

Chapter 5: Complex Numbers and Quadratic Equations

NCERT Exemplar · Class 11 · Maths

Short Answer Questions

Question 1

For a positive integer n , find the value of $(1 - i)^n \left(1 - \frac{1}{i}\right)^n$

Solution

Ans: Given, $(1 - i)^n \left(1 - \frac{1}{i}\right)^n$

The value of $i = \sqrt{-1}$ and $i^2 = -1$

Given expression,

$$= (1 - i)^n \left(1 - \frac{1}{i}\right)^n$$

$$\Rightarrow (1 - i)^n \left(\frac{i - 1}{i}\right)^n$$

$$\Rightarrow (1 - i)^n (i - 1)^n \cdot i^{-n} = (1 - i)^n (1 - i)^n (-1)^n \cdot i^{-n}$$

$$\Rightarrow \left[(1 - i)^2\right]^n (-1)^n \cdot i^{-n} = (1 + i^2 - 2i)^n (-1)^n i^{-n} [\because i^2 = -1]$$

$$\Rightarrow (1 - 1 - 2i)^n (-1)^n i^{-n} = (-2)^n \cdot i^n (-1)^n i^{-n}$$

$$\Rightarrow (-1)^{2n} \cdot 2^n = 2^n$$

$$\Rightarrow 1 \cdot 2^n = 2^n$$

Question 2

Evaluate $\sum_{n=1}^{13} (i^n + i^{n+1})$, where $n \in N$.

Solution

Ans: Given, $\sum_{n=1}^{13} (i^n + i^{n+1})$, where $n \in N$.

The value of $i = \sqrt{-1}$, $i^2 = -1$, $i^3 = -i$ and $i^4 = 1$

$$\begin{aligned} \sum_{n=1}^{13} (i^n + i^{n+1}), n \in N &= \sum_{h=1}^{13} i^n (1 + i) \\ \Rightarrow (1 + i) [i + i^2 + i^3 + i^4 + i^5 + i^6 + i^7 + i^8 + i^9 + i^{10} + i^{11} + i^{12} + i^{13}] \\ \Rightarrow (1 + i) [i^{13}] \left(\because i^n + i^{n+1} + i^{n+2} + i^{n+3} = 0, \text{ where } n \in N \text{ i.e., } \sum_{n=1}^{12} i^n = 0 \right) \\ \Rightarrow (1 + i)i &= (i^2 + i) \\ \because i &= \sqrt{-1} \\ \therefore i^2 &= -1 \\ \Rightarrow i - 1 \end{aligned}$$

Question 3

If $\left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3 = x + iy$, then find (x, y) .

Solution

Ans: Given, $\left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3 = x + iy$

If two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are equal

i.e., $z_1 = z_2 \Rightarrow x_1 + iy_1 = x_2 + iy_2$ then $x_1 = x_2$ and $y_1 = y_2$.

Given that, $\left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3 = x + y \dots (i)$

$$\therefore \left(\frac{1+i}{1-i}\right)^3 = \frac{1+i^3+3i(1+i)}{1-i^3-3i(1-i)} = \frac{1-i+3i+3i^2}{1+i-3i+3i^2}$$

$$\begin{aligned}
 &= \frac{2i - 2}{-2i - 2} = \frac{i - 1}{-i - 1} = \frac{1 - i}{1 + i} \\
 &= \frac{(1 - i)(1 - i)}{(1 + i)(1 - i)} = \frac{1 + i^2 - 2i}{1 + 1} = \frac{1 - 1 - 2i}{2} \\
 &\Rightarrow \left(\frac{1 + i}{1 - i} \right)^3 = -i \dots (ii) \\
 \text{Similarly, } &\left(\frac{1 - i}{1 + i} \right)^3 = \frac{-1}{i} = \frac{i^2}{i} = i \dots (iii)
 \end{aligned}$$

Using Eqs. (ii) and (iii) in Eq. (i), we get

$$\begin{aligned}
 -i - i &= x + iy \\
 \Rightarrow -2i &= x + iy
 \end{aligned}$$

On comparing real and imaginary part of complex number, we get

$$x = 0 \text{ and } y = -2$$

So,

$$(x, y) = (0, -2)$$

Question 4

If $\frac{(1 + i)^2}{2 - i} = x + iy$, then find the value of $x + y$.

Solution

$$\text{Ans: Given, } \frac{(1 + i)^2}{2 - i} = x + iy,$$

If two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are equal

i.e., $z_1 = z_2 \Rightarrow x_1 + iy_1 = x_2 + iy_2$ then $x_1 = x_2$ and $y_1 = y_2$.

Given that,

$$\frac{(1 + i)^2}{2 - i} = x + iy$$

$$\Rightarrow \frac{(1 + i^2 + 2i)}{2 - i} = x + iy$$

$$\Rightarrow \frac{2i}{2 - i} = x + iy$$

$$\Rightarrow \frac{2i(2 + i)}{(2 - i)(2 + i)} = x + iy$$

$$\Rightarrow \frac{4i + 2i^2}{4 - i^2} = x + iy$$

$$\Rightarrow \frac{4i - 2}{4 + 1} = x + iy$$

$$\Rightarrow \frac{-2}{5} + \frac{4i}{5} = x + iy$$

On comparing both sides, we get $x = -\frac{2}{5}$ and $y = \frac{4}{5}$

$$\Rightarrow x + y = \frac{-2}{5} + \frac{4}{5} = \frac{2}{5}$$

Question 5

If $\left(\frac{1-i}{1+i}\right)^{100} = a + ib$, then find (a, b) .

Solution

Ans: Given, $\left(\frac{1-i}{1+i}\right)^{100} = a + ib$,

The value of $i = \sqrt{-1}$, $i^2 = -1$, $i^3 = -i$ and $i^4 = 1$

If two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are equal

i.e., $z_1 = z_2 \Rightarrow x_1 + iy_1 = x_2 + iy_2$ then $x_1 = x_2$ and $y_1 = y_2$.

Given that, $\left(\frac{1-i}{1+i}\right)^{100} = a + ib$

$$\Rightarrow \left[\frac{(1-i)}{(1+i)} \cdot \frac{(1-i)}{(1-i)} \right]^{100} = a + ib$$

$$\Rightarrow \left(\frac{1 + i^2 - 2i}{1 - i^2} \right)^{100} = a + ib$$

$$\Rightarrow \left(\frac{-2i}{2} \right)^{100} = a + ib \left[\because i^2 = -1 \right]$$

$$\Rightarrow (i^4)^{25} = a + ib$$

$$\Rightarrow 1 = a + ib$$

Then, $a = 1$ and $b = 0$ $[\because i^4 = 1]$

$$\therefore (a, b) = (1, 0)$$

Question 6

If $a = \cos\theta + i\sin\theta$, find the value of $\frac{1+a}{1-a}$.

Solution

Ans: Given, $a = \cos\theta + i\sin\theta$,

The value of $i = \sqrt{-1}$, $i^2 = -1$, $i^3 = -i$ and $i^4 = 1$

To solve the above problem use the trigonometric formula, $\cos\theta = 2\cos^2\frac{\theta}{2} - 1 = 1 - 2\sin^2\frac{\theta}{2}$ and $\sin\theta = 2\sin\frac{\theta}{2} \cdot \cos\frac{\theta}{2}$

Given that, $a = \cos\theta + i\sin\theta$

$$\therefore \frac{1+a}{1-a} = \frac{1 + \cos\theta + i\sin\theta}{1 - \cos\theta - i\sin\theta}$$

$$\Rightarrow \frac{1 + 2\cos^2\frac{\theta}{2} - 1 + 2i\sin\frac{\theta}{2} \cdot \cos\frac{\theta}{2}}{1 - 1 + 2\sin^2\frac{\theta}{2} - 2i\sin\frac{\theta}{2} \cdot \cos\frac{\theta}{2}}$$

$$\Rightarrow \frac{2\cos\frac{\theta}{2} \left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2} \right)}{2\sin\frac{\theta}{2} \left(\sin\frac{\theta}{2} - i\cos\frac{\theta}{2} \right)}$$

$$\Rightarrow -\frac{2 \cos \frac{\theta}{2} \left(\cos \frac{\theta}{2} + i \sin \frac{\theta}{2} \right)}{2i \sin \frac{\theta}{2} \left(\cos \frac{\theta}{2} + i \sin \frac{\theta}{2} \right)} = -\frac{1}{i} \cot \frac{\theta}{2}$$

$$\Rightarrow \frac{+i^2}{i} \cot \frac{\theta}{2} = i \cot \frac{\theta}{2} \left[\because \frac{-1}{i} = \frac{i^2}{i} \right]$$

Question 7

If $(1 + i)z = (1 - i)\bar{z}$, then show that $z = -i\bar{z}$.

Solution

Ans: Given, $(1 + i)z = (1 - i)\bar{z}$,

The value of $i = \sqrt{-1}$, $i^2 = -1$, $i^3 = -i$ and $i^4 = 1$

We have,

$$(1 + i)z = (1 - i)\bar{z}$$

$$\Rightarrow \frac{z}{\bar{z}} = \frac{(1 - i)}{(1 + i)}$$

$$\Rightarrow \frac{z}{\bar{z}} = \frac{(1 - i)}{(1 + i)} \frac{(1 - i)}{(1 - i)}$$

$$\Rightarrow \frac{z}{\bar{z}} = \frac{1 + i^2 - 2i}{1 - i^2} [\because i^2 = -1]$$

$$\Rightarrow \frac{z}{\bar{z}} = \frac{1 - 1 - 2i}{2}$$

$$\Rightarrow \frac{z}{\bar{z}} = -i$$

$$\therefore z = i\bar{z}$$

Hence proved.

Question 8

If $z = x + iy$, then show that $z\bar{z} + 2(z + \bar{z}) + b = 0$, where $b \in \mathbb{R}$, represents a circle.

Solution

Ans: Given, $z = x + iy$

If $z = x + iy$ its conjugate is $\bar{z} = x - iy$.

If $z = x + iy$, then show that $z\bar{z} + 2(z + \bar{z}) + b = 0$, where $b \in \mathbb{R}$, represents a circle.

Given that,

$$z = x + iy$$

$$\text{Then, } \bar{z} = x - iy$$

Now,

$$z\bar{z} + 2(z + \bar{z}) + b = 0$$

$$\Rightarrow (x + iy)(x - iy) + 2(x + iy + x - iy) + b = 0$$

$$\Rightarrow x^2 + y^2 + 4x + b = 0, \text{ which is the equation of a circle.}$$

Question 9

If the real part of $\frac{\bar{z} + 2}{\bar{z} - 1}$ is 4, then show that the locus of the point representing z in the complex plane is a circle.

Solution

Ans: Given, Real part of $\frac{\bar{z} + 2}{\bar{z} - 1}$ is 4

Evaluate the value of $\frac{\bar{z} + 2}{\bar{z} - 1}$ and if it satisfies the equation of the circle the locus of the point representing z in the complex plane is a circle.

$$\text{Let } z = x + iy$$

Now,

$$\begin{aligned}
\frac{\bar{z} + 2}{\bar{z} - 1} &= \frac{x - iy + 2}{x - iy - 1} \\
&= \frac{[(x + 2) - iy][(x - 1) + iy]}{[(x - 1) - iy][(x - 1) + iy]} \\
&= \frac{(x - 1)(x + 2) - iy(x - 1) + iy(x + 2) + y^2}{(x - 1)^2 + y^2} \\
&= \frac{(x - 1)(x + 2) + y^2 + i[(x + 2)y - (x - 1)y]}{(x - 1)^2 + y^2} [\because -i^2 = 1]
\end{aligned}$$

Let,

$$\frac{(x - 1)(x + 2) + y^2}{(x - 1)^2 + y^2} + \frac{i[(x + 2)y - (x - 1)y]}{(x - 1)^2 + y^2} = x + iy$$

Taking real part,

$$\frac{(x - 1)(x + 2) + y^2}{(x - 1)^2 + y^2} = 4 \quad (\text{Given, that the real part} = 4)$$

$$\Rightarrow x^2 - x + 2x - 2 + y^2 = 4(x^2 - 2x + 1 + y^2)$$

$$\Rightarrow x^2 - x + 2x - 2 + y^2 = 4x^2 - 8x + 4 + 4y^2$$

$$\Rightarrow 3x^2 + 3y^2 - 9x + 6 = 0, \text{ which represents a circle.}$$

Hence, z lies on the circle.

Question 10

Show that the complex number z , satisfying the condition $\arg\left(\frac{z - 1}{z + 1}\right) = \frac{\pi}{4}$ lies on a circle.

Solution

Ans: Given, The complex number z First use, $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$. Also apply

$\arg(z) = \theta = \tan^{-1} \frac{y}{x}$, where $z = x + iy$ and then use the property

$$\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x - y}{1 + xy}\right)$$

Let,

$$z = x + iy$$

Given that,

$$\Rightarrow \arg(z - 1) - \arg(z + 1) = \frac{\pi}{4}$$

$$\Rightarrow \arg(x + iy - 1) - \arg(x + iy + 1) = \frac{\pi}{4}$$

$$\Rightarrow \arg(x - 1 + iy) - \arg(x + 1 + iy) = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1} \frac{y}{x-1} - \tan^{-1} \frac{y}{x+1} = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1} \left[\frac{\frac{y}{x-1} - \frac{y}{x+1}}{1 + \left(\frac{y}{x-1}\right) \left(\frac{y}{x+1}\right)} \right] = \frac{\pi}{4}$$

$$\Rightarrow \frac{y \left[\frac{x+1 - x-1}{x^2 - 1} \right]}{\frac{x^2 - 1 + y^2}{x^2 - 1}} = \tan \frac{\pi}{4}$$

$$\Rightarrow \frac{2y}{x^2 + y^2 - 1} = 1$$

$$\Rightarrow x^2 + y^2 - 1 = 2y$$

$$\Rightarrow x^2 + y^2 - 2y - 1 = 0, \text{ which represents a circle.}$$

Question 11

Solve the equation $|z| = z + 1 + 2i$.

Solution

Ans: Given, $|z| = z + 1 + 2i$

Simplify $|z| = z + 1 + 2i$ by using the formula $|z| + \sqrt{x^2 + y^2}$.

The given equation is $|z| = z + 1 + 2i \dots (i)$

Let $z = x + iy$

From Eq. (i), $|x + iy| = x + iy + 1 + 2i$

$$\Rightarrow \sqrt{x^2 + y^2} = x + iy + 1 + 2i \left[\because |z| + \sqrt{x^2 + y^2} \right]$$

$$\Rightarrow \sqrt{x^2 + y^2} = (x + 1) + i(y + 2)$$

On squaring both sides, we get

$$x^2 + y^2 = (x + 1)^2 + i^2(y + 2)^2 + 2i(x + 1)(y + 2)$$

$$\Rightarrow x^2 + y^2 = x^2 + 2x + 1 - y^2 - 4y - 4 + 2i(x + 1)(y + 2)$$

On comparing real and imaginary parts,

$$x^2 + y^2 = x^2 + 2x + 1 - y^2 - 4y - 4$$

$$\text{i.e., } 2y^2 = 2x - 4y - 3 \dots (ii)$$

and

$$2(x + 1)(y + 2) = 0$$

$$(x + 1) = 0 \text{ or } (y + 2) = 0$$

$$\Rightarrow x = -1 \text{ or } y = -2$$

For $x = -1$,

$$2y^2 = -2 - 4y - 3 [\text{using Eq. (ii)}]$$

$$2y^2 + 4y + 5 = 0$$

$$\Rightarrow y = \frac{-4 \pm \sqrt{16 - 2 \times 4 \times 5}}{4}$$

$$\Rightarrow y = \frac{-4 \pm \sqrt{-24}}{4} \notin R$$

For $y = -2$,

$$2(-2)^2 = 2x - 4(-2) - 3 [\text{using Eq. (ii)}]$$

$$\Rightarrow 8 = 2x + 8 - 3$$

$$\Rightarrow 2x = 3$$

$$\Rightarrow x = \frac{3}{2}$$

$$\therefore z = x + iy = \frac{3}{2} - 2i$$

Question 12

If $|z + 1| = z + 2(1 + i)$, then find the value of z .

Solution

Ans: Given, $|z + 1| = z + 2(1 + i)$

Take $z = x + iy$ and use the formula $|z| = \sqrt{x^2 + y^2}$.

Given that, $|z + 1| = z + 2(1 + i) \dots (i)$

Let $z = x + iy$

Then,

$$\Rightarrow |x + iy + 1| = x + iy + 2(1 + i)$$

$$\Rightarrow |x + 1 + iy| = (x + 2) + i(y + 2)$$

$$\Rightarrow \sqrt{(x + 1)^2 + y^2} = (x + 2) + i(y + 2)$$

On squaring both sides, we get

$$\Rightarrow (x + 1)^2 + y^2 = (x + 2)^2 + i^2(y + 2)^2 + 2i(x + 2)(y + 2)$$

$$\Rightarrow x^2 + 2x + 1 + y^2 = x^2 + 4x + 4 - y^2 - 4y - 4 + 2i(x + 2)(y + 2)$$

$$\Rightarrow x^2 + y^2 + 2x + 1 = x^2 - y^2 + 4x - 4y + 2i(x + 2)(y + 2)$$

On comparing real and imaginary parts, we get

$$\Rightarrow x^2 + y^2 + 2x + 1 = x^2 - y^2 + 4x - 4y$$

$$\Rightarrow 2y^2 - 2x + 4y + 1 = 0 \dots (ii)$$

$$\text{and } 2(x + 2)(y + 2) = 0$$

$$\Rightarrow x + 2 = 0 \text{ or } y + 2 = 0$$

$$\Rightarrow x = -2 \text{ or } y = -2 \dots (iii)$$

For $x = -2$,

$$2y^2 + 4 + 4y + 1 = 0 \text{ [Using Eq. (ii)]}$$

$$\Rightarrow 2y^2 + 4y + 5 = 0$$

$$\Rightarrow 16 - 4 \times 2 \times 5 < 0$$

$\therefore$ Discriminant, $D = b^2 - 4ac < 0$

$\Rightarrow 2y^2 + 4y + 5$ has no real roots.

For $y = -2$,

$$\Rightarrow 2(-2)^2 - 2x + 4(-2) + 1 = 0 \text{ [Using Eq. (ii)]}$$

$$\Rightarrow 8 - 2x - 8 + 1 = 0$$

$$\Rightarrow x = \frac{1}{2}$$

$$\therefore z = x + iy = \frac{1}{2} - 2i$$

Question 13

If $\arg(z - 1) = \arg(z + 3i)$, then find $x - 1 : y$, where $z = x + iy$.

Solution

Ans: Given, $\arg(z - 1) = \arg(z + 3i)$, where $z = x + iy$.

Apply $\arg(z) = \theta = \tan^{-1} \frac{y}{x}$, where $z = x + iy$.

Given that,

$$\arg(z - 1) = \arg(z + 3i)$$

$$\text{Let } z = x + iy$$

Now,

$$\Rightarrow \arg(x + iy - 1) = \arg(x + iy + 3i)$$

$$\Rightarrow \arg(z - 1) = \arg(z + 3i)$$

$$\Rightarrow \arg(x - 1 + iy) = \arg[x + i(y + 3)]$$

$$\Rightarrow \tan^{-1} \frac{y}{x-1} = \tan^{-1} \frac{y+3}{x}$$

$$\Rightarrow \frac{y}{x-1} = \frac{y+3}{x}$$

$$\Rightarrow xy = (x-1)(y+3)$$

$$\Rightarrow xy = xy - y + 3x - 3$$

$$\Rightarrow 3x - 3 = y$$

$$\Rightarrow \frac{3(x-1)}{y} = 1$$

$$\Rightarrow \frac{x-1}{y} = \frac{1}{3}$$

$$\therefore (x-1) : y = 1 : 3$$

Question 14

Show that $\left| \frac{z-2}{z-3} \right| = 2$ represents a circle. Find its centre and radius.

Solution

Ans: Given, $\left| \frac{z-2}{z-3} \right| = 2$

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are two complex numbers, then $\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}$, ($z_2 \neq 0$), use this

concept to solve the above problem. Also, the general equation of a circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0, \text{ with centre } (-g, -f) \text{ and radius } = \sqrt{g^2 + f^2 - c}.$$

Let $z = x + iy$

Given,

Equation is $\left| \frac{z-2}{z-3} \right| = 2$

$$\Rightarrow \left| \frac{z-2}{z-3} \right| = 2$$

$$\Rightarrow \left| \frac{x+iy-2}{x+iy-3} \right| = 2$$

$$\Rightarrow |x-2+iy| = 2|x-3+iy|$$

$$\Rightarrow \sqrt{(x-2)^2 + y^2} = 2\sqrt{(x-3)^2 + y^2} \left[\because |x+iy| = \sqrt{x^2 + y^2} \right]$$

On squaring both sides, we get

$$\Rightarrow x^2 - 4x + 4 + y^2 = 4(x^2 - 6x + 9 + y^2)$$

$$\Rightarrow 3x^2 + 3y^2 - 20x + 32 = 0$$

$$\Rightarrow x^2 + y^2 - \frac{20}{3}x + \frac{32}{3} = 0 \dots (i)$$

On comparing the above equation with $x^2 + y^2 + 2gx + 2fy + c = 0$, we get

$$\Rightarrow 2g = \frac{-20}{3}$$

$$\Rightarrow g = \frac{-10}{3}$$

and

$$\Rightarrow 2f = 0$$

$$\Rightarrow f = 0 \text{ and } c = \frac{32}{3}$$

$$\therefore \text{Centre} = (-g, -f) = \left(\frac{10}{3}, 0 \right)$$

Also,

radius

$$(r) = \sqrt{\left(\frac{10}{3} \right)^2 + 0 - \frac{32}{3}} \left[\because r = \sqrt{g^2 + f^2 - c} \right]$$

$$= \frac{1}{3} \sqrt{(100 - 96)} = \frac{2}{3}$$

Question 15

If $\frac{z-1}{z+1}$ is a purely imaginary number, then find the value of $|z|$.

Solution

Ans: Let,

$$z = x + iy$$

$$\Rightarrow \frac{z-1}{z+1} = \frac{x+iy-1}{x+iy+1} z \neq -1$$

$$\begin{aligned}\Rightarrow \frac{x-1+iy}{x+1+iy} &= \frac{(x-1+iy)(x+1-iy)}{(x+1+iy)(x+1-iy)} \\ &= \frac{(x^2-1) + iy(x+1) - iy(x-1) - i^2y^2}{(x+1)^2 - (iy)^2}\end{aligned}$$

$$\Rightarrow \frac{z-1}{z+1} = \frac{(x^2-1) + y^2 + i[y(x+1) - y(x-1)]}{(x+1)^2 + y^2}$$

Given that, $\frac{z-1}{z+1}$ is a purely imaginary number.

Then,

$$\Rightarrow \frac{(x^2-1) + y^2}{(x+1)^2 + y^2} = 0$$

$$\Rightarrow x^2 - 1 + y^2 = 0$$

$$\Rightarrow x^2 + y^2 = 1$$

$$\Rightarrow \sqrt{x^2 + y^2} = \sqrt{1} \left[\because |z| = \sqrt{x^2 + y^2} \right]$$

$$\therefore |z| = 1$$

Question 16

z_1 and z_2 are two complex numbers such that $|z_1| = |z_2|$ and $\arg(z_1) + \arg(z_2) = \pi$, then show that $z_1 = -\bar{z}_2$.

Solution

Ans: Given, $|z_1| = |z_2|$ and $\arg(z_1) + \arg(z_2) = \pi$

$\arg(z) = \theta = \tan^{-1} \frac{y}{x}$, where $z = x + iy$.

Let $z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2 (\cos \theta_2 + i \sin \theta_2)$ are two complex numbers.

Given that, $|z_1| = |z_2|$

and $\arg(z_1) + \arg(z_2) = \pi$

If $|z_1| = |z_2|$

$r_1 = r_2$

and if $\arg(z_1) + \arg(z_2) = \pi$

$\theta_1 + \theta_2 = \pi$

$\theta_1 = \pi - \theta_2$

Now, $z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$

$\Rightarrow z_1 = r_2 [\cos(\pi - \theta_2) + i \sin(\pi - \theta_2)] [\because r_1 = r_2 \text{ and } \theta_1 = (\pi - \theta_2)]$

$\Rightarrow z_1 = r_2 (-\cos \theta_2 + i \sin \theta_2)$

$\Rightarrow z_1 = -r_2 (\cos \theta_2 - i \sin \theta_2)$

$\Rightarrow z_1 = -[r_2 (\cos \theta_2 - i \sin \theta_2)]$

$\Rightarrow z_1 = -\bar{z}_2 [\because \bar{z}_2 = r_2 (\cos \theta_2 - i \sin \theta_2)]$

Question 17

If $|z_1| = 1$ ($z_1 \neq -1$) and $z_2 = \frac{z_1 - 1}{z_1 + 1}$, then show that the real part of z_2 is zero.

Solution

Ans: Given, $|z_1| = 1$ ($z_1 \neq -1$) and $z_2 = \frac{z_1 - 1}{z_1 + 1}$.

Take $z_1 = x + iy$ and use the formula $|z| = \sqrt{x^2 + y^2}$.

Let,

$$z_1 = x + iy$$

$$\Rightarrow |z_1| = \sqrt{x^2 + y^2} = 1 \quad [\because |z_1| = 1, \text{ given}] \dots (i)$$

Now,

$$\begin{aligned} z_2 &= \frac{z_1 - 1}{z_1 + 1} = \frac{x + iy - 1}{x + iy + 1} \\ &= \frac{x - 1 + iy}{x + 1 + iy} = \frac{(x - 1 + iy)(x + 1 - iy)}{(x + 1 + iy)(x + 1 - iy)} \\ &= \frac{x^2 - 1 + iy(x + 1) - iy(x - 1) - i^2 y^2}{(x + 1)^2 - i^2 y^2} \\ &= \frac{x^2 - 1 + ixy + y - ixy + iy + y^2}{(x + 1)^2 + y^2} \\ &= \frac{x^2 + y^2 - 1 + 2iy}{(x + 1)^2 + y^2} = \frac{1 - 1 + 2iy}{(x + 1)^2 + y^2} \quad [\because x^2 + y^2 = 1] \\ &= 0 + \frac{2yi}{(x + 1)^2 + y^2} \end{aligned}$$

Hence, the real part of

$$z_2$$

is zero.

Question 18

If z_1, z_2 and z_3, z_4 are two pairs of conjugate complex numbers, then find $\arg\left(\frac{z_1}{z_4}\right) + \arg\left(\frac{z_2}{z_3}\right)$.

Solution

Ans: Given, z_1, z_2 and z_3, z_4 are two pairs of conjugate complex numbers.

First let, $z = r(\cos \theta + i \sin \theta)$ then conjugate of z i.e., $\bar{z} = r(\cos \theta - i \sin \theta)$. Use the

property $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$.

Let $z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$

Then,

$$z_2 = \bar{z}_1$$

$$\Rightarrow r_1 (\cos \theta_1 - i \sin \theta_1) = r_1 [\cos(-\theta_1) + i \sin(-\theta_1)]$$

Also,

Let $z_3 = r_2 (\cos \theta_2 + i \sin \theta_2)$,

Then,

$$z_4 = \bar{z}_3 = r_2 (\cos \theta_2 - i \sin \theta_2)$$

$$\arg\left(\frac{z_1}{z_4}\right) + \arg\left(\frac{z_2}{z_3}\right) = \arg(z_1) - \arg(z_4) + \arg(z_2) - \arg(z_3)$$

$$= \theta_1 - (-\theta_2) + (-\theta_1) - \theta_2 [\because \arg(z) = \theta]$$

$$= \theta_1 + \theta_2 - \theta_1 - \theta_2 = 0$$

Question 19

If $|z_1| = |z_2| = L = |z_n| = 1$, then show that $|z_1 + z_2 + z_3 + L + z_n| = \left| \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} + L + \frac{1}{z_n} \right|$.

Solution

Ans: Given, $|z_1| = |z_2| = \dots = |z_n| = 1$

Using formula $|z|^2 = z \cdot \bar{z}$ and $|z_1 + z_2 + z_3 + \dots + z_n| = |z_1| + |z_2| + |z_3| + \dots + |z_n|$

Given that,

$$|z_1| = |z_2| = \dots = |z_n| = 1$$

$$\Rightarrow |z_1|^2 = |z_2|^2 = \dots = |z_n|^2 = 1$$

$$\Rightarrow z_1 \bar{z}_1 = z_2 \bar{z}_2 = z_3 \bar{z}_3 = \dots = z_n \bar{z}_n = 1$$

$$\Rightarrow z_1 = \frac{1}{\bar{z}_1}, z_2 = \frac{1}{\bar{z}_2} = \dots = z_n = \frac{1}{\bar{z}_n}$$

Now,

$$\begin{aligned} & |z_1 + z_2 + z_3 + z_4 + \dots + z_n| \\ &= \left| \frac{z_1 \bar{z}_1}{\bar{z}_1} + \frac{z_2 \bar{z}_2}{\bar{z}_2} + \frac{z_3 \bar{z}_3}{\bar{z}_3} + \dots + \frac{z_n \bar{z}_n}{\bar{z}_n} \right| \left[\because z_1 \bar{z}_1 = 1, \text{ where } z_1 = \frac{1}{\bar{z}_1}, z_1 = \frac{\bar{z}}{\bar{z} - \bar{z}}, z_1 = \bar{z} \right] \\ &= \left| \frac{|z_1|^2}{\bar{z}_1} + \frac{|z_2|^2}{\bar{z}_2} + \frac{|z_3|^2}{\bar{z}_3} + \dots + \frac{|z_n|^2}{\bar{z}_n} \right| \\ &= \left| \frac{1}{\bar{z}_1} + \frac{1}{\bar{z}_2} + \frac{1}{\bar{z}_3} + \dots + \frac{1}{\bar{z}_n} \right| = \sqrt{\frac{1}{\bar{z}_1} + \frac{1}{\bar{z}_2} + \frac{1}{\bar{z}_3}} \\ &= \left| \frac{1}{z_1} + \frac{1}{z_2} + \dots + \frac{1}{z_n} \right| \end{aligned}$$

Hence proved.

Question 20

If the complex numbers z_1 and z_2 , $\arg(z_1) - \arg(z_2) = 0$, then show that $|z_1 - z_2| = |z_1| - |z_2|$.

Solution

Ans: Given, $\arg(z_1) - \arg(z_2) = 0$.

Apply $\arg(z) = \theta$, where $z = x + iy$ and $z = r(\cos \theta + i \sin \theta)$.

Let,

$$z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$$

$$z_2 = r_2 (\cos \theta_2 + i \sin \theta_2)$$

$$\Rightarrow \arg(z_1) = \theta_1 \text{ and } \arg(z_2) = \theta_2$$

Given that,

$$\arg(z_1) - \arg(z_2) = 0$$

$$\theta_1 - \theta_2 = 0$$

$$\Rightarrow \theta_1 = \theta_2$$

$$z_2 = r_2 (\cos \theta_1 + i \sin \theta_1) [\because \theta_1 = \theta_2]$$

$$z_1 - z_2 = (r_1 \cos \theta_1 - r_2 \cos \theta_1) + i (r_1 \sin \theta_1 - r_2 \sin \theta_1)$$

$$|z_1 - z_2| = \sqrt{(r_1 \cos \theta_1 - r_2 \cos \theta_1)^2 + (r_1 \sin \theta_1 - r_2 \sin \theta_1)^2}$$

$$= \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos^2 \theta_1 - 2r_1 r_2 \sin^2 \theta_1}$$

$$= \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 (\sin^2 \theta_1 + \cos^2 \theta_1)}$$

$$= \sqrt{r_1^2 + r_2^2 - 2r_1 r_2} = \sqrt{(r_1 - r_2)^2}$$

$$|z_1 - z_2| = r_1 - r_2 [\because r = |z|]$$

$$= |z_1| - |z_2|$$

Hence proved.

Question 21

Solve the system of equations $\operatorname{Re}(z^2) = 0, |z| = 2$.

Solution

Ans: Given, $\operatorname{Re}(z^2) = 0, |z| = 2$

Let $z = x + iy$ and use the formula $|z| = \sqrt{x^2 + y^2}$.

Given that, $\operatorname{Re}(z^2) = 0, |z| = 2$

Let $z = x + iy$

$$|z| = \sqrt{x^2 + y^2}$$

$$\therefore \sqrt{x^2 + y^2} = 2 \quad \text{(On squaring both sides)}$$

$$\Rightarrow x^2 + y^2 = 4 \dots (i)$$

$$\text{and } \operatorname{Re}(z) = x$$

$$\text{Also, } z = x + iy$$

$$\Rightarrow z^2 = x^2 + 2ixy - y^2 \quad (\text{On squaring both sides})$$

$$\Rightarrow z^2 = (x^2 - y^2) + 2ixy$$

$$\Rightarrow \operatorname{Re}(z^2) = x^2 - y^2 \quad [\because \operatorname{Re}(z^2) = 0]$$

$$\Rightarrow x^2 - y^2 = 0 \dots (ii)$$

On adding Eq. (i) and (ii),

$$x^2 + y^2 + x^2 - y^2 = 4 + 0$$

$$\Rightarrow 2x^2 = 4$$

$$\Rightarrow x^2 = 2$$

$$\Rightarrow x = \pm\sqrt{2}$$

Putting the value of x in equation (i), we get

$$y = \pm\sqrt{2}$$

$$\therefore z = x + iy$$

$$\Rightarrow z = \sqrt{2} \pm i\sqrt{2}, -\sqrt{2} \pm i\sqrt{2}$$

Question 22

Find the complex number satisfying the equation $z + \sqrt{2}|(z + 1)| + i = 0$.

Solution

$$\text{Ans: Given, } z + \sqrt{2}|(z + 1)| + i = 0$$

Let $z = x + iy$ and use the formula $|z| = \sqrt{x^2 + y^2}$. Compare the real and imaginary parts of LHS and RHS.

$$\text{Given equation is } z + \sqrt{2}|(z + 1)| + i = 0 \quad \dots (i)$$

Let $z = x + iy$

On substituting the value of z in equation (i), we get

$$\Rightarrow x + iy + \sqrt{2}|x + y + 1| + i = 0$$

$$\Rightarrow x + i(1 + y) + \sqrt{2} \left[\sqrt{(x + 1)^2 + y^2} \right] = 0$$

$$\Rightarrow x + i(1 + y) + \sqrt{2}\sqrt{(x^2 + 2x + 1 + y^2)} = 0$$

On comparing real and imaginary parts of LHS with RHS, we get

$$\Rightarrow x + \sqrt{2}\sqrt{x^2 + 2x + 1 + y^2} = 0 \quad (\text{real part})$$

$$\Rightarrow x^2 = 2(x^2 + 2x + 1 + y^2)$$

$$\Rightarrow x^2 + 4x + 2y^2 + 2 = 0 \dots (ii)$$

$$1 + y = 0 \quad (\text{imaginary part})$$

$$\Rightarrow y = -1$$

For $y = -1$,

$$\Rightarrow x^2 + 4x + 2 + 2 = 0 \text{ [Using eq. (ii)]}$$

$$\Rightarrow x^2 + 4x + 4 = 0$$

$$\Rightarrow (x + 2)^2 = 0$$

$$\Rightarrow x + 2 = 0$$

$$\Rightarrow x = -2$$

$$\therefore z = x + iy = -2 - i$$

Question 23

Write the complex number $z = \frac{1 - i}{\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}}$ in polar form.

Solution

Ans: Given, $z = \frac{1-i}{\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}}$

Use trigonometrically identity to convert complex number into polar form.

Given that,

$$\begin{aligned}
 z &= \frac{1-i}{\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}} \left(\text{Multiplying and dividing numerator by } \sqrt{2} \right) \\
 &\Rightarrow \frac{-\sqrt{2} \left[\frac{-1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right]}{\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}} = \frac{-\sqrt{2} \left[\cos \left(\pi - \frac{\pi}{4} \right) + i \sin \left(\pi - \frac{\pi}{4} \right) \right]}{\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}} \\
 &\Rightarrow \frac{-\sqrt{2} \left[\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right]}{\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}} \\
 &\Rightarrow -\sqrt{2} \left[\cos \left(\frac{3\pi}{4} - \frac{\pi}{3} \right) + i \sin \left(\frac{3\pi}{4} - \frac{\pi}{3} \right) \right] \\
 &\Rightarrow -\sqrt{2} \left[\cos \frac{5\pi}{12} + i \sin \frac{5\pi}{12} \right]
 \end{aligned}$$

Question 24

If z and w are two complex numbers such that $|zw| = 1$ and $\arg(z) - \arg(w) = \frac{\pi}{2}$, then show that $\bar{z}w = -i$.

Solution

Ans: Given, z and w are two complex numbers such that $|zw| = 1$ and $\arg(z) - \arg(w) = \frac{\pi}{2}$.

Let $z = r_1 (\cos \theta_1 + i \sin \theta_1)$ and $w = r_2 (\cos \theta_2 + i \sin \theta_2)$ and $|zw| = |z||w|$.

Let $z = r_1 (\cos \theta_1 + i \sin \theta_1)$ and $w = r_2 (\cos \theta_2 + i \sin \theta_2)$

Also, $|zw| = |z||w| = r_1 r_2 = 1$

$$\therefore r_1 r_2 = 1$$

$$\text{Further, } \arg(z) = \theta_1 \text{ and } \arg(w) = \theta_2$$

But,

$$\arg(z) - \arg(w) = \frac{\pi}{2}$$

$$\Rightarrow \theta_1 - \theta_2 = \frac{\pi}{2}$$

$$\Rightarrow \arg\left(\frac{z}{w}\right) = \frac{\pi}{2}$$

Now,

$$\text{to prove } \bar{z}w = -i$$

$$\text{LHS} = \bar{z}w$$

$$= r_1 (\cos \theta_1 - i \sin \theta_1) r_2 (\cos \theta_2 + i \sin \theta_2)$$

$$= r_1 r_2 [\cos(\theta_2 - \theta_1) + i \sin(\theta_2 - \theta_1)]$$

$$= \left[r_1 r_2 \cos\left(\frac{-\pi}{2}\right) + i \sin\left(\frac{-\pi}{2}\right) \right]$$

$$= 1[0 - i] = -i = \text{RHS}$$

Question 25

Fill in the blanks.

(i) For any two complex numbers z_1, z_2 and any real numbers a, b , $|az_1 - bz_2|^2 + |bz_1 + az_2|^2 = \dots$

(ii) The value of $\sqrt{-25} \times \sqrt{-9}$ is ...

(iii) The number $\frac{(1-i)^3}{1-i^3}$ is equal to ...

(iv) The sum of the series $i + i^2 + i^3 + \dots$ upto 1000 terms is ...

(v) Multiplicative inverse of $1 + i$ is ...

(vi) If z_1 and z_2 are complex numbers such that $z_1 + z_2$ is a real number, then $z_1 = \dots$

(vii) $\arg(z) + \arg \bar{z}$ where, is ...

(viii) If $|z + 4| \leq 3$, then the greatest and least values of $|z + 1|$ are ... and

(ix) If $\left| \frac{z-2}{z+2} \right| = \frac{\pi}{6}$, then the locus of z is ...

(x) If $|z| = 4$ and $\arg(z) = \frac{5\pi}{6}$, then $z = \dots$

Solution

$$\begin{aligned} \text{Ans: } & |az_1 - bz_2|^2 + |bz_1 + az_2|^2 \\ &= |az_1|^2 + |bz_2|^2 - 2 \operatorname{Re}(az_1 \cdot b\bar{z}_2) + |bz_1|^2 + |az_2|^2 + 2 \operatorname{Re}(az_1 \cdot b\bar{z}_2) \\ &= (a^2 + b^2) |z_1|^2 + (a^2 + b^2) |z_2|^2 \\ &= (a^2 + b^2) (|z_1|^2 + |z_2|^2) \end{aligned}$$

$$\text{Ans: } \sqrt{-25} \times \sqrt{-9}$$

$$\Rightarrow i\sqrt{25} \times i\sqrt{9} = i^2(5 \times 3)$$

$$\Rightarrow -15$$

$$\text{Ans: } \frac{(1-i)^3}{1-i^3} = \frac{(1-i)^3}{(1-i)(1+i+i^2)}$$

$$\Rightarrow \frac{(1-i)^2}{i} = \frac{1+i^2-2i}{i}$$

$$\Rightarrow \frac{-2i}{i} = -2$$

Ans: $i + i^2 + i^3 + \dots$ upto 1000 terms,

$$= i + i^2 + i^3 + i^4 + \dots + i^{1000} = 0$$

$$\left[\because i^n + i^{n+1} + i^{n+2} + i^{n+3} = 0, \text{ where } n \in N \text{ i.e., } \sum_{n=1}^{1000} i^n = 0 \right]$$

Ans: Multiplicative inverse of $1 + i$

$$\frac{1}{1+i} = \frac{1-i}{1-i^2}$$

$$\Rightarrow \frac{1}{2}(1-i)$$

Ans: Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2) \text{ which is real.}$$

$$\text{If } z_1 + z_2 \text{ is real, then } y_1 + y_2 = 0$$

$$\Rightarrow y_1 = -y_2$$

$$\therefore z_2 = x_2 - i_1$$

$$\Rightarrow z_2 = \bar{z}_1 [\text{when } x_1 = x_2]$$

$$\text{Ans: } \arg(z) + \arg(\bar{z}), (\bar{z} \neq 0)$$

$$\Rightarrow \theta + (-\theta) = 0$$

$$\text{Ans: Given that, } |z + 4| \leq 3 \text{ For the greatest value of } |z + 1|.$$

$$\Rightarrow |z + 1| = |z + 4 - 3| \leq |z + 4| + |-3|$$

$$\Rightarrow |z + 4 - 3| \leq 3 + 3$$

$$\Rightarrow |z + 4 - 3| \leq 6$$

So, greatest value of $|z + 1| = 6$ For, now, least value of $|z + 1|$, we know that the least value of the modulus of a complex number is zero. So, the least value of $|z + 1|$ is zero.

$$\text{Ans: Given that, } \left| \frac{z - 2}{z + 2} \right| = \frac{\pi}{6}$$

$$\Rightarrow \frac{|x + iy - 2|}{|x + iy + 2|} = \frac{\pi}{6}$$

$$\Rightarrow \frac{|x - 2 + iy|}{|x + 2 + i|} = \frac{\pi}{6}$$

$$\Rightarrow 6|x - 2 + iy| = \pi|x + 2 + iy|$$

$$\Rightarrow 6\sqrt{(x - 2)^2 + y^2} = \pi\sqrt{(x + 2)^2 + y^2}$$

$$\Rightarrow 36[x^2 + 4 - 4x + y^2] = \pi^2[x^2 + 4x + 4 + y^2]$$

$$\Rightarrow (36 - \pi^2)x^2 + (36 - \pi^2)y^2 - (144 + 4\pi^2)x + 144 + 4\pi^2 = 0, \text{ which is a circle.}$$

$$\text{Ans: Given that, } |z| = 4 \text{ and } \arg(z) = \frac{5\pi}{6}$$

Let,

$$z = x + iy = r(\cos \theta + i \sin \theta)$$

$$|z| = r = 4 \text{ and } \arg(z) = \theta$$

$$\therefore \tan \theta = \frac{5\pi}{6}$$

$$\Rightarrow z = 4 \left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right)$$

$$\Rightarrow 4 \left[\cos \left(\pi - \frac{\pi}{6} \right) + i \sin \left(\pi - \frac{\pi}{6} \right) \right]$$

$$\Rightarrow 4 \left[-\cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right]$$

$$\Rightarrow 4 \left[\frac{-\sqrt{3}}{2} + i \frac{1}{2} \right] = -2\sqrt{3} + 2i$$

Question 26

State true or false for the following.

- (i) The order relation is defined on the set of complex numbers.
- (ii) Multiplication of a non-zero complex number by $-i$ rotates the point about origin through a right angle in the anti-clockwise direction.
- (iii) For any complex number z , the minimum value of $|z| + |z - 1|$ is 1.
- (iv) The locus represented by $|z - 1| = |z - i|$ is a line perpendicular to the join of the points $(1, 0)$ and $(0, 1)$.
- (v) If z is a complex number such that $\operatorname{Re}(z) = 0$, then, $\operatorname{Im}(z^2) = 0$
- (vi) The inequality $|z - 4| < |z - 2|$ represents the region given by $x > 3$.
- (vii) Let z_1 and z_2 be two complex numbers such that $|z_1 + z_2| = |z_1| + |z_2|$, then $\arg(z_1 - z_2) = 0$
- (viii) 2 is not a complex number.

Solution

Ans: False

We can compare two complex numbers when they are purely real. Otherwise comparison of complex number is not possible.

Ans: False

Ans: True

Let, $z = x + iy$

$$\Rightarrow |z| + |z - 1| = \sqrt{x^2 + y^2} + \sqrt{(x - 1)^2 + y^2}$$

If $x = 0, y = 0$, then the value of $|z| + |z - 1| = 1$.

Ans: True

Let, $z = x + iy$

$$|z - 1| = |z - i|$$

Then,

$$\Rightarrow |x - 1 + iy| = |x - i(1 - y)|$$

$$\Rightarrow (x - 1)^2 + y^2 = x^2 + (1 - y)^2$$

$$\Rightarrow x^2 - 2x + 1 + y^2 = x^2 + 1 + y^2 - 2y$$

$$\Rightarrow -2x + 1 = 1 - 2y$$

$$\Rightarrow -2x + 2y = 0$$

$$\Rightarrow x - y = 0$$

Equation of a line through the points $(1, 0)$ and $(0, 1)$, ...(i)

$$y - 0 = \frac{1 - 0}{0 - 1}(x - 1)$$

$$\Rightarrow y = -(x - 1) \Rightarrow x + y = 1 \quad \text{.....(ii)}$$

which is perpendicular to the line $x - y = 0$.

Ans: False

Let $z = x + iy, z \neq 0$ and $\operatorname{Re}(z) = 0$

i.e., $x = 0$

$\therefore z = iy$

$\operatorname{Im}(z^2) = i^2 y^2 = -y^2$ which is real.

Ans: True

Given inequality, $|z - 4| < |z - 2|$

Let $z = x + iy$

$$\therefore |x - 4 + iy| < |x - 2 + iy|$$

$$\Rightarrow \sqrt{(x - 4)^2 + y^2} < \sqrt{(x - 2)^2 + y^2}$$

$$\Rightarrow (x - 4)^2 + y^2 < (x - 2)^2 + y^2$$

$$\Rightarrow x^2 - 8x + 16 + y^2 < x^2 - 4x + 4 + y^2$$

$$\Rightarrow -8x + 16 < -4x + 4$$

$$\Rightarrow -8x < -4x - 12$$

$$\Rightarrow -4x < -12$$

$$\Rightarrow 4x > 12$$

$$\Rightarrow x > 3$$

Ans: False

Let, $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

Given that,

$$|z_1 + z_2| = |z_1| + |z_2|$$

$$\Rightarrow |x_1 + iy_1 + x_2 + iy_2| = |x_1 + iy_1| + |x_2 + iy_2|$$

$$\Rightarrow \sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2} = \sqrt{(x_1^2 + y_1^2)} + \sqrt{(x_2^2 + y_2^2)}$$

On squaring both sides, we get

$$(x_1 + x_2)^2 + (y_1 + y_2)^2 = (x_1^2 + y_1^2) + (x_2^2 + y_2^2) + 2\sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}$$

$$\Rightarrow x_1^2 + x_2^2 + 2x_1x_2 + y_1^2 + y_2^2 + 2y_1y_2 = x_1^2 + y_1^2 + x_2^2 + y_2^2 + 2\sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}$$

$$\Rightarrow 2x_1x_2 + 2y_1y_2 = 2\sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}$$

$$\Rightarrow x_1x_2 + y_1y_2 = \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}$$

On squaring both sides, we get

$$x_1^2 x_2^2 + y_1^2 y_2^2 + 2x_1 x_2 y_1 y_2 = x_1^2 x_2^2 + y_1^2 x_2^2 + x_1^2 y_2^2 + y_1^2 y_2^2$$

$$\Rightarrow (x_1 y_2 - x_2 y_1)^2 = 0$$

$$\Rightarrow x_1 y_2 = x_2 y_1$$

$$\Rightarrow \frac{y_1}{x_1} = \frac{y_2}{x_2}$$

$$\Rightarrow \left(\frac{y_1}{x_1} \right) - \left(\frac{y_2}{x_2} \right) = 0$$

$$\Rightarrow \arg(z_1) - \arg(z_2) = 0$$

Ans: True

We know that, 2 is a real number.

Since, 2 is not a complex number.

Question 27

Match the statements of Column A and Column B.

Column A	Column B
i. The polar form of $i + \sqrt{3}$	a. Perpendicular bisector of segment joining $(-2, 0)$ and $(2, 0)$.
ii. The amplitude $-1 + \sqrt{-3}$ is	b. On or outside the circle having centre at $(0, -4)$ and radius 3
iii. If $ z + 2 = z - 2 $, then locus of z is	c. $\frac{2\pi}{3}$
iv. If $ z + 2i = z - 2i $, then locus z is	d. Perpendicular bisector of segment joining $(0, -2)$ and $(0, 2)$
v. Region represented by	e. $2 \left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right)$

vi. Region represented by $ z + 4 \leq 3$ is	f. On or inside the circle having centre $(-4, 0)$ and radius 3 units.
vii. Conjugate of $\frac{1 + 2i}{1 - i}$ lies in	g. First quadrant
viii. Reciprocal of $1 - i$ lies in	h. Third quadrant

Solution

Ans: (i) Given,

$$z = i + \sqrt{3} = r(\cos \theta + \sin \theta)$$

$$\therefore r \cos \theta = \sqrt{3}, r \sin \theta = 1$$

$$\Rightarrow r^2 = 1 + 3 = 4 [\because r > 0]$$

$$\Rightarrow r = 2$$

$$\Rightarrow \tan \alpha = \left| \frac{r \sin \theta}{r \cos \theta} \right| = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \tan \alpha = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \alpha = \frac{\pi}{6}$$

$$\therefore x > 0, y > 0$$

$$\text{and } \arg(z) = \theta = \frac{\pi}{6}$$

$$\text{So the polar form of } z \text{ is } 2 \left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right).$$

(ii) Given that,

$$z = -1 + \sqrt{-3} = -1 + i\sqrt{3}$$

$$\therefore \tan \alpha = \left| \frac{\sqrt{3}}{-1} \right| = \sqrt{3}$$

$$\Rightarrow \tan \alpha = \tan \frac{\pi}{3}$$

$$\Rightarrow \alpha = \frac{\pi}{3}$$

$$\because x < 0, y > 0$$

$$\Rightarrow \theta = \pi - \alpha$$

$$= \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

(iii) Given that,

$$|z + 2| = |z - 2|$$

$$\Rightarrow |x + 2 + iy| = |x - 2 + iy|$$

$$\Rightarrow (x + 2)^2 + y^2 = (x - 2)^2 + y^2$$

$$\Rightarrow x^2 + 4x + 4 = x^2 - 4x + 4$$

$$\Rightarrow 8x = 0$$

$$\therefore x = 0$$

It is a straight line which is a perpendicular bisector of segment joining the points $(-2, 0)$ and $(2, 0)$.

(iv) Given that,

$$|z + 2i| = |z - 2i|$$

$$\Rightarrow |x + i(y + 2)| = |x + i(y - 2)|$$

$$\Rightarrow x^2 + (y + 2)^2 = x^2 + (y - 2)^2$$

$$\Rightarrow 4y = 0$$

$$\Rightarrow y = 0$$

It is a straight line, which is a perpendicular bisector of segment joining $(0, -2)$ and $(0, 2)$.

(v) Given that,

$$|z + 4i| \geq 3 = |x + iy + 4i| \geq 3$$

$$\Rightarrow |x + i(y + 4)| \geq 3$$

$$\Rightarrow \sqrt{x^2 + (y + 4)^2} \geq 3$$

$$\Rightarrow x^2 + (y + 4)^2 \geq 9$$

$$\Rightarrow x^2 + y^2 + 8y + 16 \geq 9$$

$$\Rightarrow x^2 + y^2 + 8y + 7 \geq 0$$

Which represents a circle. On or outside having center $(0, -4)$ and radius 3.

(vi) Given that,

$$|z + 4| \leq 3$$

$$\Rightarrow |x + iy + 4| \leq 3$$

$$\Rightarrow |x + 4 + iy| \leq 3$$

$$\Rightarrow \sqrt{(x + 4)^2 + y^2} \leq 3$$

$$\Rightarrow (x + 4)^2 + y^2 \leq 9$$

$$\Rightarrow x^2 + 8x + 16 + y^2 \leq 9$$

$$\Rightarrow x^2 + 8x + y^2 + 7 \leq 0$$

It represents the region which is on or inside the circle having the center $(-4, 0)$ and radius 3.

(vii) Given that,

$$z = \frac{1 + 2i}{1 - i} = \frac{(1 + 2i)(1 + i)}{(1 - i)(1 + i)}$$

$$\Rightarrow \frac{1 + 2i + i + 2i^2}{1 - i^2} = \frac{1 - 2 + 3i}{1 + 1}$$

$$\Rightarrow \frac{-1 + 3i}{2}$$

$$\therefore \bar{z} = \frac{-1}{2} - \frac{3i}{2}$$

Hence, $\left(\frac{-1}{2}, \frac{-3}{2}\right)$ lies in the third quadrant.

(viii) Given that,

$$z = 1 - i$$

$$\therefore \frac{1}{z} = \frac{1}{1 - i}$$

$$\Rightarrow \frac{1+i}{(1-i)(1+i)} = \frac{1+i}{1-i^2}$$

$$\Rightarrow \frac{1}{2}(1+i)$$

Hence, $\left(\frac{1}{2}, \frac{1}{2}\right)$ lies in first quadrant.

Column A	Column B
i.	c.
ii.	e.
iii.	b.
iv.	d.
v.	a.
vi.	f.
vii.	h.
viii.	g.

Question 28

What is the conjugate of $\frac{2-i}{(1-2i)^2}$?

Solution

Ans: Given, $\frac{2-i}{(1-2i)^2}$

If $z = x + iy$ and $\bar{z} = x - iy$.

Given that,

$$\begin{aligned}z &= \frac{2-i}{(1-2i)^2} \\ \Rightarrow \frac{2-i}{1+4i^2-4i} &= \frac{2-i}{1-4-4i} \\ \Rightarrow \frac{2-i}{-3-4i} &= \frac{(2-i)}{-(3+4i)} \\ \Rightarrow -\left[\frac{(2-i)(3-4i)}{(3+4i)(3-4i)}\right] &= -\left(\frac{6-8i-3i+4i^2}{9+16}\right) \\ \Rightarrow -\frac{(-11i+2)}{25} &= \frac{-1}{25}(2-11i) \\ \Rightarrow z &= \frac{1}{25}(-2+11i) \\ \Rightarrow \bar{z} &= \frac{1}{25}(-2-11i) = \frac{-2}{25} - \frac{11}{25}i\end{aligned}$$

Question 29

If $|z_1| = |z_2|$, is it necessary that $z_1 = z_2$.

Solution

Ans: Given, $|z_1| = |z_2|$

If $z = x + iy$ then $|z| = \sqrt{x^2 + y^2}$.

Given that,

$$|z_1| = |z_2|$$

Let,

$$z_1 = x_1 + iy_1 \text{ and } z_2 = x_2 + iy_2$$

$$\Rightarrow |x_1 + iy_1| = |x_2 + iy_2|$$

$$\Rightarrow x_1^2 + y_1^2 = x_2^2 + y_2^2$$

$$\Rightarrow x_1^2 = x_2^2, y_1^2 = y_2^2$$

$$\Rightarrow x_1 = \pm x_2, y_1 = \pm y_2$$

$$\Rightarrow z_1 = x_1 + iy_1 \text{ or } z_1 = \pm x_2 \pm iy_2$$

Hence, it is not necessary that $z_1 = z_2$.

Question 30

If $\frac{(a^2 + 1)^2}{2a - i} = x + iy$, then what is the value of $x^2 + y^2$?

Solution

Ans: Given, $\frac{(a^2 + 1)^2}{2a - i} = x + iy$

Rationalize the given expression and compare the real and imaginary parts.

Given that,

$$\frac{(a^2 + 1)^2}{2a - i} = x + iy$$

$$\Rightarrow \frac{(a^2 + 1)^2}{2a - i} \times \frac{2a + i}{2a + i} = x + iy$$

$$\Rightarrow \frac{(a^2 + 1)^2 (2a + i)}{4a^2 + 1} = x + iy$$

$$\Rightarrow \frac{2a(a^2 + 1)^2 + i(a^2 + 1)^2}{4a^2 + 1} = x + iy$$

On comparing the real and imaginary parts, we get

$$\Rightarrow x = \frac{2a(a^2 + 1)^2}{4a^2 + 1} \text{ and } y = \frac{(a^2 + 1)^2}{4a^2 + 1}$$

$$\therefore x^2 + y^2 = 4a^2 \left[\frac{(a^2 + 1)^2}{4a^2 + 1} \right]^2 + \left[\frac{(a^2 + 1)^2}{4a^2 + 1} \right]^2$$

$$= \frac{(4a^2 + 1)(a^2 + 1)^4}{(4a^2 + 1)^2} = \frac{(a^2 + 1)^4}{(4a^2 + 1)}$$

Question 31

Find the value of z , if $|z| = 4$ and $\arg(z) = \frac{5\pi}{6}$.

Solution

Ans: Given, $|z| = 4$ and $\arg(z) = \frac{5\pi}{6}$

If $z = x + iy$ then it's polar form is $z = r(\cos \theta + i \sin \theta)$ where $\arg(z) = \theta$.

Let,

$$z = r(\cos \theta + i \sin \theta)$$

also $|z| = r = 4$ and $\theta = \frac{5\pi}{6} [\because \arg(z) = \theta]$

$$\therefore z = 4 \left[\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right] [\because z = r(\cos \theta + i \sin \theta)]$$

$$\Rightarrow 4 \left[\cos \left(\pi - \frac{\pi}{6} \right) + i \sin \left(\pi - \frac{\pi}{6} \right) \right]$$

$$\Rightarrow 4 \left[-\cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right]$$

$$\Rightarrow 4 \left[-\frac{\sqrt{3}}{2} + i \frac{1}{2} \right] = -2\sqrt{3} + 2i$$

Question 32

Find the value of $\left| (1+i) \frac{(2+i)}{(3+i)} \right|$

Solution

Ans: Given, $\left| (1+i) \frac{(2+i)}{(3+i)} \right|$

First, convert the given expression in the formula $a + ib$, then use $|a + ib| = \sqrt{a^2 + b^2}$.

Given that,

$$\begin{aligned} & \left| (1+i) \frac{(2+i)}{(3+i)} \right| \\ \Rightarrow & \left| \frac{(2+i+2i+i^2)}{(3+i)} \right| = \left| \frac{2+3i-1}{3+i} \right| \\ \Rightarrow & \left| \frac{1+3i}{3+i} \right| = \left| \frac{(1+3i)(3-i)}{(3+i)(3-i)} \right| \\ \Rightarrow & \left| \frac{3+9i-i-3i^2}{9-i^2} \right| = \left| \frac{3+8i+3}{9+1} \right| \\ \Rightarrow & \left| \frac{6+8i}{10} \right| = \sqrt{\frac{6^2}{100} + \frac{8^2}{100}} \\ \Rightarrow & \sqrt{\frac{36+64}{100}} = \sqrt{\frac{100}{100}} = 1 \end{aligned}$$

Question 33

Find the principal argument of $(1 + i\sqrt{3})^2$

Solution

Ans: Given, $(1 + i\sqrt{3})^2$

Let $z = a + ib$, then the polar form of z is $r(\cos \theta + i \sin \theta)$, where $r = |z| = \sqrt{a^2 + b^2}$ and $\tan \theta = \frac{b}{a}$. Here, θ is argument or amplitude of z i.e., $\arg(z) = \theta$. The principal argument is a unique value of θ such that $-\pi \leq \theta \leq \pi$.

Given that,

$$z = (1 + i\sqrt{3})^2$$

$$\Rightarrow z = 1 - 3 + 2i\sqrt{3} = -2 + i2\sqrt{3}$$

$$\Rightarrow \tan \alpha = \left| \frac{2\sqrt{3}}{-2} \right| = |-\sqrt{3}| = \sqrt{3} \left[\because \tan \alpha = \left| \frac{\text{Im}(z)}{\text{Re}(z)} \right| \right]$$

$$\Rightarrow \tan \alpha = \tan \frac{\pi}{3}$$

$$\Rightarrow \alpha = \frac{\pi}{3}$$

$$\because \text{Re}(z) < 0 \text{ and } \text{Im}(z) > 0$$

$$\Rightarrow \arg(z) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

Question 34

Where does z lie, if $\left| \frac{z - 5i}{z + 5i} \right| = 1$.

Solution

Ans: Given, $\left| \frac{z - 5i}{z + 5i} \right| = 1$

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ then $|z_1| = \sqrt{x_1^2 + y_1^2}$ and $|z_2| = \sqrt{x_2^2 + y_2^2}$.

Also, use the modulus property i.e., $\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}$.

Let $z = x + iy$

Given that, $\left| \frac{z - 5i}{z + 5i} \right| = \left| \frac{x + iy - 5i}{x + iy + 5i} \right|$

$$\Rightarrow \left| \frac{z - 5i}{z + 5i} \right| = \frac{|x + i(y - 5)|}{|x + i(y + 5)|} \left[\because \left| \frac{z - 5i}{z + 5i} \right| = 1 \right]$$

$$\Rightarrow \left| \frac{z - 5i}{z + 5i} \right| = \frac{\sqrt{x^2 + (y - 5)^2}}{\sqrt{x^2 + (y + 5)^2}}$$

On squaring both sides, we get

$$x^2 + (y - 5)^2 = x^2 + (y + 5)^2$$

$$\Rightarrow -10y = +10y$$

$$\Rightarrow 20y = 0$$

$$\therefore y = 0$$

So, z lies on real axis.

Choose the correct answer from the given four options indicated against each of the Exercises from 35 to 50 (M.C.Q)

Question 35

$z = \sin x + i \cos 2x$ and $\cos x - i \sin 2x$ are conjugate to each other for

(A) $(x = n\pi)$

(B) $x = \left(n + \frac{1}{2}\right) \frac{\pi}{2}$

(C) $x = 0$

(D) No value of x

Solution

Ans: Option C is correct answer

$z = \sin x + i \cos 2x$ and $\cos x - i \sin 2x$ are conjugate to each other.

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ then $|z_1| = \sqrt{x_1^2 + y_1^2}$ and $|z_2| = \sqrt{x_2^2 + y_2^2}$.

Also, use the modulus property i.e., $\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}$.

Let $z = \sin x + i \cos 2x$ (i)

and $\bar{z} = \sin x - i \cos 2x$ (ii)

Given that,

$$\bar{z} = \cos x - i \sin 2x$$

$$\therefore \sin x - i \cos 2x = \cos x - i \sin 2x$$

$$\Rightarrow \sin x = \cos x \text{ and } \cos 2x = \sin 2x$$

$$\Rightarrow \tan x = 1 \text{ and } \tan 2x = 1$$

$$\Rightarrow \tan x = \tan \frac{\pi}{4} \text{ and } \tan 2x = \tan \frac{\pi}{4}$$

$$\Rightarrow x = n\pi + \frac{\pi}{4} \text{ and } 2x = n\pi + \frac{\pi}{4}$$

$$\Rightarrow 2x - x = 0$$

$$\Rightarrow x = 0$$

Question 36

The real value of α for which the expression $\frac{1 - i \sin \alpha}{1 + 2i \sin \alpha}$ is purely real is

(A) $(n + 1) \frac{\pi}{2}$

(B) $(2n + 1) \frac{\pi}{2}$

(C) $n\pi$

(D) None of these

Solution

Ans: Option C is correct answer

$$\frac{1 - i \sin \alpha}{1 + 2i \sin \alpha}$$

First, convert the given expansion into $a + ib$ form and then check whether the complex number $a + ib$ is purely real.

Given expression,

$$z = \frac{1 - i \sin \alpha}{1 + 2i \sin \alpha}$$

$$\begin{aligned}
&= \frac{(1 - i \sin \alpha)(1 - 2i \sin \alpha)}{(1 + 2i \sin \alpha)(1 - 2i \sin \alpha)} \\
&\Rightarrow \frac{1 - i \sin \alpha - 2i \sin \alpha + 2i^2 \sin^2 \alpha}{1 - 4i^2 \sin^2 \alpha} \\
&\Rightarrow \frac{1 - 3i \sin \alpha - 2\sin^2 \alpha}{1 + 4\sin^2 \alpha} \\
&\Rightarrow \frac{1 - 2\sin^2 \alpha}{1 + 4\sin^2 \alpha} - \frac{3i \sin \alpha}{1 + 4\sin^2 \alpha}
\end{aligned}$$

It is given that z is a purely real.

$$\therefore \frac{-3 \sin \alpha}{1 + 4\sin^2 \alpha} = 0$$

$$\Rightarrow -3 \sin \alpha = 0$$

$$\Rightarrow \sin \alpha = 0$$

$$\alpha = n\pi$$

Question 37

If $z = x + iy$ lies in the third quadrant, then $\frac{\bar{z}}{z}$ also lies in the third quadrant, if

(A) $x > y > 0$

(B) $x < y < 0$

(C) $y < x < 0$

(D) $y > x > 0$

Solution

Ans: Option B is correct answer

Given, $z = x + iy$ lies in third quadrant. $x < 0$ and $y < 0$.

If $z = x + iy$ then $\bar{z} = x - iy$.

Given that, $z = x + iy$ lies in third quadrant. $x < 0$ and $y < 0$.

Now,

$$\frac{\bar{z}}{z} = \frac{x - iy}{x + iy}$$
$$\Rightarrow \frac{(x - iy)(x - iy)}{(x + iy)(x - iy)} = \frac{x^2 - y^2 - 2ixy}{x^2 + y^2}$$

and

$$\frac{\bar{z}}{z} = \frac{x^2 - y^2}{x^2 + y^2} - \frac{2ixy}{x^2 + y^2}$$

Since, $\frac{\bar{z}}{z}$ also lies in the third quadrant.

$$\therefore \frac{x^2 - y^2}{x^2 + y^2} < 0 \text{ and } \frac{-2xy}{x^2 + y^2} < 0$$

$$x^2 - y^2 < 0 \text{ and } -2xy < 0$$

$$x^2 < y^2 \text{ and } xy > 0$$

So,

$$x < y < 0$$

Question 38

The value of $(z + 3)(\bar{z} + 3)$ is equivalent to

(A) $|z + 3|^2$

(B) $|z - 3|$

(C) $z^2 + 3$

(D) None of these

Solution

Ans: Option A is correct answer

Given, $(z + 3)(\bar{z} + 3)$

If $z = x + iy$ then $\bar{z} = x - iy$.

Given that, $(z + 3)(\bar{z} + 3)$

Let

$$z = x + iy$$

$$\Rightarrow (z + 3)(\bar{z} + 3) = (x + iy + 3)(x + 3 - iy)$$

$$\Rightarrow (x + 3)^2 - (iy)^2 = (x + 3)^2 + y^2$$

$$\Rightarrow |x + 3 + iy|^2 = |z + 3|^2$$

Question 39

If $\left(\frac{1+i}{1-i}\right)^x = 1$, then

(A) $x = 2n + 1$

(B) $x = 4n$

(C) $x = 2n$

(D) $x = 4n + 1$

Solution

Ans: Option B is correct answer

$$\left(\frac{1+i}{1-i}\right)^x = 1$$

$$i^{4n} = 1, n \in N$$

Given that,

$$\left(\frac{1+i}{1-i}\right)^x = 1$$

$$\Rightarrow \left[\frac{(1+i)(1+i)}{(1-i)(1+i)}\right]^x = 1$$

$$\Rightarrow \left[\frac{1+2i+i^2}{1-i^2}\right]^x = 1$$

$$\Rightarrow \left[\frac{2i}{1+1} \right]^x = 1$$

$$\Rightarrow \left[\frac{2i}{2} \right]^x = 1$$

$$\Rightarrow i^x = 1$$

$$\Rightarrow i^x = (i^{4n}) [\because i^{4n} = 1, n \in N]$$

$$\Rightarrow x = 4n$$

Question 40

A real value of x satisfies the equation $\left(\frac{3-4ix}{3+4ix} \right) = \alpha - i\beta (\alpha, \beta \in R)$, if $\alpha^2 + \beta^2$ is equal to

(A)1

(B)-1

(C)2

(D)-2

Solution

Ans: Option A is correct answer

$$\left(\frac{3-4ix}{3+4ix} \right) = \alpha - i\beta (\alpha, \beta \in R)$$

If $z = x + iy$ then $\bar{z} = x - iy$.

$$(a+b)(a-b) = a^2 - b^2$$

Given equation,

$$\left(\frac{3-4ix}{3+4ix} \right) = \alpha - i\beta (\alpha, \beta \in R)$$

$$\Rightarrow \left[\frac{3-4ix}{3+4ix} \right] = \alpha - i\beta$$

$$\text{Now, } (\alpha - i\beta) = \frac{(3-4ix)(3-4ix)}{(3+4ix)(3-4ix)} = \frac{9 + 16i^2x^2 - 24ix}{9 - 16i^2x^2}$$

$$\Rightarrow \alpha - i\beta = \frac{9 - 16x^2 - 24ix}{9 + 16x^2}$$

$$\Rightarrow \alpha - i\beta = \frac{9 - 16x^2}{9 + 16x^2} - \frac{i24x}{9 + 16x^2} \quad \dots(i)$$

$$\therefore \alpha + i\beta = \frac{9 - 16x^2}{9 + 16x^2} + \frac{i24x}{9 + 16x^2} \quad \dots(ii)$$

$$\text{So, } (\alpha - i\beta)(\alpha + i\beta) = \left(\frac{9 - 16x^2}{9 + 16x^2} \right)^2 - \left(\frac{i24x}{9 + 16x^2} \right)^2$$

$$\therefore \alpha^2 + \beta^2 = \frac{81 + 256x^4 - 288x^2 + 576x^2}{(9 + 16x^2)^2}$$

$$\Rightarrow \frac{81 + 256x^4 + 288x^2}{(9 + 16x^2)^2}$$

$$\Rightarrow \frac{(9 + 16x^2)^2}{(9 + 16x^2)^2} = 1$$

Question 41

Which of the following is correct for any two complex numbers z_1 and z_2 ?

(A) $|z_1 z_2| = |z_1| |z_2|$

(B) $|z_1 + z_2| = |z_1| + |z_2|$

(C) $\arg(z_1 z_2) = \arg(z_1) \cdot \arg(z_2)$

Solution

Ans: Option A is correct answer

complex numbers z_1 and z_2 .

Let $z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2 (\cos \theta_2 + i \sin \theta_2)$ are two complex numbers.

Let $z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$

$$\Rightarrow |z_1| = r_1 \dots (i)$$

and $z_2 = r_2 (\cos \theta_2 + i \sin \theta_2)$

$$\Rightarrow |z_2| = r_2 \dots (ii)$$

Now,

$$z_1 z_2 = r_1 r_2 [\cos \theta_1 \cos \theta_2 + i \sin \theta_1 \cos \theta_2 + i \cos \theta_1 \sin \theta_2 + i^2 \sin \theta_1 \sin \theta_2]$$

$$= r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]$$

$$\Rightarrow |z_1 z_2| = r_1 r_2$$

$$\therefore |z_1 z_2| = |z_1| |z_2| \text{ [Using equation (i) and (ii)]}$$

Question 42

The point represented by the complex number $(2 - i)$ is rotated about

origin through an angle $\frac{\pi}{2}$ in the clockwise direction, the new position of point is

(A) $1 + 2i$

(B) $-1 - 2i$

(C) $2 + i$

(D) $-1 + 2i$

Solution

Ans: Option B is correct answer

complex number $(2 - i)$

Here, $z = r e^{i\alpha}$ is a complex number, where modulus is r and argument $(\theta + \alpha)$. If $P(z)$ rotates in clockwise sense through an angle α , then its new position will be $z(\theta - i\alpha)$

Given that, $z = 2 - i$

It is rotated about origin through an angle $\frac{\pi}{2}$ in the clockwise direction.

$$\therefore \text{New position} = z e^{-i\pi/2}$$

$$\Rightarrow (2 - i) e^{-i\pi/2} = (2 - i) \left[\cos\left(\frac{-\pi}{2}\right) + i \sin\left(\frac{-\pi}{2}\right) \right]$$

$$\Rightarrow (2 - i)[0 - i] = -2i - 1 = -1 - 2i$$

Question 43

If $x, y \in R$, then $x + iy$ is a non-real complex number, if

(A) $x = 0$

(B) $y = 0$

(C) $y = 0$

(D) $y \neq 0$

Solution

Ans: Option D is correct answer

Given, $x, y \in R$

$x + iy$ is non-real complex number if and only if $y \neq 0$.

Given that,

$$x, y \in R$$

Then, $x + iy$ is non-real complex number if and only if $y \neq 0$.

Question 44

If $a + ib = c + id$, if

(A) $a^2 + c^2 = 0$

(B) $b^2 + c^2 = 0$

(C) $b^2 + d^2 = 0$

(D) $a^2 + b^2 = c^2 + d^2$

Solution

Ans: Option D is correct answer

$$a + ib = c + id$$

If two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are equal, then

$$|z_1| = |z_2|$$

$$\Rightarrow \sqrt{x_1^2 + y_1^2} = \sqrt{x_2^2 + y_2^2}$$

Given that, $a + ib = c + id$

$$\Rightarrow |a + ib| = |c + id|$$

$$\Rightarrow \sqrt{a^2 + b^2} = \sqrt{c^2 + d^2}$$

On squaring both sides, we get

$$a^2 + b^2 = c^2 + d^2$$

Question 45

The complex number z which satisfies the condition $\left| \frac{i + z}{i - z} \right| = 1$ lies on

(A) circle $x^2 + y^2 = 1$

(B) the x - axis

(C) the y - axis

(D) the line $x + y = 1$

Solution

Ans: Option B is correct answer

$$\left| \frac{i + z}{i - z} \right| = 1$$

Let z be a complex number and $z = x + iy$ then $|z| = \sqrt{x^2 + y^2}$.

$$\text{Given that, } \left| \frac{i + z}{i - z} \right| = 1$$

Let $z = x + iy$

$$\therefore \left| \frac{x + i(y + 1)}{-x - i(y - 1)} \right| = 1$$

$$\Rightarrow \frac{x^2 + (y+1)^2}{x^2 + (y-1)^2} = 1$$

$$\Rightarrow x^2 + (y+1)^2 = x^2 + (y-1)^2$$

$$\Rightarrow 4y = 0$$

$$\Rightarrow y = 0$$

So, z lies on x -axis (real axis).

Question 46

If z is a complex number, then

(A) $|z^2| > |z|$

(B) $|z^2| = |z|^2$

(C) $|z^2| < |z|^2$

(D) $|z^2| \geq |z|^2$

Solution

Ans: Option B is correct answer

z is a complex number.

If z is a complex number, then $z = x + iy$ $|z| = \sqrt{x^2 + y^2}$ and $|z|^2 = x^2 + y^2$

If z is a complex number, then $z = x + iy$ $|z| = \sqrt{x^2 + y^2}$ and $|z|^2 = x^2 + y^2$

$$\Rightarrow |z|^2 = x^2 + y^2 \dots (i)$$

$$\text{and } z^2 = (x + iy)^2 = x^2 + i^2 y^2 + i2xy$$

$$z^2 = x^2 - y^2 + i2xy$$

$$\Rightarrow |z^2| = \sqrt{(x^2 - y^2)^2 + (2xy)^2}$$

$$\Rightarrow |z^2| = \sqrt{x^4 + y^4 - 2x^2y^2 + 4x^2y^2}$$

$$\Rightarrow |z^2| = \sqrt{x^4 + y^4 + 2x^2y^2} = \sqrt{(x^2 + y^2)^2}$$

$$\Rightarrow |z^2| = x^2 + y^2 \dots (ii)$$

From Eqs. (i) and (ii), $|z|^2 = |z^2|$

Question 47

$|z_1 + z_2| = |z_1| + |z_2|$ is possible, if

(A) $z_2 = \bar{z}_1$

(B) $z_2 = \frac{1}{z_1}$

(C) $\arg(z_1) = \arg(z_2)$

(D) $|z_1| = |z_2|$

Solution

Ans: Option C is correct answer

$$|z_1 + z_2| = |z_1| + |z_2|$$

Let z be a complex number and $z = x + iy$ then $|z| = \sqrt{x^2 + y^2}$.

Given that, $|z_1 + z_2| = |z_1| + |z_2|$

$$\Rightarrow |r_1(\cos \theta_1 + i \sin \theta_1) + r_2(\cos \theta_2 + i \sin \theta_2)| = |r_1(\cos \theta_1 + i \sin \theta_1)| + |r_2(\cos \theta_2 + i \sin \theta_2)|$$

$$\Rightarrow |(r_1 \cos \theta_1 + r_2 \cos \theta) + i(r_1 \sin \theta_1 + r_2 \sin \theta_2)| = r_1 + r_2$$

$$\Rightarrow \sqrt{r_1^2 \cos^2 \theta_1 + r_2^2 \cos^2 \theta_2 + 2r_1 r_2 \cos \theta_1 \cos \theta_2 + r_1^2 \sin^2 \theta_1 + r_2^2 \sin^2 \theta_2}$$

$$\sqrt{+2r_1 r_2 \sin \theta_1 \sin \theta_2} = r_1 + r_2$$

$$\Rightarrow \sqrt{r_1^2 + r_2^2 + 2r_1 r_2 [\cos(\theta_1 - \theta_2)]} = r_1 + r_2$$

On squaring both sides, we get

$$r_1^2 + r_2^2 + 2r_1 r_2 \cos(\theta_1 - \theta_2) = r_1^2 + r_2^2 + 2r_1 r_2$$

$$\Rightarrow 2r_1 r_2 [1 - \cos(\theta_1 - \theta_2)] = 0$$

$$\Rightarrow 1 - \cos(\theta_1 - \theta_2) = 0$$

$$\Rightarrow \cos(\theta_1 - \theta_2) = 1$$

$$\Rightarrow \cos(\theta_1 - \theta_2) = \cos 0^\circ$$

$$\Rightarrow \theta_1 - \theta_2 = 0^\circ$$

$$\Rightarrow \theta_1 = \theta_2$$

$$\therefore \arg(z_1) = \arg(z_2)$$

Question 48

The real value of θ for which the expression $\frac{1 + i\cos\theta}{1 - 2i\cos\theta}$ is a real number is

(A) $n\pi + \frac{\pi}{4}$

(B) $n\pi + (-1)^n \frac{\pi}{4}$

(C) $2n\pi \pm \frac{\pi}{2}$

(D) None of these

Solution

Ans: Option C is correct answer

$$\frac{1 + i\cos\theta}{1 - 2i\cos\theta}$$

Simplify the given equation and use the formula $\cos \frac{\pi}{2} = 0$.

Given expression,

$$= \frac{1 + i\cos\theta}{1 - 2i\cos\theta} = \frac{(1 + i\cos\theta)(1 + 2i\cos\theta)}{(1 - 2i\cos\theta)(1 + 2i\cos\theta)}$$

$$\Rightarrow \frac{1 + i\cos\theta + 2i\cos\theta + 2i^2\cos^2\theta}{1 - 4i^2\cos^2\theta}$$

$$\Rightarrow \frac{1 + 3i\cos\theta - 2\cos^2\theta}{1 + 4\cos^2\theta}$$

For real value of θ ,

$$\frac{3 \cos \theta}{1 + 4 \cos^2 \theta} = 0$$

$$\Rightarrow 3 \cos \theta = 0$$

$$\Rightarrow \cos \theta = \cos \frac{\pi}{2}$$

$$\Rightarrow \theta = 2n\pi \pm \frac{\pi}{2}$$

Question 49

The value of $\arg(x)$, when $x < 0$ is

(A) 0

(B) $\frac{\pi}{2}$

(C) π

(D) None of these

Solution

Ans: Option C is correct answer

$$x < 0$$

Let z be a complex number and $z = x + iy$ then $|z| = \sqrt{x^2 + y^2}$.

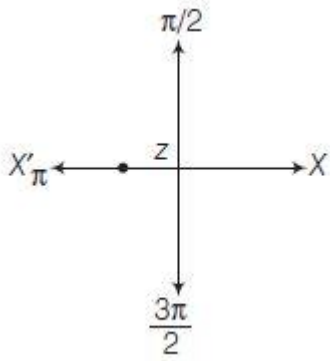
Let

$$z = x + 0i \text{ and } x < 0$$

$$|z| = \sqrt{(-1)^2 + (0^2)} = 1$$

Since, the point $(x, 0)$ represent $z = x + 0i$ lies on the negative side of real axis.

$$\therefore \text{Principal } \arg(z) = \pi$$



Question 50

If $f(z) = \frac{7-z}{1-z^2}$, where $z = 1 + 2i$, then $|f(z)|$ is equal to

- (A) $\frac{|z|}{2}$
- (B) $|z|$
- (C) $2|z|$
- (D) None of these

Solution

Ans: Option A is correct answer

$$f(z) = \frac{7-z}{1-z^2}, \text{ where } z = 1 + 2i$$

Let z be a complex number and $z = x + iy$ then $|z| = \sqrt{x^2 + y^2}$.

$$\text{Let } z = 1 + 2i$$

$$\Rightarrow |z| = \sqrt{1+4} = \sqrt{5}$$

Now,

$$\begin{aligned} f(z) &= \frac{7-z}{1-z^2} = \frac{7-1-2i}{1-(1+2i)^2} \\ &\Rightarrow \frac{6-2i}{1-1-4i^2-4i} = \frac{6-2i}{4-4i} \end{aligned}$$

$$\Rightarrow \frac{(3-i)(2+2i)}{(2-2i)(2+2i)} = \frac{6-2i+6i-2i^2}{4-4i^2}$$

$$\Rightarrow \frac{6+4i+2}{4+4} = \frac{8+4i}{8}$$

$$\Rightarrow 1 + \frac{1}{2}i$$

$$f(z) = 1 + \frac{1}{2}i$$

$$\therefore |f(z)| = \sqrt{1 + \frac{1}{4}}$$

$$\Rightarrow \sqrt{\frac{4+1}{4}} = \frac{\sqrt{5}}{2} = \frac{|z|}{2}$$

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