6}{7}$$

$$\text{Now, } \hat{r} = \pm \left(\frac{2}{7} \hat{i} + \frac{3}{7} \hat{j} - \frac{6}{7} \hat{k} \right)$$

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|}$$

$$\Rightarrow \vec{r} = \pm \left(\frac{2}{7} \hat{i} + \frac{3}{7} \hat{j} - \frac{6}{7} \hat{k} \right) \cdot 14 = \pm (4\hat{i} + 6\hat{j} - 12\hat{k})$$

Thus, the required direction cosines are $\frac{2}{7}, \frac{3}{7}, \frac{-6}{7}$ and the components of $\vec{r}$ are $4\hat{i}, 6\hat{j}$ and $-12\hat{k}$.

Thus, the required direction cosines are $\frac{2}{7}, \frac{3}{7}, \frac{-6}{7}$ and the components of $\vec{r}$ are $4\hat{i}, 6\hat{j}$ and $-12\hat{k}$.

Solution

Solution coming soon.

Question 8

Find a vector of magnitude 6, which is perpendicular to both the vectors $2\hat{i} - \hat{j} + 2\hat{k}$ and $4\hat{i} - \hat{j} + 3\hat{k}$.

Solution

Let $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = 4\hat{i} - \hat{j} + 3\hat{k}$

We know that unit vector perpendicular to $\vec{a}$ and $\vec{b} = \frac{(\vec{a} \times \vec{b})}{|\vec{a} \times \vec{b}|}$

$$\begin{aligned}\vec{a} \times \vec{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 2 \\ 4 & -1 & 3 \end{vmatrix} \\ &= \hat{i}(-3 + 2) - \hat{j}(6 - 8) + \hat{k}(-2 + 4) = -\hat{i} + 2\hat{j} + 2\hat{k}\end{aligned}$$

$$\therefore |\vec{a} \times \vec{b}| = \sqrt{(-1)^2 + (2)^2 + (2)^2} = \sqrt{1 + 4 + 4} = \sqrt{9} = 3$$

$$\text{So, } \frac{(\vec{a} \times \vec{b})}{|\vec{a} \times \vec{b}|} = \frac{-\hat{i} + 2\hat{j} + 2\hat{k}}{3} = \frac{1}{3}(-\hat{i} + 2\hat{j} + 2\hat{k})$$

$$\begin{aligned}\text{Now the vector of magnitude 6} &= \frac{1}{3}(-\hat{i} + 2\hat{j} + 2\hat{k}) \cdot 6 \\ &= 2(-\hat{i} + 2\hat{j} + 2\hat{k}) = -2\hat{i} + 4\hat{j} + 4\hat{k}\end{aligned}$$

Thus, the required vector is $-2\hat{i} + 4\hat{j} + 4\hat{k}$.

$$2\hat{i} - \hat{j} + \hat{k} \text{ and } 3\hat{i} + 4\hat{j} - \hat{k}.$$

Question 9

Find the angle between the vectors

Solution

Let $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = 3\hat{i} + 4\hat{j} - \hat{k}$ and let θ be the angle between $\vec{a}$ and $\vec{b}$.

$$\begin{aligned} \text{Now, } \cos \theta &= \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} = \frac{(2\hat{i} - \hat{j} + \hat{k}) \cdot (3\hat{i} + 4\hat{j} - \hat{k})}{\sqrt{4+1+1} \cdot \sqrt{9+16+1}} \\ \therefore &= \frac{6-4-1}{\sqrt{6} \cdot \sqrt{26}} \Rightarrow \frac{1}{2\sqrt{3} \cdot \sqrt{13}} = \frac{1}{2\sqrt{39}} \\ \therefore \theta &= \cos^{-1} \frac{1}{2\sqrt{39}} \Rightarrow \theta = \cos^{-1} \left(\frac{1}{156} \right) \end{aligned}$$

Thus, the required value of θ is $\cos^{-1} \left(\frac{1}{156} \right)$.

Question 10

If $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, show that $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$. Interpret the result geometrically?

Solution

Given that $\vec{a} + \vec{b} + \vec{c} = 0$

So, $\vec{a} \times (\vec{a} + \vec{b} + \vec{c}) = \vec{a} \times 0$

$$\vec{a} \times \vec{a} + \vec{a} \times \vec{b} + \vec{a} \times \vec{c} = 0$$

$$\vec{0} + \vec{a} \times \vec{b} + \vec{a} \times \vec{c} = 0 \quad (\vec{a} \times \vec{a} = 0)$$

$$\vec{a} \times \vec{b} - \vec{c} \times \vec{a} = 0 \quad (\vec{a} \times \vec{c} = -\vec{c} \times \vec{a})$$

$$\Rightarrow \vec{a} \times \vec{b} = \vec{c} \times \vec{a} \quad \dots(i)$$

Now, $\vec{a} + \vec{b} + \vec{c} = 0$

$$\vec{b} \times (\vec{a} + \vec{b} + \vec{c}) = \vec{b} \times 0$$

$$\vec{b} \times \vec{a} + \vec{b} \times \vec{b} + \vec{b} \times \vec{c} = 0$$

$$\vec{b} \times \vec{a} + \vec{0} + \vec{b} \times \vec{c} = 0 \quad (\because \vec{b} \times \vec{b} = 0)$$

$$-(\vec{a} \times \vec{b}) + \vec{b} \times \vec{c} = 0$$

$$\therefore \vec{b} \times \vec{c} = \vec{a} \times \vec{b} \quad \dots(ii)$$

From eq. (i) and (ii) we get

$$\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}.$$

- Hence proved.

Question 11

Find the sine of the angle between the vectors $\vec{a} = 3\hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = 2\hat{i} - 2\hat{j} + 4\hat{k}$.

Solution

Given that $\vec{a} = 3\hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = 2\hat{i} - 2\hat{j} + 4\hat{k}$

We know that $|\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}| \sin \theta$

So,

$$\begin{aligned}\vec{a} \times \vec{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 2 \\ 2 & -2 & 4 \end{vmatrix} \\ &= \hat{i}(4 + 4) - \hat{j}(12 - 4) + \hat{k}(-6 - 2) \\ &= 8\hat{i} - 8\hat{j} - 8\hat{k}\end{aligned}$$

$$\begin{aligned}|\vec{a} \times \vec{b}| &= \sqrt{(8)^2 + (-8)^2 + (-8)^2} \\ &= \sqrt{64 + 64 + 64} = \sqrt{192} = \sqrt{64 \times 3} = 8\sqrt{3}\end{aligned}$$

$$|\vec{a}| = \sqrt{(3)^2 + (1)^2 + (2)^2} = \sqrt{9 + 1 + 4} = \sqrt{14}$$

$$\begin{aligned}|\vec{b}| &= \sqrt{(2)^2 + (-2)^2 + (4)^2} = \sqrt{4 + 4 + 16} \\ &= \sqrt{24} = 2\sqrt{6}\end{aligned}$$

$$\begin{aligned}\text{Now, } \sin \theta &= \frac{|\vec{a} \times \vec{b}|}{|\vec{a}| |\vec{b}|} = \frac{8\sqrt{3}}{\sqrt{14} \cdot 2\sqrt{6}} \\ &= \frac{4\sqrt{3}}{\sqrt{84}} = \frac{4\sqrt{3}}{2\sqrt{21}} = \frac{2}{\sqrt{7}}\end{aligned}$$

$$\text{Thus, } \sin \theta = \frac{2}{\sqrt{7}}.$$

Thus, $\sin \theta = 2/\sqrt{7}$

Question 12

If A, B, C, D are the points with position vectors $\hat{i} + \hat{j} - \hat{k}$, $2\hat{i} - \hat{j} + 3\hat{k}$, $2\hat{i} - 3\hat{k}$, $3\hat{i} - 2\hat{j} + \hat{k}$, respectively, find the projection of $\overline{AB}$ along $\overline{CD}$.

Solution

We have,

$$\text{Position vector of A} = \hat{i} + \hat{j} - \hat{k}$$

$$\text{Position vector of B} = 2\hat{i} - \hat{j} + 3\hat{k}$$

$$\text{Position vector of C} = 2\hat{i} - 3\hat{k}$$

$$\text{Position vector of D} = 3\hat{i} - 2\hat{j} + \hat{k}$$

$$\overline{AB} = \text{P.V of B} - \text{P.V of A}$$

$$= (2\hat{i} - \hat{j} + 3\hat{k}) - (\hat{i} + \hat{j} - \hat{k}) = \hat{i} - 2\hat{j} + 4\hat{k}$$

$$\overline{CD} = \text{P.V. of D} - \text{P.V. of C}$$

$$= (3\hat{i} - 2\hat{j} + \hat{k}) - (2\hat{i} - 3\hat{k}) = \hat{i} - 2\hat{j} + 4\hat{k}$$

$$\begin{aligned} \text{Projection of } \overline{AB} \text{ on } \overline{CD} &= \frac{\overline{AB} \cdot \overline{CD}}{|\overline{CD}|} \\ &= \frac{(\hat{i} - 2\hat{j} + 4\hat{k}) \cdot (\hat{i} - 2\hat{j} + 4\hat{k})}{\sqrt{(1)^2 + (-2)^2 + (4)^2}} \\ &= \frac{1 + 4 + 16}{\sqrt{1 + 4 + 16}} = \frac{21}{\sqrt{21}} = \sqrt{21} \end{aligned}$$

Thus, the required projection = $\sqrt{21}$.

Question 13

Using vectors, find the area of the triangle ABC with vertices A(1, 2, 3), B(2, -1, 4) and C(4, 5, -1).

Solution

Given vertices A(1, 2, 3), B(2, -1, 4) and C(4, 5, -1).

$$\overrightarrow{AB} = (2-1)\hat{i} + (-1-2)\hat{j} + (4-3)\hat{k}$$

$$\overrightarrow{AB} = \hat{i} - 3\hat{j} + \hat{k}$$

$$\overrightarrow{AC} = (4-1)\hat{i} + (5-2)\hat{j} + (-1-3)\hat{k} = 3\hat{i} + 3\hat{j} - 4\hat{k}$$

Now,

$$\begin{aligned}\text{Area of } \triangle ABC &= \frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 1 \\ 3 & 3 & -4 \end{vmatrix} \\ &= \frac{1}{2} [\hat{i}(12-3) - \hat{j}(-4-3) + \hat{k}(3+9)] \\ &= \frac{1}{2} |9\hat{i} + 7\hat{j} + 12\hat{k}| = \frac{1}{2} \sqrt{(9)^2 + (7)^2 + (12)^2} \\ &= \frac{1}{2} \sqrt{81 + 49 + 144} = \frac{1}{2} \sqrt{274}\end{aligned}$$

Thus, the required area is $\frac{\sqrt{274}}{2}$.

$$\overrightarrow{AB} = (2-1)\hat{i} + (-1-2)\hat{j} + (4-3)\hat{k}$$

$$\overrightarrow{AB} = \hat{i} - 3\hat{j} + \hat{k}$$

$$\overrightarrow{AC} = (4-1)\hat{i} + (5-2)\hat{j} + (-1-3)\hat{k} = 3\hat{i} + 3\hat{j} - 4\hat{k}$$

Now,

$$\begin{aligned} \text{Area of } \triangle ABC &= \frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 1 \\ 3 & 3 & -4 \end{vmatrix} \\ &= \frac{1}{2} [\hat{i}(12-3) - \hat{j}(-4-3) + \hat{k}(3+9)] \\ &= \frac{1}{2} |9\hat{i} + 7\hat{j} + 12\hat{k}| = \frac{1}{2} \sqrt{(9)^2 + (7)^2 + (12)^2} \\ &= \frac{1}{2} \sqrt{81 + 49 + 144} = \frac{1}{2} \sqrt{274} \end{aligned}$$

Thus, the required area is $\frac{\sqrt{274}}{2}$.

Question 14

Using vectors, prove that the parallelogram on the same base and between the same parallels are equal in area.

Solution

Let's consider ABCD and ABFE be two parallelograms on the same base AB and between same parallel lines AB and DF.

$$\text{Let } \overrightarrow{AB} = \vec{a} \text{ and } \overrightarrow{AD} = \vec{b}$$

$$\therefore \text{Area of parallelogram ABCD} = |\vec{a} \times \vec{b}|$$

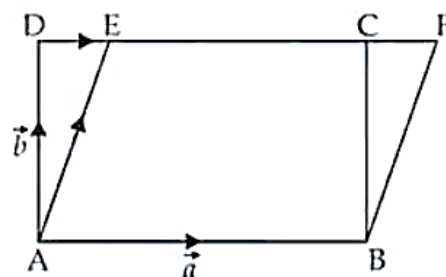
$$\text{Now, Area of parallelogram ABFE} = |\overrightarrow{AB} \times \overrightarrow{AE}|$$

$$= |\vec{a} \times (\overrightarrow{AD} + \overrightarrow{DE})| = |\vec{a} \times (\vec{b} + K\vec{a})|$$

$$= |(\vec{a} \times \vec{b}) + K(\vec{a} \times \vec{a})| = |\vec{a} \times \vec{b}| + 0$$

$$= |\vec{a} \times \vec{b}|$$

$$[\because \vec{a} \times \vec{a} = 0]$$



– Hence proved.

Long Answer (L.A.)

Question 15

Prove that in any triangle ABC, , where a, b, c are the magnitudes of the sides opposite to the vertices A, B, C, respectively.

Solution

In triangle ABC, the components of c are $c \cos A$ and $c \sin A$.

$$\therefore \overline{CD} = b - c \cos A$$

In $\triangle BDC$,

$$a^2 = CD^2 + BD^2$$

$$a^2 = (b - c \cos A)^2 + (c \sin A)^2$$

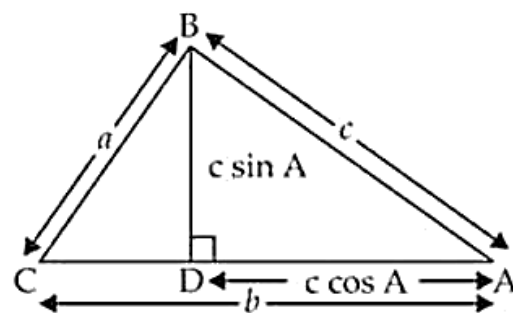
$$a^2 = b^2 + c^2 \cos^2 A - 2bc \cos A + c^2 \sin^2 A$$

$$a^2 = b^2 + c^2(\cos^2 A + \sin^2 A) - 2bc \cos A$$

$$a^2 = b^2 + c^2 - 2bc \cos A \Rightarrow 2bc \cos A = b^2 + c^2 - a^2$$

$$\text{Thus, } \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

- Hence proved.



Question 16

$\frac{1}{2} [\vec{b} \times \vec{c} + \vec{c} \times \vec{a} + \vec{a} \times \vec{b}]$ If $\vec{a}, \vec{b}, \vec{c}$ determine the vertices of a triangle, show that gives the

vector area of the triangle. Hence deduce the condition that the three points $\vec{a}, \vec{b}, \vec{c}$ are collinear. Also find the unit vector normal to the plane of the triangle.

Solution

As, $\vec{a}, \vec{b}$ and $\vec{c}$ are the vertices of ΔABC

So, $\overrightarrow{AB} = \vec{b} - \vec{a}, \overrightarrow{BC} = \vec{c} - \vec{b}$

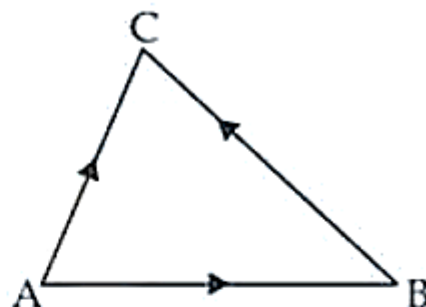
And, $\overrightarrow{AC} = \vec{c} - \vec{a}$

$$\therefore \text{Area of } \Delta ABC = \frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AC}|$$

$$= \frac{1}{2} |(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})|$$

$$= \frac{1}{2} |\vec{b} \times \vec{c} - \vec{b} \times \vec{a} - \vec{a} \times \vec{c} + \vec{a} \times \vec{a}|$$

$$= \frac{1}{2} |\vec{b} \times \vec{c} + \vec{a} \times \vec{b} + \vec{c} \times \vec{a}|$$



$$\left[\begin{array}{l} \because \vec{a} \times \vec{b} = -\vec{b} \times \vec{a} \\ \vec{c} \times \vec{a} = -\vec{a} \times \vec{c} \\ \vec{a} \times \vec{a} = \vec{0} \end{array} \right]$$

If three vectors are collinear, area of $\Delta ABC = 0$

$$\text{So, } \frac{1}{2} |\vec{b} \times \vec{c} + \vec{a} \times \vec{b} + \vec{c} \times \vec{a}| = 0$$

$$\therefore |\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}| = 0$$

which is the condition of collinearity of $\vec{a}, \vec{b}$ and $\vec{c}$.

Let $\hat{n}$ be the unit vector normal to the plane of the ΔABC

$$\begin{aligned} \text{So, } \hat{n} &= \frac{\overrightarrow{AB} \times \overrightarrow{AC}}{|\overrightarrow{AB} \times \overrightarrow{AC}|} \\ &= \frac{\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}}{|\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|} \end{aligned}$$

Question 17

$$\frac{|\vec{a} \times \vec{b}|}{2}$$

Show that area of the parallelogram whose diagonals are given by $\vec{a}$ and $\vec{b}$ is . Also find the area of the parallelogram whose diagonals are $2\hat{i} - \hat{j} + \hat{k}$ and $\hat{i} + 3\hat{j} - \hat{k}$.

Solution

Let's take ABCD to be a parallelogram such that

$$\overline{AB} = \vec{p}, \overline{AD} = \vec{q} = \overline{BC}$$

So by law of triangle, we get

$$\overline{AC} = \vec{a} = \vec{p} + \vec{q} \quad \dots(i)$$

$$\text{and } \overline{BD} = \vec{b} = -\vec{p} + \vec{q} \quad \dots(ii)$$

Adding eq. (i) and (ii) we get,

$$\vec{a} + \vec{b} = 2\vec{q} \Rightarrow \vec{q} = \left(\frac{\vec{a} + \vec{b}}{2} \right)$$

Subtracting eq. (ii) from eq. (i) we get

$$\vec{a} - \vec{b} = 2\vec{p} \Rightarrow \vec{p} = \left(\frac{\vec{a} - \vec{b}}{2} \right)$$

So,

$$\vec{p} \times \vec{q} = \frac{1}{4} (\vec{a} + \vec{b}) \times (\vec{a} - \vec{b}) = \frac{1}{4} (\vec{a} \times \vec{a} - \vec{a} \times \vec{b} + \vec{b} \times \vec{a} - \vec{b} \times \vec{b})$$

$$= \frac{1}{4} (-\vec{a} \times \vec{b} + \vec{b} \times \vec{a}) \quad \left[\begin{array}{l} \because \vec{a} \times \vec{a} = 0 \\ \vec{b} \times \vec{b} = 0 \end{array} \right]$$

$$= \frac{1}{4} (\vec{a} \times \vec{b} + \vec{a} \times \vec{b}) = \frac{1}{4} \cdot 2 (\vec{a} \times \vec{b}) = \frac{|\vec{a} \times \vec{b}|}{2}$$

And, the area of the parallelogram ABCD = $|\vec{p} \times \vec{q}| = \frac{1}{2} |\vec{a} \times \vec{b}|$

Now area of parallelogram whose diagonals are $2\hat{i} - \hat{j} + \hat{k}$

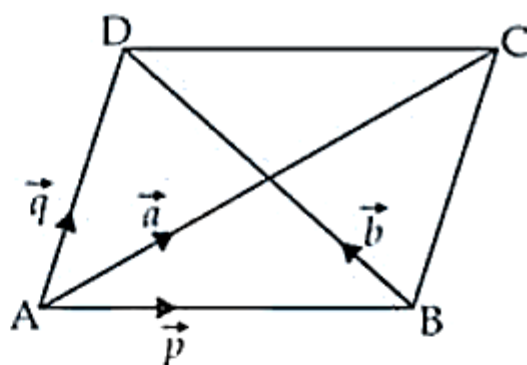
$$\text{and } \hat{i} + 3\hat{j} - \hat{k} = \frac{1}{2} |(2\hat{i} - \hat{j} + \hat{k}) \times (\hat{i} + 3\hat{j} - \hat{k})|$$

$$= - \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 1 \\ 1 & 3 & 1 \end{vmatrix}$$

$$= \frac{1}{2} |\hat{i}(1-3) - \hat{j}(-2-1) + \hat{k}(6+1)| = \frac{1}{2} |-2\hat{i} + 3\hat{j} + 7\hat{k}|$$

$$= \frac{1}{2} \sqrt{(-2)^2 + (3)^2 + (7)^2} = \frac{1}{2} \sqrt{4+9+49}$$

1 —



$$= \frac{1}{2} \sqrt{62} \text{ sq. units}$$

Thus, the required area is $\frac{1}{2} \sqrt{62}$ sq. units.

$$\overline{AB} = \vec{p}, \overline{AD} = \vec{q} = \overline{BC}$$

So by law of triangle, we get

$$\overline{AC} = \vec{a} = \vec{p} + \vec{q} \quad \dots(i)$$

$$\text{and } \overline{BD} = \vec{b} = -\vec{p} + \vec{q} \quad \dots(ii)$$

Adding eq. (i) and (ii) we get,

$$\vec{a} + \vec{b} = 2\vec{q} \Rightarrow \vec{q} = \left(\frac{\vec{a} + \vec{b}}{2} \right)$$

Subtracting eq. (ii) from eq. (i) we get

$$\vec{a} - \vec{b} = 2\vec{p} \Rightarrow \vec{p} = \left(\frac{\vec{a} - \vec{b}}{2} \right)$$

So,

$$\vec{p} \times \vec{q} = \frac{1}{4} (\vec{a} + \vec{b}) \times (\vec{a} - \vec{b}) = \frac{1}{4} (\vec{a} \times \vec{a} - \vec{a} \times \vec{b} + \vec{b} \times \vec{a} - \vec{b} \times \vec{b})$$

$$= \frac{1}{4} (-\vec{a} \times \vec{b} + \vec{b} \times \vec{a}) \quad \left[\begin{array}{l} \because \vec{a} \times \vec{a} = 0 \\ \vec{b} \times \vec{b} = 0 \end{array} \right]$$

$$= \frac{1}{4} (\vec{a} \times \vec{b} + \vec{a} \times \vec{b}) = \frac{1}{4} \cdot 2 (\vec{a} \times \vec{b}) = \frac{|\vec{a} \times \vec{b}|}{2}$$

And, the area of the parallelogram ABCD = $|\vec{p} \times \vec{q}| = \frac{1}{2} |\vec{a} \times \vec{b}|$

Now area of parallelogram whose diagonals are $2\hat{i} - \hat{j} + \hat{k}$

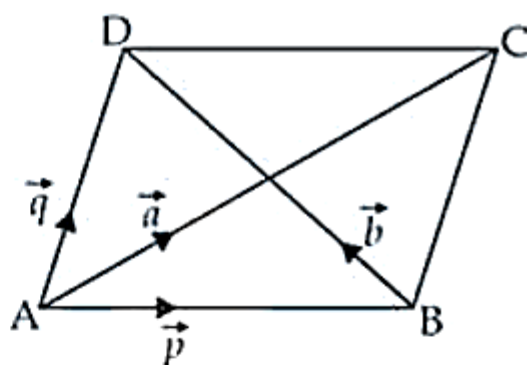
$$\text{and } \hat{i} + 3\hat{j} - \hat{k} = \frac{1}{2} |(2\hat{i} - \hat{j} + \hat{k}) \times (\hat{i} + 3\hat{j} - \hat{k})|$$

$$= - \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 1 \\ 1 & 3 & 1 \end{vmatrix}$$

$$= \frac{1}{2} |\hat{i}(1-3) - \hat{j}(-2-1) + \hat{k}(6+1)| = \frac{1}{2} |-2\hat{i} + 3\hat{j} + 7\hat{k}|$$

$$= \frac{1}{2} \sqrt{(-2)^2 + (3)^2 + (7)^2} = \frac{1}{2} \sqrt{4+9+49}$$

1 —



$$= \frac{1}{2} \sqrt{62} \text{ sq. units}$$

Thus, the required area is $\frac{1}{2} \sqrt{62}$ sq. units.

Question 18

If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{j} - \hat{k}$, find a vector $\vec{c}$ such that
 $\vec{a} \times \vec{c} = \vec{b}$ and $\vec{a} \cdot \vec{c} = 3$.

Solution

Let $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

Also given that $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{j} - \hat{k}$

As, $\vec{a} \times \vec{c} = \vec{b}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ c_1 & c_2 & c_3 \end{vmatrix} = \hat{j} - \hat{k}$$

$$= \hat{i}(c_3 - c_2) - \hat{j}(c_3 - c_1) + \hat{k}(c_2 - c_1) = \hat{j} - \hat{k}$$

On comparing the like terms, we get

$$c_3 - c_2 = 0 \quad \dots(i)$$

$$c_1 - c_3 = 1 \quad \dots(ii)$$

and $c_2 - c_1 = -1 \quad \dots(iii)$

Now for $\vec{a} \cdot \vec{c} = 3$

$$(\hat{i} + \hat{j} + \hat{k}) \cdot (c_1\hat{i} + c_2\hat{j} + c_3\hat{k}) = 3$$

$$\therefore c_1 + c_2 + c_3 = 3 \quad \dots(iv)$$

Adding eq. (ii) and eq. (iii) we get,

$$c_2 - c_3 = 0 \quad \dots(v)$$

From (iv) and (v) we get

$$c_1 + 2c_2 = 3 \quad \dots(vi)$$

From (iii) and (vi) we get

$$\begin{array}{r} c_1 + 2c_2 = 3 \\ + \quad -c_1 + c_2 = -1 \\ \hline 3c_2 = 2 \end{array}$$

We have, $c_2 = \frac{2}{3}$

$$c_3 - c_2 = 0 \Rightarrow c_3 - \frac{2}{3} = 0$$

$$\therefore c_3 = \frac{2}{3}$$

Now, $c_2 - c_1 = -1 \Rightarrow \frac{2}{3} - c_1 = -1$

$$\Rightarrow c_1 = 1 + \frac{2}{3} = \frac{5}{3}$$

$$\therefore \vec{c} = \frac{5}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}$$

$$\text{Thus, } \vec{c} = \frac{1}{3}(5\hat{i} + 2\hat{j} + 2\hat{k}).$$

Let $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

Also given that $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{j} - \hat{k}$

As, $\vec{a} \times \vec{c} = \vec{b}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ c_1 & c_2 & c_3 \end{vmatrix} = \hat{j} - \hat{k}$$

$$= \hat{i}(c_3 - c_2) - \hat{j}(c_3 - c_1) + \hat{k}(c_2 - c_1) = \hat{j} - \hat{k}$$

On comparing the like terms, we get

$$c_3 - c_2 = 0 \quad \dots(i)$$

$$c_1 - c_3 = 1 \quad \dots(ii)$$

and $c_2 - c_1 = -1 \quad \dots(iii)$

Now for $\vec{a} \cdot \vec{c} = 3$

$$(\hat{i} + \hat{j} + \hat{k}) \cdot (c_1\hat{i} + c_2\hat{j} + c_3\hat{k}) = 3$$

$$\therefore c_1 + c_2 + c_3 = 3 \quad \dots(iv)$$

Adding eq. (ii) and eq. (iii) we get,

$$c_2 - c_3 = 0 \quad \dots(v)$$

From (iv) and (v) we get

$$c_1 + 2c_2 = 3 \quad \dots(vi)$$

From (iii) and (vi) we get

$$\begin{array}{r} c_1 + 2c_2 = 3 \\ + \quad -c_1 + c_2 = -1 \\ \hline 3c_2 = 2 \end{array}$$

We have, $c_2 = \frac{2}{3}$

$$c_3 - c_2 = 0 \Rightarrow c_3 - \frac{2}{3} = 0$$

$$\therefore c_3 = \frac{2}{3}$$

Now, $c_2 - c_1 = -1 \Rightarrow \frac{2}{3} - c_1 = -1$

$$\Rightarrow c_1 = 1 + \frac{2}{3} = \frac{5}{3}$$

$$\therefore \vec{c} = \frac{5}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}$$

$$\text{Thus, } \vec{c} = \frac{1}{3}(5\hat{i} + 2\hat{j} + 2\hat{k}).$$

Objective Type Questions

Choose the correct answer from the given four options in each of the Exercises from 19 to 33 (M.C.Q)

Question 19

The vector in the direction of the vector $\hat{i} - 2\hat{j} + 2\hat{k}$ that has magnitude 9 is

- (A) $\hat{i} - 2\hat{j} + 2\hat{k}$ (B) $\frac{\hat{i} - 2\hat{j} + 2\hat{k}}{3}$
- (C) $3(\hat{i} - 2\hat{j} + 2\hat{k})$ (D) $9(\hat{i} - 2\hat{j} + 2\hat{k})$

Solution

The correct option is (C).

$$\text{Let } \vec{a} = \hat{i} - 2\hat{j} + 2\hat{k}$$

$$\begin{aligned} \text{Unit vector in the direction of } \vec{a} &= \frac{\vec{a}}{|\vec{a}|} \\ &= \frac{\hat{i} - 2\hat{j} + 2\hat{k}}{\sqrt{(1)^2 + (-2)^2 + (2)^2}} = \frac{\hat{i} - 2\hat{j} + 2\hat{k}}{\sqrt{1+4+4}} = \frac{\hat{i} - 2\hat{j} + 2\hat{k}}{3} \end{aligned}$$

$$\therefore \text{Vector of magnitude 9} = \frac{9(\hat{i} - 2\hat{j} + 2\hat{k})}{3} = 3(\hat{i} - 2\hat{j} + 2\hat{k})$$

Question 20

The position vector of the point which divides the join of points $2\vec{a} - 3\vec{b}$ and $\vec{a} + \vec{b}$ in the ratio 3 : 1 is

- (A) $\frac{3\vec{a} - 2\vec{b}}{2}$ (B) $\frac{7\vec{a} - 8\vec{b}}{4}$ (C) $\frac{3\vec{a}}{4}$ (D) $\frac{5\vec{a}}{4}$

Solution

The correct option is (D).

The given vectors are in the ratio 3: 1

So, the position vector of the required point c which divides the join of the given vectors $\vec{a}$ and $\vec{b}$ is

$$\begin{aligned}\vec{c} &= \frac{m_1x_2 + m_2x_1}{m_1 + m_2} \\ &= \frac{1 \cdot (2\vec{a} - 3\vec{b}) + 3(\vec{a} + \vec{b})}{3 + 1} = \frac{2\vec{a} - 3\vec{b} + 3\vec{a} + 3\vec{b}}{4} \\ &= \frac{5\vec{a}}{4} = \frac{5}{4} \vec{a}\end{aligned}$$

Question 21

The vector having initial and terminal points as (2, 5, 0) and (-3, 7, 4), respectively is

- (A) $-\hat{i} + 12\hat{j} + 4\hat{k}$ (B) $5\hat{i} + 2\hat{j} - 4\hat{k}$
(C) $-5\hat{i} + 2\hat{j} + 4\hat{k}$ (D) $\hat{i} + \hat{j} + \hat{k}$

Solution

The correct option is (C).

Let A and B be two points whose coordinates are given as (2, 5, 0) and (-3, 7, 4)

So, we have

$$\overrightarrow{AB} = (-3 - 2)\hat{i} + (7 - 5)\hat{j} + (4 - 0)\hat{k}$$

$$\overrightarrow{AB} = -5\hat{i} + 2\hat{j} + 4\hat{k}$$

Question 22

The angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitudes $\sqrt{3}$ and 4, respectively, and

$$\vec{a} \cdot \vec{b} = 2\sqrt{3}$$
 is

- (A) $\frac{\pi}{6}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{2}$ (D) $\frac{5\pi}{2}$

Solution

The correct option is (B).

Here, given that $|\vec{a}| = \sqrt{3}$, $|\vec{b}| = 4$ and $\vec{a} \cdot \vec{b} = 2\sqrt{3}$

So, from scalar product, we know that

$$\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$$

$$2\sqrt{3} = \sqrt{3} \cdot 4 \cdot \cos \theta$$

$$\cos \theta = \frac{2\sqrt{3}}{\sqrt{3} \cdot 4} = \frac{1}{2}$$

$$\therefore \theta = \frac{\pi}{3}$$
