

Chapter 3: Trigonometric Functions

NCERT Exemplar · Class 11 · Maths

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Question 1

Prove that

$$\frac{\tan A + \sec A - 1}{\tan A - \sec A + 1} = \frac{1 + \sin A}{\cos A}$$

Solution

According to the question,

$$\begin{aligned} \text{LHS} &= \frac{\tan A + \sec A - 1}{\tan A - \sec A + 1} \\ &= \frac{\frac{\sin A}{\cos A} + \frac{1}{\cos A} - 1}{\frac{\sin A}{\cos A} - \frac{1}{\cos A} + 1} \\ &= \frac{\frac{\sin A + 1 - \cos A}{\cos A}}{\frac{\sin A - 1 + \cos A}{\cos A}} \\ &= \frac{\sin A + 1 - \cos A}{\sin A - 1 + \cos A} \end{aligned}$$

Using the identity,

$\sin^2 A + \cos^2 A = 1$, we get,

$\sin A + (1 - \cos A)$.

$$\begin{aligned}
\therefore \text{LHS} &= \frac{\sin A + (1 - \cos A)}{\sin A - (1 - \cos A)} \times \frac{\sin A + (1 - \cos A)}{\sin A + (1 - \cos A)} \\
&= \frac{\{\sin A + (1 - \cos A)\}^2}{\sin^2 A - (1 - \cos A)^2} \\
&= \frac{\sin^2 A + (1 - \cos A)^2 + 2 \sin A (1 - \cos A)}{\sin^2 A - (1 - \cos A)^2} \\
&= \frac{(\sin^2 A + \cos^2 A) + 1 - 2 \cos A + 2 \sin A (1 - \cos A)}{\sin^2 A - \{1 + \cos^2 A - 2 \cos A\}} \\
&= \frac{(1) + 1 - 2 \cos A + 2 \sin A (1 - \cos A)}{(\sin^2 A - 1) - \cos^2 A + 2 \cos A} \\
&= \frac{2(1 - \cos A) + 2 \sin A (1 - \cos A)}{(-\cos^2 A) - \cos^2 A + 2 \cos A} \\
&= \frac{2(1 + \sin A)(1 - \cos A)}{-2 \cos^2 A + 2 \cos A} \\
&= \frac{2 \cos A (1 - \cos A)}{(1 + \sin A)} = \text{RHS}
\end{aligned}$$

Hence, L.H.S = R.H.S

Question 2

If $[2\sin\alpha / (1+\cos\alpha+\sin\alpha)] = y$, then prove that $[(1-\cos\alpha+\sin\alpha) / (1+\sin\alpha)]$ is also equal to y .

$$\left[\text{Hint : Express } \frac{1 - \cos \alpha + \sin \alpha}{1 + \sin \alpha} = \frac{1 - \cos \alpha + \sin \alpha}{1 + \sin \alpha} \cdot \frac{1 + \cos \alpha + \sin \alpha}{1 + \cos \alpha + \sin \alpha} \right]$$

Solution

According to the question,

$$y = 2\sin\alpha / (1+\cos\alpha+\sin\alpha)$$

Multiplying numerator and denominator by $(1 - \cos \alpha + \sin \alpha)$,

We get,

$$\Rightarrow y = \frac{2 \sin \alpha}{1 + \cos \alpha + \sin \alpha} \times \frac{1 - \cos \alpha + \sin \alpha}{1 - \cos \alpha + \sin \alpha}$$

$$= \frac{2 \sin \alpha}{(1 + \sin \alpha) + \cos \alpha} \times \frac{(1 + \sin \alpha) - \cos \alpha}{(1 + \sin \alpha) - \cos \alpha}$$

Using $(a + b)(a - b) = a^2 - b^2$, we get:

$$= \frac{2 \sin \alpha \{(1 + \sin \alpha) - \cos \alpha\}}{(1 + \sin \alpha)^2 - \cos^2 \alpha}$$

$$= \frac{2 \sin \alpha (1 + \sin \alpha) - 2 \sin \alpha \cos \alpha}{1 + \sin^2 \alpha + 2 \sin \alpha - \cos^2 \alpha}$$

Since, $1 - \cos^2 \alpha = \sin^2 \alpha$

$$\therefore y = \frac{2 \sin \alpha (1 + \sin \alpha - \cos \alpha)}{\sin^2 \alpha + 2 \sin \alpha + \sin^2 \alpha}$$

$$= \frac{2 \sin \alpha (1 + \sin \alpha - \cos \alpha)}{2 \sin \alpha (1 + \sin \alpha)}$$

$$\Rightarrow y = \frac{(1 + \sin \alpha - \cos \alpha)}{(1 + \sin \alpha)}$$

$$\Rightarrow y = \frac{(1 - \cos \alpha + \sin \alpha)}{(1 + \sin \alpha)}$$

Hence Proved

Question 3

If $m \sin \theta = n \sin (\theta + 2\alpha)$, then prove that

$$\tan (\theta + \alpha) \cot \alpha = (m + n)/(m - n)$$

[Hints: Express $\sin(\theta + 2\alpha) / \sin \theta = m/n$ and apply componendo and dividend]

Solution

According to the question,

$$m \sin \theta = n \sin (\theta + 2\alpha)$$

To prove:

$$\tan (\theta + \alpha) \cot \alpha = (m + n)/(m - n)$$

Proof:

$$m \sin \theta = n \sin (\theta + 2\alpha)$$

$$\Rightarrow \sin(\theta + 2\alpha) / \sin \theta = m/n$$

Applying componendo-dividendo rule, we have,

$$\Rightarrow \frac{\sin(\theta+2\alpha)+\sin \theta}{\sin(\theta+2\alpha)-\sin \theta} = \frac{m+n}{m-n}$$

By transformation formula of T-ratios,

We know that,

$$\sin A + \sin B = 2 \sin ((A+B)/2) \cos ((A - B)/2)$$

And,

$$\sin A - \sin B = 2 \cos ((A+B)/2) \sin ((A - B)/2)$$

On applying the formula, we get,

$$\begin{aligned} \frac{2 \sin \left(\frac{2\theta + 2\alpha}{2} \right) \cos \left(\frac{\theta + 2\alpha - \theta}{2} \right)}{2 \cos \left(\frac{2\theta + 2\alpha}{2} \right) \sin \left(\frac{\theta + 2\alpha - \theta}{2} \right)} &= \frac{m+n}{m-n} \\ \frac{\sin(\theta+\alpha) \cos(\alpha)}{\cos(\theta+\alpha) \sin(\alpha)} &= \frac{m+n}{m-n} \\ \{\because \tan x = (\sin x)/(\cos x)\} \\ \Rightarrow \tan(\theta + \alpha) \cot \alpha &= \frac{m+n}{m-n} \end{aligned}$$

Therefore, $\tan(\theta + \alpha) \cot \alpha = (m + n)/(m - n)$

Hence Proved

Question 4

If $\cos(\alpha + \beta) = \frac{4}{5}$ and $\sin(\alpha - \beta) = \frac{5}{13}$, where α lie between 0 and $\pi/4$, find value of $\tan 2\alpha$
 [Hint: Express $\tan 2\alpha$ as $\tan(\alpha + \beta + \alpha - \beta)$]

Solution

According to the question,

$$\cos(\alpha + \beta) = 4/5 \dots(i)$$

We know that,

$$\sin x = \sqrt{1 - \cos^2 x}$$

Therefore,

$$\sin(\alpha + \beta) = \sqrt{1 - \cos^2(\alpha + \beta)}$$

$$\Rightarrow \sin(\alpha + \beta) = \sqrt{1 - (4/5)^2} = 3/5 \dots(ii)$$

Also,

$$\sin(\alpha - \beta) = 5/13 \text{ \{given\} } \dots(iii)$$

we know that,

$$\cos x = \sqrt{1 - \sin^2 x}$$

Therefore,

$$\cos(\alpha - \beta) = \sqrt{1 - \sin^2(\alpha - \beta)}$$

$$\Rightarrow \cos(\alpha - \beta) = \sqrt{1 - (5/13)^2} = 12/13 \dots(iv)$$

Therefore,

$$\tan 2\alpha = \tan(\alpha + \beta + \alpha - \beta)$$

We know that,

$$\begin{aligned} \tan(x + y) &= \frac{\tan x + \tan y}{1 - \tan x \tan y} \\ \therefore \tan 2\alpha &= \frac{\tan(\alpha + \beta) + \tan(\alpha - \beta)}{1 - \tan(\alpha + \beta) \tan(\alpha - \beta)} \\ &= \frac{\frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} + \frac{\sin(\alpha - \beta)}{\cos(\alpha - \beta)}}{1 - \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} \times \frac{\sin(\alpha - \beta)}{\cos(\alpha - \beta)}} \end{aligned}$$

From equation i, ii, iii and iv we have,

$$\begin{aligned}
 \Rightarrow \tan 2\alpha &= \frac{\frac{\frac{3}{5} + \frac{5}{13}}{\frac{3}{5} \times \frac{5}{13}}}{1 - \frac{\frac{3}{5} + \frac{5}{13}}{\frac{3}{5} \times \frac{5}{13}}} \\
 &= \frac{\frac{3}{4} + \frac{5}{12}}{1 - \frac{3}{4} \times \frac{5}{12}} \\
 &= \frac{\frac{9+5}{12}}{1 - \frac{15}{48}} \\
 \Rightarrow \tan 2\alpha &= \frac{14}{12 \left(\frac{33}{48} \right)} \\
 &= \frac{56}{33}
 \end{aligned}$$

Hence, $\tan 2\alpha = 56/33$

Question 5

If $\tan x = b/a$ then find the value of

$$\sqrt{\frac{a+b}{a-b}} + \sqrt{\frac{a-b}{a+b}}.$$

Solution

According to the question,

$$\tan x = b/a$$

Let,

$$\begin{aligned}
 y &= \sqrt{\frac{a+b}{a-b}} + \sqrt{\frac{a-b}{a+b}} \\
 \therefore y &= \sqrt{\frac{a(1+\frac{b}{a})}{a(1-\frac{b}{a})}} + \sqrt{\frac{a(1-\frac{b}{a})}{a(1+\frac{b}{a})}} \\
 &= \sqrt{\frac{(1+\tan x)}{(1-\tan x)}} + \sqrt{\frac{(1-\tan x)}{(1+\tan x)}} \\
 &= \frac{\sqrt{1+\tan x}}{\sqrt{1-\tan x}} + \frac{\sqrt{1-\tan x}}{\sqrt{1+\tan x}} \\
 &= \frac{(\sqrt{1+\tan x})^2 + (\sqrt{1-\tan x})^2}{(\sqrt{1-\tan x})(\sqrt{1+\tan x})} \\
 &= \frac{1+\tan x + 1-\tan x}{\sqrt{1-\tan^2 x}} = \frac{2}{\sqrt{1-\tan^2 x}} \\
 \therefore y &= \frac{\sqrt{\frac{a+b}{a-b}} + \sqrt{\frac{a-b}{a+b}}}{2} = \frac{2}{\sqrt{1-\tan^2 x}} \\
 &= \frac{2}{\sqrt{1-\frac{\sin^2 \theta}{\cos^2 \theta}}} \\
 &= \frac{2}{\frac{\sqrt{\cos^2 \theta - \sin^2 \theta}}{\cos \theta}} \\
 \therefore \cos^2 \theta - \sin^2 \theta &= \cos 2\theta \\
 &= \frac{2\cos \theta}{\sqrt{\cos 2\theta}}
 \end{aligned}$$

Question 6

Prove that $\cos \theta \cos \theta/2 - \cos 3\theta \cos 9\theta/2 = \sin 7\theta \sin 4\theta$

[Hint: Express L.H.S. = $\frac{1}{2} [2\cos \theta \cos \theta/2 - 2\cos 3\theta \cos 9\theta / 2]$

Solution

Using transformation formula, we get,

$$2 \cos A \cos B = \cos(A+B) + \cos(A-B)$$

$$-2 \sin A \sin B = \cos(A+B) - \cos(A-B)$$

Multiplying and dividing the expression by 2.

$$\therefore \text{LHS} = \frac{1}{2} \left(2 \cos \theta \cos \frac{\theta}{2} - 2 \cos 3\theta \cos \frac{9\theta}{2} \right)$$

Applying transformation formula, we get,

$$\text{LHS} = \frac{1}{2} \left(\cos \left(\theta + \frac{\theta}{2} \right) + \cos \left(\theta - \frac{\theta}{2} \right) - \left\{ \cos \left(3\theta + \frac{9\theta}{2} \right) + \cos \left(3\theta - \frac{9\theta}{2} \right) \right\} \right)$$

$$\Rightarrow \text{LHS} = \frac{1}{2} \left(\cos \frac{3\theta}{2} + \cos \frac{\theta}{2} - \cos \left(\frac{15\theta}{2} \right) - \cos \left(-\frac{3\theta}{2} \right) \right)$$

$$\Rightarrow \text{LHS} = \frac{1}{2} \left(\cos \frac{3\theta}{2} + \cos \frac{\theta}{2} - \cos \frac{15\theta}{2} - \cos \frac{3\theta}{2} \right) \{ \because \cos(-x) = \cos x \}$$

$$\Rightarrow \text{LHS} = \frac{1}{2} \left(\cos \frac{\theta}{2} - \cos \frac{15\theta}{2} \right)$$

$$\Rightarrow \text{LHS} = \frac{1}{2} \left(2 \sin \left(\frac{\frac{\theta}{2} + \frac{15\theta}{2}}{2} \right) \sin \left(\frac{\frac{15\theta}{2} - \frac{\theta}{2}}{2} \right) \right)$$

$$\Rightarrow \text{LHS} = \frac{1}{2} \left(2 \sin \left(\frac{8\theta}{2} \right) \sin \left(\frac{7\theta}{2} \right) \right)$$

$$\therefore \text{LHS} = \sin 4\theta \sin \left(\frac{7\theta}{2} \right) = \text{RHS}$$

Hence,

$$\cos \theta \cos \frac{\theta}{2} - \cos 3\theta \cos \frac{9\theta}{2} = \sin 4\theta \sin \left(\frac{7\theta}{2} \right)$$

Question 7

If $a \cos \theta + b \sin \theta = m$ and $a \sin \theta - b \cos \theta = n$, then show that $a^2 + b^2 = m^2 + n^2$.

Solution

According to the question,

$$a \cos \theta + b \sin \theta = m \dots (i)$$

$$a \sin \theta - b \cos \theta = n \dots (ii)$$

Squaring and adding equation 1 and 2, we get,

$$(a \cos \theta + b \sin \theta)^2 + (a \sin \theta - b \cos \theta)^2 = m^2 + n^2$$

$$\Rightarrow a^2 \cos^2 \theta + b^2 \sin^2 \theta + 2ab \sin \theta \cos \theta + a^2 \sin^2 \theta + b^2 \cos^2 \theta - 2ab \sin \theta \cos \theta = m^2 + n^2$$

$$\Rightarrow a^2 \cos^2 \theta + b^2 \sin^2 \theta + a^2 \sin^2 \theta + b^2 \cos^2 \theta = m^2 + n^2$$

$$\Rightarrow a^2 (\sin^2 \theta + \cos^2 \theta) + b^2 (\sin^2 \theta + \cos^2 \theta) = m^2 + n^2$$

Using, $\sin^2 \theta + \cos^2 \theta = 1$,

We get,

$$\Rightarrow a^2 + b^2 = m^2 + n^2$$

Question 8

Find the value of $\tan 22^\circ 30'$.

$$\tan \frac{\theta}{2} = \frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}} = \frac{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{2 \cos^2 \frac{\theta}{2}} = \frac{\sin \theta}{1 + \cos \theta}$$

[Hint: Let $\theta = 45^\circ$, use

Solution

Let, $\theta = 45^\circ$

As we need to find: $\tan 22^\circ 30' = \tan (\theta/2)$

We know that,

$$\sin \theta = \cos \theta = 1/\sqrt{2} \text{ (for } \theta = 45^\circ \text{)}$$

Since,

$$\tan \frac{\theta}{2} = \frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}}$$

Multiplying $2 \cos \theta/2$ in numerator and denominator, we get,

$$\Rightarrow \tan \frac{\theta}{2} = \frac{2 \cos \frac{\theta}{2} \sin \frac{\theta}{2}}{2 \cos^2 \frac{\theta}{2}}$$

By applying formula of T-ratios of multiple angles-

$$\sin 2x = 2 \sin x \cos x$$

$$\cos 2x = 2 \cos^2 x - 1 \text{ or } 1 + \cos 2x = 2 \cos^2 x$$

$$\therefore \tan \frac{\theta}{2} = \frac{\sin \theta}{1 + \cos \theta}$$

$$\Rightarrow \tan 22^\circ 30' = \frac{\sin 45^\circ}{1 + \cos 45^\circ}$$

$$= \frac{\frac{1}{\sqrt{2}}}{1 + \frac{1}{\sqrt{2}}} = \frac{1}{\sqrt{2} + 1}$$

By rationalizing the term, we get,

$$\Rightarrow \tan 22^\circ 30' = \frac{1}{\sqrt{2} + 1} \times \frac{\sqrt{2} - 1}{\sqrt{2} - 1} = \frac{\sqrt{2} - 1}{(\sqrt{2})^2 - 1}$$

Therefore, $\tan 22^\circ 30' = \sqrt{2} - 1$

Question 9

Prove that $\sin 4A = 4\sin A \cos^3 A - 4\cos A \sin^3 A$.

Solution

$$\sin 4A = \sin (2A + 2A)$$

We know that,

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$\text{Therefore, } \sin 4A = \sin 2A \cos 2A + \cos 2A \sin 2A$$

$$\Rightarrow \sin 4A = 2 \sin 2A \cos 2A$$

From T-ratios of multiple angle,

We get,

$$\sin 2A = 2 \sin A \cos A \text{ and } \cos 2A = \cos^2 A - \sin^2 A$$

$$\Rightarrow \sin 4A = 2(2 \sin A \cos A)(\cos^2 A - \sin^2 A)$$

$$\Rightarrow \sin 4A = 4 \sin A \cos^3 A - 4 \cos A \sin^3 A$$

$$\text{Hence, } \sin 4A = 4 \sin A \cos^3 A - 4 \cos A \sin^3 A$$

Question 10

If $\tan \theta + \sin \theta = m$ and $\tan \theta - \sin \theta = n$, then prove that $m^2 - n^2 = 4 \sin \theta \tan \theta$

[Hint: $m + n = 2\tan\theta$, $m - n = 2 \sin \theta$, then use $m^2 - n^2 = (m + n)(m - n)$]

Solution

According to the question,

$$\tan \theta + \sin \theta = m \dots (i)$$

$$\tan \theta - \sin \theta = n \dots (ii)$$

Adding equation i and ii,

$$2 \tan \theta = m + n \dots(\text{iii})$$

Subtracting equation ii from i,

We get,

$$2 \sin \theta = m - n \dots(\text{iv})$$

Multiplying equations (iii) and (iv),

$$2 \sin \theta (2 \tan \theta) = (m + n)(m - n)$$

$$\Rightarrow 4 \sin \theta \tan \theta = m^2 - n^2$$

Hence,

$$m^2 - n^2 = 4 \sin \theta \tan \theta$$

Question 11

If $\tan(A + B) = p$, $\tan(A - B) = q$, then show that $\tan 2A = (p + q) / (1 - pq)$.

[Hint: Use $2A = (A + B) + (A - B)$]

Solution

We know that,

$$\tan 2A = \tan(A + B + A - B)$$

And also,

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$\therefore \tan 2A = \frac{\tan(A+B) + \tan(A-B)}{1 - \tan(A+B) \tan(A-B)}$$

Substituting the values given in question,

$$\Rightarrow \tan 2A = \frac{p+q}{1-pq}$$

$$\text{Hence, } \tan 2A = \frac{p+q}{1-pq}$$

Question 12

If $\cos \alpha + \cos \beta = 0 = \sin \alpha + \sin \beta$, then prove that $\cos 2\alpha + \cos 2\beta = -2\cos(\alpha + \beta)$.

[Hint: $\cos \alpha + \cos \beta)^2 - (\sin \alpha + \sin \beta)^2 = 0$]

Solution

According to the question,

$$\cos \alpha + \cos \beta = 0 = \sin \alpha + \sin \beta \dots (i)$$

$$\text{Since, LHS} = \cos 2\alpha + \cos 2\beta$$

We know that,

$$\cos 2x = \cos^2 x - \sin^2 x$$

Therefore,

$$\text{LHS} = \cos^2 \alpha - \sin^2 \alpha + (\cos^2 \beta - \sin^2 \beta)$$

$$\Rightarrow \text{LHS} = \cos^2 \alpha + \cos^2 \beta - (\sin^2 \alpha + \sin^2 \beta)$$

Also, since,

$$a^2 + b^2 = (a+b)^2 - 2ab$$

$$\Rightarrow \text{LHS} = (\cos \alpha + \cos \beta)^2 - 2\cos \alpha \cos \beta - (\sin \alpha + \sin \beta)^2 + 2\sin \alpha \sin \beta$$

From equation (i),

$$\Rightarrow \text{LHS} = 0 - 2\cos \alpha \cos \beta - 0 + 2\sin \alpha \sin \beta$$

$$\Rightarrow \text{LHS} = -2(\cos \alpha \cos \beta - \sin \alpha \sin \beta)$$

$$\therefore \cos (\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\text{Therefore, LHS} = -2 \cos (\alpha + \beta) = \text{RHS}$$

$$\text{Hence, } \cos 2\alpha + \cos 2\beta = -2\cos (\alpha + \beta)$$

Question 13

$$\text{If } \frac{\sin(x+y)}{\sin(x-y)} = \frac{a+b}{a-b}, \text{ then show that } \frac{\tan x}{\tan y} = \frac{a}{b}.$$

[Hint: Use componendo and Dividendo]

Solution

According to the question,

$$\frac{\sin(x+y)}{\sin(x-y)} = \frac{a+b}{a-b}$$

Since, $\sin(A+B) = \sin A \cos B + \cos A \sin B$

$$\frac{\sin(x+y)}{\sin(x-y)} = \frac{a+b}{a-b}$$

$$\Rightarrow \frac{\sin x \cos y + \cos x \sin y}{\sin x \cos y - \cos x \sin y} = \frac{a+b}{a-b}$$

Applying componendo-dividendo rule,

We get,

$$\frac{(\sin x \cos y + \cos x \sin y) + (\sin x \cos y - \cos x \sin y)}{(\sin x \cos y + \cos x \sin y) - (\sin x \cos y - \cos x \sin y)} = \frac{(a+b) + (a-b)}{(a+b) - (a-b)}$$

$$\Rightarrow \frac{2 \sin x \cos y}{2 \cos x \sin y} = \frac{2a}{2b}$$

$$\Rightarrow \left(\frac{\sin x}{\cos x} \right) \left(\frac{\cos y}{\sin y} \right) = \frac{a}{b}$$

Since, $\tan A = (\sin A)/\cos(A)$

$$\Rightarrow \tan x \left(\frac{1}{\tan y} \right) = \frac{a}{b}$$

$$\Rightarrow \frac{\tan x}{\tan y} = \frac{a}{b}$$

Question 14

$$\tan \theta = \frac{\sin \alpha - \cos \alpha}{\sin \alpha + \cos \alpha},$$

If $\tan \theta = \frac{\sin \alpha - \cos \alpha}{\sin \alpha + \cos \alpha}$ then show that $\sin \alpha + \cos \alpha = \sqrt{2} \cos \theta$.

[Hint: Express $\tan \theta = \tan(\alpha - \pi/2)$ $\theta = \alpha - \pi/4$]

Solution

We know that,

$$\tan \theta = \frac{\sin \alpha - \cos \alpha}{\sin \alpha + \cos \alpha}$$

$$\Rightarrow \tan \theta = \frac{\cos \alpha \left(\frac{\sin \alpha}{\cos \alpha} - 1 \right)}{\cos \alpha \left(\frac{\sin \alpha}{\cos \alpha} + 1 \right)}$$

Since, $\tan A = (\sin A)/(\cos A)$

$$\Rightarrow \tan \theta = (\tan \alpha - 1) / (\tan \alpha + 1)$$

Since, $\tan \pi/4 = 1$

$$\Rightarrow \tan \theta = \frac{(\tan \alpha - \tan \frac{\pi}{4})}{(1 + \tan \frac{\pi}{4} \cdot \tan \alpha)}$$

We know that,

$$\tan(x-y) = (\tan x - \tan y) / (1 + \tan x \cdot \tan y)$$

$$\text{Therefore, } \tan \theta = \tan (\alpha - \pi/4)$$

$$\Rightarrow \theta = \alpha - \pi/4$$

$$\Rightarrow \alpha = \theta + \pi/4 \dots(i)$$

To prove,

$$\sin \alpha + \cos \alpha = \sqrt{2} \cos \theta$$

$$\therefore \text{LHS} = \sin \alpha + \cos \alpha$$

From equation (i)

$$\Rightarrow \text{LHS} = \sin(\theta + \pi/4) + \cos(\theta + \pi/4)$$

$$\therefore \sin(x+y) = \sin x \cos y + \cos x \sin y$$

$$\text{And, } \cos(x+y) = \cos x \cos y - \sin x \sin y$$

$$\text{Therefore, LHS} = \sin \theta \cos(\pi/4) + \sin(\pi/4)\cos \theta + \cos \theta \cos(\pi/4) - \sin(\pi/4)\sin \theta$$

$$\therefore \sin(\pi/4) = \cos(\pi/4) = 1/\sqrt{2}$$

$$\Rightarrow \text{LHS} = \sin \theta (1/\sqrt{2}) + (1/\sqrt{2}) \cos \theta + \cos \theta (1/\sqrt{2}) - \sin \theta (1/\sqrt{2})$$

$$\Rightarrow \text{LHS} = 2 \cos \theta (1/\sqrt{2})$$

$$\Rightarrow \text{LHS} = \sqrt{2} \cos \theta = \text{RHS}$$

$$\text{Therefore, } \sin \alpha + \cos \alpha = \sqrt{2} \cos \theta$$

Question 15

If $\sin \theta + \cos \theta = 1$, then find the general value of θ .

Solution

According to the question,

$$\sin \theta + \cos \theta = 1$$

$$\text{As, } \sin \theta + \cos \theta = 1$$

$$\Rightarrow \sqrt{2} \left(\frac{1}{\sqrt{2}} \sin \theta + \frac{1}{\sqrt{2}} \cos \theta \right) = 1$$

We know that,

$$\sin(\pi/4) = \cos(\pi/4) = 1/\sqrt{2}$$

$$\Rightarrow \sqrt{2} \left(\sin \theta \cos \frac{\pi}{4} + \sin \frac{\pi}{4} \cos \theta \right) = 1$$

We know that,

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$\Rightarrow \sin \left(\frac{\pi}{4} + \theta \right) = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \sin \left(\frac{\pi}{4} + \theta \right) = \sin \frac{\pi}{4}$$

Since we know,

$$\text{If } \sin \theta = \sin \alpha \Rightarrow \theta = n\pi + (-1)^n \alpha$$

We get,

$$\theta + \pi/4 = n\pi + (-1)^n (\pi/4)$$

$$\Rightarrow \theta = n\pi + (\pi/4)((-1)^n - 1)$$

Question 16

Find the most general value of θ satisfying the equation $\tan \theta = -1$ and $\cos \theta = 1/\sqrt{2}$

Solution

According to the question,

We have,

$$\tan \theta = -1$$

$$\text{And } \cos \theta = 1/\sqrt{2}.$$

$$\Rightarrow \theta = -\pi/4$$

So, we know that,

θ lies in IV quadrant.

$$\theta = 2\pi - \pi/4 = 7\pi/4$$

So, general solution is $\theta = 7\pi/4 + 2n\pi, n \in \mathbb{Z}$

Question 17

If $\cot \theta + \tan \theta = 2 \operatorname{cosec} \theta$, then find the general value of θ .

Solution

According to the question,

$$\Rightarrow \frac{\cos \theta}{\sin \theta} + \frac{\sin \theta}{\cos \theta} = 2 \operatorname{cosec} \theta$$

Since,

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\Rightarrow \frac{\cos^2 \theta + \sin^2 \theta}{\sin \theta \cos \theta} = 2 \operatorname{cosec} \theta$$

$$\Rightarrow 1 = 2 \operatorname{cosec} \theta \sin \theta \cos \theta$$

We know that,

$$\sin \theta \operatorname{cosec} \theta = 1$$

$$\Rightarrow 1 = 2 \cos \theta$$

$$\Rightarrow \cos \theta = 1/2 = \cos(\pi/3)$$

Hence,

The solution of $\cos x = \cos \alpha$ can be given by,

$$x = 2m\pi \pm \alpha \quad \forall m \in \mathbb{Z}$$

$$\Rightarrow \theta = 2n\pi \pm \pi/3, n \in \mathbb{Z}$$

Question 18

If $2\sin^2 \theta = 3\cos \theta$, where $0 \leq \theta \leq 2\pi$, then find the value of θ .

Solution

According to the question,

$$2\sin^2 \theta = 3\cos \theta$$

We know that,

$$\sin^2 \theta = 1 - \cos^2 \theta$$

Given that,

$$2 \sin^2 \theta = 3 \cos \theta$$

$$2 - 2 \cos^2 \theta = 3 \cos \theta$$

$$2 \cos^2 \theta + 3 \cos \theta - 2 = 0$$

$$(\cos \theta + 2)(2 \cos \theta - 1) = 0$$

Therefore,

$$\cos \theta = \frac{1}{2} = \cos \pi/3$$

$$\theta = \pi/3 \text{ or } 2\pi - \pi/3$$

$$\theta = \pi/3, 5\pi/3$$

$$\text{Therefore, } 2(1 - \cos^2 \theta) = 3 \cos \theta$$

$$\Rightarrow 2 - 2 \cos^2 \theta = 3 \cos \theta$$

$$\Rightarrow 2 \cos^2 \theta + 3 \cos \theta - 2 = 0$$

$$\Rightarrow 2 \cos^2 \theta + 4 \cos \theta - \cos \theta - 2 = 0$$

$$\Rightarrow 2 \cos \theta (\cos \theta + 2) + 1 (\cos \theta + 2) = 0$$

$$\Rightarrow (2 \cos \theta + 1)(\cos \theta + 2) = 0$$

Since, $\cos \theta \in [-1, 1]$, for any value θ .

$$\text{So, } \cos \theta \neq -2$$

Therefore,

$$2 \cos \theta - 1 = 0$$

$$\Rightarrow \cos \theta = \frac{1}{2}$$

$$= \pi/3 \text{ or } 2\pi - \pi/3$$

$$\theta = \pi/3, 5\pi/3$$

Question 19

If $\sec x \cos 5x + 1 = 0$, where $0 < x \leq \pi/2$, then find the value of x .

Solution

According to the question,

$$\sec x \cos 5x = -1$$

$$\Rightarrow \cos 5x = -1/\sec x$$

We know that,

$$\sec x = 1/\cos x$$

$$\Rightarrow \cos 5x + \cos x = 0$$

By transformation formula of T-ratios,

We know that,

$$\begin{aligned}\cos A + \cos B &= 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) \\ \Rightarrow 2\cos\left(\frac{5x+x}{2}\right)\cos\left(\frac{5x-x}{2}\right) &= 0\end{aligned}$$

$$\Rightarrow 2\cos 3x \cos 2x = 0$$

$$\Rightarrow \cos 3x = 0 \text{ or } \cos 2x = 0$$

$$\because 0 < x \leq \pi/2$$

Therefore, $0 < 2x \leq \pi$ or $0 < 3x \leq 3\pi/2$

Therefore, $2x = \pi/2$

$$\Rightarrow x = \pi/4$$

$$3x = \pi/2$$

$$\Rightarrow x = \pi/6$$

$$\text{Or } 3x = 3\pi/2$$

$$\Rightarrow x = \pi/2$$

Hence, $x = \pi/6, \pi/4, \pi/2$.

Question 20

If $\sin(\theta + \alpha) = a$ and $\sin(\theta + \beta) = b$, then prove that $\cos 2(\alpha - \beta) - 4ab \cos(\alpha - \beta) = 1 - 2a^2 - 2b^2$

Solution

According to the question,

$$\sin(\theta + \alpha) = a \text{ and } \sin(\theta + \beta) = b$$

$$\text{LHS} = \cos 2(\alpha - \beta) - 4ab \cos(\alpha - \beta)$$

$$\text{Using } \cos 2x = 2\cos^2 x - 1,$$

Let us solve,

$$\Rightarrow \text{LHS} = 2\cos^2(\alpha - \beta) - 1 - 4ab \cos(\alpha - \beta)$$

$$\Rightarrow \text{LHS} = 2\cos(\alpha - \beta) \{\cos(\alpha - \beta) - 2ab\} - 1$$

Since,

$$\cos(\alpha - \beta) = \cos\{(\theta + \alpha) - (\theta + \beta)\}$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

$$\Rightarrow \cos(\alpha - \beta) = \cos(\theta + \alpha)\cos(\theta + \beta) + \sin(\theta + \alpha)\sin(\theta + \beta)$$

Since,

$$\sin(\theta + \alpha) = a$$

$$\Rightarrow \cos(\theta + \alpha) = \sqrt{1 - \sin^2(\theta + \alpha)} = \sqrt{1 - a^2}$$

Similarly,

$$\cos(\theta + \beta) = \sqrt{1 - b^2}$$

Therefore,

$$\cos(\alpha - \beta) = \sqrt{1 - a^2}\sqrt{1 - b^2} + ab$$

Therefore,

$$\text{LHS} = 2\{ab + \sqrt{1 - a^2}\sqrt{1 - b^2}\}\{ab + \sqrt{1 - a^2}\sqrt{1 - b^2} - 2ab\} - 1$$

$$\Rightarrow \text{LHS} = 2\{\sqrt{1 - a^2}\sqrt{1 - b^2} + ab\}\{\sqrt{1 - a^2}\sqrt{1 - b^2} - ab\} - 1$$

Using $(x + y)(x - y) = x^2 - y^2$

$$\Rightarrow \text{LHS} = 2\{(1-a^2)(1-b^2) - a^2b^2\} - 1$$

$$\Rightarrow \text{LHS} = 2\{1 - a^2 - b^2 + a^2b^2\} - 1$$

$$\Rightarrow \text{LHS} = 2 - 2a^2 - 2b^2 - 1$$

$$\Rightarrow \text{LHS} = 1 - 2a^2 - 2b^2 = \text{RHS}$$

Therefore,

We get,

$$\cos 2(\alpha - \beta) - 4ab \cos (\alpha - \beta) = 1 - 2a^2 - 2b^2$$

Question 21

If $\cos (\theta + \phi) = m \cos (\theta - \phi)$, then prove that $\tan \theta = ((1 - m)/(1 + m)) \cot \phi$

[Hint: Express $\cos (\theta + \phi) / \cos (\theta - \phi) = m/1$ and apply Componendo and Dividendo] Solution:

According to the question,

$$\cos (\theta + \phi) = m \cos (\theta - \phi)$$

$$\therefore \cos (\theta + \phi) = m \cos (\theta - \phi)$$

$$\Rightarrow \frac{\cos(\theta-\phi)}{\cos(\theta+\phi)} = \frac{1}{m}$$

Applying componendo – dividendo, we get,

$$\Rightarrow \frac{\cos(\theta-\phi)+\cos(\theta+\phi)}{\cos(\theta-\phi)-\cos(\theta+\phi)} = \frac{1+m}{1-m}$$

From transformation formula, we know that,

$$\cos(A+B) + \cos(A-B) = 2\cos A \cos B$$

$$\cos(A-B) - \cos(A+B) = 2\sin A \sin B$$

$$\Rightarrow \frac{2 \cos \theta \cos \phi}{2 \sin \theta \sin \phi} = \frac{1+m}{1-m}$$

Since, $(\cos \theta) / (\sin \theta) = \cot \theta$

$$\Rightarrow \cot \theta \cot \phi = \frac{1+m}{1-m}$$

$$\Rightarrow \left(\frac{1-m}{1+m}\right) \cot \phi = \frac{1}{\cot \theta}$$

$$\Rightarrow \tan \theta = \left(\frac{1-m}{1+m}\right) \cot \phi$$

22. Find the value of the expression

$$3 \left[\sin^4 \left(\frac{3\pi}{2} - \alpha \right) + \sin^4 (3\pi + \alpha) \right] - 2 \left[\sin^6 \left(\frac{\pi}{2} + \alpha \right) + \sin^6 (5\pi - \alpha) \right]$$

Solution

According to the question,

$$\text{Let, } y = 3[\sin^4 (3\pi/2 - \alpha) + \sin^4 (3\pi + \alpha)] - 2[\sin^6 (\pi/2 + \alpha) + \sin^6 (5\pi - \alpha)]$$

We know that,

$$\sin(3\pi/2 - \alpha) = -\cos \alpha$$

$$\sin(3\pi + \alpha) = -\sin \alpha$$

$$\sin(\pi/2 + \alpha) = \cos \alpha$$

$$\sin(5\pi - \alpha) = \sin \alpha$$

Therefore,

$$y = 3[(-\cos \alpha)^4 + (-\sin \alpha)^4] - 2[\cos^6 \alpha + \sin^6 \alpha]$$

$$\Rightarrow y = 3 [\cos^4 \alpha + \sin^4 \alpha] - 2[\sin^6 \alpha + \cos^6 \alpha]$$

$$\Rightarrow y = 3[(\sin^2 \alpha + \cos^2 \alpha)^2 - 2\sin^2 \alpha \cos^2 \alpha] - 2[(\sin^2 \alpha)^3 + (\cos^2 \alpha)^3]$$

Since, we know that,

$$\sin^2 \alpha + \cos^2 \alpha = 1$$

Also, we know that,

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

$$\Rightarrow y = 3[1 - 2\sin^2 \alpha \cos^2 \alpha] - 2[(\sin^2 \alpha + \cos^2 \alpha)(\cos^4 \alpha + \sin^4 \alpha - \sin^2 \alpha \cos^2 \alpha)]$$

$$\Rightarrow y = 3[1 - 2\sin^2 \alpha \cos^2 \alpha] - 2[\cos^4 \alpha + \sin^4 \alpha - \sin^2 \alpha \cos^2 \alpha]$$

$$\Rightarrow y = 3[1 - 2\sin^2 \alpha \cos^2 \alpha] - 2[(\sin^2 \alpha + \cos^2 \alpha)^2 - 2\sin^2 \alpha \cos^2 \alpha - \sin^2 \alpha \cos^2 \alpha]$$

$$\Rightarrow y = 3[1 - 2\sin^2 \alpha \cos^2 \alpha] - 2[1 - 3\sin^2 \alpha \cos^2 \alpha]$$

$$\Rightarrow y = 3 - 6\sin^2 \alpha \cos^2 \alpha - 2 + 6 \sin^2 \alpha \cos^2 \alpha$$

$$\Rightarrow y = 1$$

Question 22

Find the value of the expression

$$3 \left[\sin^4 \left(\frac{3\pi}{2} - \alpha \right) + \sin^4 (3\pi + \alpha) \right] - 2 \left[\sin^6 \left(\frac{\pi}{2} + \alpha \right) + \sin^6 (5\pi - \alpha) \right].$$

Solution

$$1$$

Question 23

If $a \cos 2\theta + b \sin 2\theta = c$ has α and β as its roots, then prove that $\tan \alpha + \tan \beta = 2b/(a + c)$

[Hint: Use the identities $\cos 2\theta = (1 - \tan^2 \theta)/(1 + \tan^2 \theta)$ and $\sin 2\theta = 2\tan \theta/(1 + \tan^2 \theta)$]

Solution

According to the question,

$$a \cos 2\theta + b \sin 2\theta = c$$

α and β are the roots of the equation.

Using the formula of multiple angles,

We know that,

$$\cos 2\theta = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} \text{ and } \sin 2\theta = \frac{2 \tan \theta}{1 + \tan^2 \theta}$$

$$\therefore a \left(\frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} \right) + b \left(\frac{2 \tan \theta}{1 + \tan^2 \theta} \right) - c = 0$$

$$\Rightarrow a(1 - \tan^2 \theta) + 2b \tan \theta - c(1 + \tan^2 \theta) = 0$$

$$\Rightarrow (-c - a)\tan^2 \theta + 2b \tan \theta - c + a = 0 \dots (i)$$

We know that,

The sum of roots of a quadratic equation, $ax^2 + bx + c = 0$ is given by $(-b/a)$

Therefore,

$$\tan \alpha + \tan \beta = -2b/-(c + a) = 2b/(c + a)$$

$$\text{Hence, } \tan \alpha + \tan \beta = 2b/(c + a)$$

Question 24

If $x = \sec \phi - \tan \phi$ and $y = \operatorname{cosec} \phi + \cot \phi$, then show that $xy + x - y + 1 = 0$.

[Hint: Find $xy + 1$ and then show $\tan x - y = -(xy + 1)$]

Solution

According to the question,

$$x = \sec \phi - \tan \phi \text{ and } y = \operatorname{cosec} \phi + \cot \phi$$

$$\text{Given that, LHS} = xy + x - y + 1$$

$$\begin{aligned} &= (\sec \phi - \tan \phi)(\operatorname{cosec} \phi + \cot \phi) + (\sec \phi - \tan \phi) - (\operatorname{cosec} \phi + \cot \phi) + 1 \\ &= \sec \phi \operatorname{cosec} \phi + \cot \phi \sec \phi - \tan \phi \cot \phi - \tan \phi \operatorname{cosec} \phi \\ &\quad + \sec \phi - \tan \phi - (\operatorname{cosec} \phi + \cot \phi) + 1 \\ &= \frac{1}{\sin \phi \cos \phi} + \frac{1}{\sin \phi} - 1 - \sec \phi + \sec \phi - \tan \phi - \left(\frac{1}{\sin \phi} + \frac{\cos \phi}{\sin \phi} \right) + 1 \\ &= \frac{1}{\sin \phi \cos \phi} + \frac{1}{\sin \phi} - \tan \phi - \left(\frac{1}{\sin \phi} + \frac{\cos \phi}{\sin \phi} \right) \\ &= \frac{1}{\sin \phi \cos \phi} - \frac{\cos \phi}{\sin \phi} - \frac{\cos \phi}{\sin \phi} \\ &= \frac{1}{\sin \phi \cos \phi} - \left(\frac{\cos \phi}{\sin \phi} + \frac{\sin \phi}{\cos \phi} \right) \\ &= \frac{1}{\sin \phi \cos \phi} - \left(\frac{\cos^2 \phi + \sin^2 \phi}{\sin \phi \cos \phi} \right) \\ &\text{Since, } \sin^2 \theta + \cos^2 \theta = 1 \\ &= \frac{1}{\sin \phi \cos \phi} - \left(\frac{1}{\sin \phi \cos \phi} \right) = 0 \end{aligned}$$

$$\text{Thus, LHS} = xy + x - y + 1 = 0$$

Question 25

If θ lies in the first quadrant and $\cos \theta = 8/17$, then find the value of $\cos(30^\circ + \theta) + \cos(45^\circ - \theta) + \cos(120^\circ - \theta)$

Solution

According to the question,

$$\cos \theta = 8/17$$

$$\sin \theta = \pm\sqrt{1 - \cos^2 \theta}$$

Since, θ lies in first quadrant, only positive sign can be considered.

$$\Rightarrow \sin \theta = \sqrt{1 - 64/289} = 15/17$$

$$\text{Let, } y = \cos(30^\circ + \theta) + \cos(45^\circ - \theta) + \cos(120^\circ - \theta)$$

We know that,

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

Therefore,

$$y = \cos 30^\circ \cos \theta - \sin 30^\circ \sin \theta + \cos 45^\circ \cos \theta + \sin 45^\circ \sin \theta + \cos 120^\circ \cos \theta + \sin 120^\circ \sin \theta$$

Substituting values of $\cos 30^\circ$, $\sin 30^\circ$, $\cos 120^\circ$, $\sin 120^\circ$ and $\cos 45^\circ$

$$\begin{aligned} \Rightarrow y &= \frac{\sqrt{3}}{2} \cdot \frac{8}{17} - \frac{1}{2} \cdot \frac{15}{17} + \frac{1}{\sqrt{2}} \cdot \frac{8}{17} + \frac{1}{\sqrt{2}} \cdot \frac{15}{17} + \left(-\frac{1}{2}\right) \left(\frac{8}{17}\right) + \left(\frac{\sqrt{3}}{2}\right) \left(\frac{15}{17}\right) \\ &= \frac{8\sqrt{3}}{34} - \frac{15}{34} + \frac{8 + 15}{17\sqrt{2}} - \frac{8}{34} + \frac{15\sqrt{3}}{34} \\ &= \frac{23\sqrt{3}}{34} + \frac{23}{17\sqrt{2}} - \frac{23}{34} \\ \Rightarrow y &= \frac{23}{34}(\sqrt{3} + \sqrt{2} - 1) \end{aligned}$$

Question 26

Find the value of the expression $\cos^4(\pi/8) + \cos^4(3\pi/8) + \cos^4(5\pi/8) + \cos^4(7\pi/8)$.

[Hint: Simplify the expression to

$$2 \left(\cos^4 \frac{\pi}{8} + \cos^4 \frac{3\pi}{8} \right) = 2 \left[\left(\cos^2 \frac{\pi}{8} + \cos^2 \frac{3\pi}{8} \right)^2 - 2 \cos^2 \frac{\pi}{8} \cos^2 \frac{3\pi}{8} \right]$$

Solution

According to the question,

$$\text{Let } y = \cos^4(\pi/8) + \cos^4(3\pi/8) + \cos^4(5\pi/8) + \cos^4(7\pi/8).$$

$$\Rightarrow y = \cos^4(\pi/8) + \cos^4(3\pi/8) + \cos^4(\pi - 3\pi/8) + \cos^4(\pi - \pi/8).$$

Since we know that, $\cos(\pi - x) = -\cos x$, we get,

$$\begin{aligned} y &= \cos^4 \frac{\pi}{8} + \cos^4 \frac{3\pi}{8} + \cos^4 \frac{5\pi}{8} + \cos^4 \frac{7\pi}{8} \\ &= \cos^4 \frac{\pi}{8} + \cos^4 \frac{3\pi}{8} + \cos^4 \left(\pi - \frac{3\pi}{8} \right) + \cos^4 \left(\pi - \frac{\pi}{8} \right) \end{aligned}$$

Since, we know that,

$$\begin{aligned} \cos(\pi - x) &= -\cos x \\ &= \cos^4 \frac{\pi}{8} + \cos^4 \frac{3\pi}{8} + \cos^4 \left(\frac{3\pi}{8} \right) + \cos^4 \left(\frac{\pi}{8} \right) \\ &= 2 \left(\cos^4 \frac{\pi}{8} + \cos^4 \frac{3\pi}{8} \right) \\ &= 2 \left(\cos^4 \frac{\pi}{8} + \cos^4 \left(\frac{\pi}{2} - \frac{\pi}{8} \right) \right) \\ &= 2 \left(\cos^4 \frac{\pi}{8} + \sin^4 \frac{\pi}{8} \right) \\ &= 2 \left[\left(\cos^2 \frac{\pi}{8} + \sin^2 \frac{\pi}{8} \right)^2 - 2 \cos^2 \frac{\pi}{8} \cdot \sin^2 \frac{\pi}{8} \right] \\ &= 2 \left[1 - 2 \cos^2 \frac{\pi}{8} \cdot \sin^2 \frac{\pi}{8} \right] \\ &= 2 - \left(2 \cos \frac{\pi}{8} \cdot \sin \frac{\pi}{8} \right)^2 \\ &= 2 - \left(\sin \frac{2\pi}{8} \right)^2 \\ &= 2 - \left(\frac{1}{\sqrt{2}} \right)^2 \\ &= 2 - \frac{1}{2} \\ &= 3/2 \end{aligned}$$

$$= 2 - (1/\sqrt{2})^2$$

$$= 2 - \frac{1}{2}$$

$$= 3/2$$

Question 27

Find the general solution of the equation

$$5\cos^2\theta + 7\sin^2\theta - 6 = 0$$

Solution

According to the question,

$$5\cos^2\theta + 7\sin^2\theta - 6 = 0$$

We know that,

$$\sin^2\theta = 1 - \cos^2\theta$$

$$\text{Therefore, } 5\cos^2\theta + 7(1 - \cos^2\theta) - 6 = 0$$

$$\Rightarrow 5\cos^2\theta + 7 - 7\cos^2\theta - 6 = 0$$

$$\Rightarrow -2\cos^2\theta + 1 = 0$$

$$\Rightarrow \cos^2\theta = \frac{1}{2}$$

$$\text{Therefore, } \cos \theta = \pm 1/\sqrt{2}$$

$$\text{Therefore, } \cos \theta = \cos \pi/4 \text{ or } \cos \theta = \cos 3\pi/4$$

Since, solution of $\cos x = \cos \alpha$ is given by

$$x = 2m\pi \pm \alpha \quad \forall m \in \mathbb{Z}$$

$$\theta = n\pi \pm \pi/4, n \in \mathbb{Z}$$

Question 28

Find the general solution of the equation $\sin x - 3\sin 2x + \sin 3x = \cos x - 3\cos 2x + \cos 3x$

Solution

According to the question,

$$\sin x - 3\sin 2x + \sin 3x = \cos x - 3\cos 2x + \cos 3x$$

Grouping $\sin x$ and $\sin 3x$ in LHS and, $\cos x$ and $\cos 3x$ in RHS,

We get,

$$\sin x + \sin 3x - 3\sin 2x = \cos x + \cos 3x - 3\cos 2x$$

Applying transformation formula,

$$\cos A + \cos B = 2\cos \left(\frac{A+B}{2}\right) \cos \left(\frac{A-B}{2}\right)$$

$$\sin A + \sin B = 2 \sin \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right)$$

$\Rightarrow$

$$2 \sin \left(\frac{3x+x}{2} \right) \cos \left(\frac{3x-x}{2} \right) - 3 \sin 2x = 2 \cos \left(\frac{3x+x}{2} \right) \cos \left(\frac{3x-x}{2} \right) - 3 \cos 2x$$

$$\Rightarrow 2 \sin 2x \cos x - 3 \sin 2x = 2 \cos 2x \cos x - 3 \cos 2x$$

$$\Rightarrow 2 \sin 2x \cos x - 3 \sin 2x - 2 \cos 2x \cos x + 3 \cos 2x = 0$$

$$\Rightarrow 2 \cos x (\sin 2x - \cos 2x) - 3(\sin 2x - \cos 2x) = 0$$

$$\Rightarrow (\sin 2x - \cos 2x)(2 \cos x - 3) = 0$$

$$\Rightarrow \cos x = 3/2 \text{ or } \sin 2x = \cos 2x$$

As $\cos x \in [-1, 1]$

Hence, no value of x exists for which $\cos x = 3/2$

Therefore, $\sin 2x = \cos 2x$

$$\Rightarrow \tan 2x = 1 = \tan \pi/4$$

We know solution of $\tan x = \tan \alpha$ is given by,

$$x = n\pi + \alpha, n \in \mathbb{Z}$$

$$\text{Therefore, } 2x = n\pi + (\pi/4)$$

$$\Rightarrow x = n\pi/2 + (\pi/8), n \in \mathbb{Z}$$

Question 29

Find the general solution of the equation $(\sqrt{3} - 1) \cos \theta + (\sqrt{3} + 1) \sin \theta = 2$

[Hint: Put $\sqrt{3} - 1 = r \sin \alpha$, $\sqrt{3} + 1 = r \cos \alpha$ which gives $\tan \alpha = \tan((\pi/4) - (\pi/6))$ $\alpha = \pi/12$]

Solution

$$\text{Let, } r \sin \alpha = \sqrt{3} - 1 \text{ and } r \cos \alpha = \sqrt{3} + 1$$

$$\text{Therefore, } r = \sqrt{(\sqrt{3} - 1)^2 + (\sqrt{3} + 1)^2} = \sqrt{8} = 2\sqrt{2}$$

$$\text{And, } \tan \alpha = (\sqrt{3} - 1) / (\sqrt{3} + 1)$$

$$\text{Therefore, } r(\sin \alpha \cos \theta + \cos \alpha \sin \theta) = 2$$

$$\Rightarrow r \sin (\theta+\alpha)=2$$

$$\Rightarrow \sin (\theta+\alpha)=1 / \sqrt{2}$$

$$\Rightarrow \sin (\theta+\alpha)=\sin (\pi / 4)$$

$$\Rightarrow \theta+\alpha=n \pi+(-1)^n(\pi / 4), n \in \mathbb{Z}$$

$$\Rightarrow \theta=n \pi+(-1)^n(\pi / 4)-(\pi / 12), n \in \mathbb{Z}$$

Objective Type Questions

Question 30

If $\sin \theta + \operatorname{cosec} \theta = 2$, then $\sin^2 \theta + \operatorname{cosec}^2 \theta$ is equal to

- A. 1
- B. 4
- C. 2
- D. None of these

Solution

C. 2

Explanation:

According to the question,

$$\sin \theta + \operatorname{cosec} \theta = 2$$

Squaring LHS and RHS,

We get,

$$\Rightarrow (\sin \theta + \operatorname{cosec} \theta)^2 = 2^2$$

$$\Rightarrow \sin^2 \theta + \operatorname{cosec}^2 \theta + 2 \sin \theta \operatorname{cosec} \theta = 4$$

$$\Rightarrow \sin^2 \theta + \operatorname{cosec}^2 \theta + 2 \sin \theta (1/\sin) \theta = 4$$

$$\Rightarrow \sin^2 \theta + \operatorname{cosec}^2 \theta + 2 = 4$$

$$\Rightarrow \sin^2 \theta + \operatorname{cosec}^2 \theta = 2$$

Thus, option (C) 2 is the correct answer.

Question 31

If $f(x) = \cos^2 x + \sec^2 x$, then

A. $f(x) < 1$

B. $f(x) = 1$

C. $2 < f(x) < 1$

D. $f(x) \geq 2$

[Hint: A.M $\geq$ G.M.]

Solution

D. $f(x) \geq 2$

Explanation:

According to the question,

We have, $f(x) = \cos^2 x + \sec^2 x$

We know that, A.M $\geq$ G.M.

$$\Rightarrow \frac{\cos^2 x + \sec^2 x}{2} \geq \sqrt{\cos^2 x \sec^2 x}$$

$$\Rightarrow \frac{\cos^2 x + \sec^2 x}{2} \geq \sqrt{\cos^2 x \frac{1}{\cos^2 x}}$$

$$\Rightarrow \frac{\cos^2 x + \sec^2 x}{2} \geq 1$$

$$\Rightarrow \cos^2 x + \sec^2 x \geq 2$$

$$\Rightarrow f(x) \geq 2$$

Thus, option (D) $f(x) \geq 2$ is the correct answer.

Question 32

If $\tan \theta = 1/2$ and $\tan \phi = 1/3$, then the value of $\theta + \phi$ is

A. $\pi/6$

B. π

C. 0

D. $\pi/4$

Solution

D. $\pi/4$

Explanation:

According to the question,

$$\tan \theta = \frac{1}{2} \text{ and } \tan \phi = \frac{1}{3}$$

We know that,

$$\begin{aligned}\tan(\theta + \phi) &= \frac{\tan \theta + \tan \phi}{1 - \tan \theta \tan \phi} \\ &= \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2} \times \frac{1}{3}} = \frac{\frac{5}{6}}{\frac{5}{6}} = 1 \\ \Rightarrow \tan(\theta + \phi) &= \tan \frac{\pi}{4} \\ \Rightarrow \theta + \phi &= \frac{\pi}{4}\end{aligned}$$

Thus, option (D) $\pi/4$ is the correct answer.

Question 33

Which of the following is not correct?

A. $\sin \theta = -1/5$

B. $\cos \theta = 1$

C. $\sec \theta = 1/2$

D. $\tan \theta = 20$

Solution

C. $\sec \theta = 1/2$

Explanation:

According to the question,

We know that,

a) $\sin \theta = -1/5$ is correct since $\sin \theta \in [-1, 1]$

b) $\cos \theta = 1$ is correct since $\cos \theta \in [-1, 1]$

c) $\sec \theta = \frac{1}{2}$

$$\Rightarrow (1/\cos \theta) = \frac{1}{2}$$

$\Rightarrow \cos \theta = 2$ is incorrect since $\cos \theta \in [-1, 1]$

d) $\tan \theta = 20$ is correct since $\tan \theta \in \mathbb{R}$.

Thus, option (C) $\sec \theta = \frac{1}{2}$ is the correct answer.

Question 34

The value of $\tan 1^\circ \tan 2^\circ \tan 3^\circ \dots \tan 89^\circ$ is

A. 0

B. 1

C. $\frac{1}{2}$

D. Not defined

Solution

B. 1

Explanation:

According to the question,

$$\tan 1^\circ \tan 2^\circ \tan 3^\circ \dots \tan 89^\circ$$

$$= \tan 1^\circ \tan 2^\circ \dots \tan 45^\circ \tan (90-44^\circ) \tan (90-43^\circ) \dots \tan (90-1^\circ)$$

$$= \tan 1^\circ \tan 2^\circ \dots \tan 45^\circ \cot 44^\circ \cot 43^\circ \dots \cot 1^\circ [\because \tan (90-\theta) = \cot \theta]$$

$$= \tan 1^\circ \cot 1^\circ \tan 2^\circ \cot 2^\circ \dots \tan 45^\circ \dots \tan 89^\circ \cot 89^\circ$$

$$= 1.1 \dots 1 = 1$$

Thus, option (B) 1 is the correct answer.

Question 35

The value of $(1 - \tan^2 15^\circ)/(1 + \tan^2 15^\circ)$ is

A. 1

B. $\sqrt{3}$

C. $\sqrt{3}/2$

D. 2

Solution

C. $\sqrt{3}/2$

Explanation:

According to the question,

Let $\theta = 15^\circ \Rightarrow 2\theta = 30^\circ$

Now, since we know that,

$$\begin{aligned}\cos 2\theta &= \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} \\ \Rightarrow \cos 30^\circ &= \frac{1 - \tan^2 15^\circ}{1 + \tan^2 15^\circ} \\ \Rightarrow \frac{\sqrt{3}}{2} &= \frac{1 - \tan^2 15^\circ}{1 + \tan^2 15^\circ}\end{aligned}$$

Thus, option (C) $\sqrt{3}/2$ is the correct answer.

Question 36

The value of $\cos 1^\circ \cos 2^\circ \cos 3^\circ \dots \cos 179^\circ$ is

A. $1/\sqrt{2}$

B. 0

C. 1

D. -1

Solution

B. 0

Explanation:

According to the question,

Since $\cos 90^\circ = 0$

We get,

$$\Rightarrow \cos 1^\circ \cos 2^\circ \cos 3^\circ \dots \cos 90^\circ \dots \cos 179^\circ = 0$$

Thus, option (B) 0 is the correct answer.

Question 37

If $\tan \theta = 3$ and θ lies in third quadrant, then the value of $\sin \theta$ is

- A. $1/\sqrt{10}$
- B. $-1/\sqrt{10}$
- C. $-3/\sqrt{10}$
- D. $3/\sqrt{10}$

Solution

C. $-3/\sqrt{10}$

Explanation:

According to the question,

Given that, $\tan \theta = 3$ and θ lies in third quadrant

$$\Rightarrow \cot \theta = 1/3$$

We know that,

$$\operatorname{Cosec}^2 \theta = 1 + \cot^2 \theta$$

$$= 1 + \left(\frac{1}{3}\right)^2 = 1 + \frac{1}{9} = \frac{10}{9}$$

$$\Rightarrow \sin^2 \theta = \frac{9}{10}$$

$$\Rightarrow \sin \theta = \pm \frac{3}{\sqrt{10}}$$

$$\Rightarrow \sin \theta = -\frac{3}{\sqrt{10}}, \text{ since } \theta \text{ lies in third quadrant.}$$

Thus, option (C) $-3/\sqrt{10}$ is the correct answer.

Question 38

The value of $\tan 75^\circ - \cot 75^\circ$ is equal to

- A. $2\sqrt{3}$

B. $2 + \sqrt{3}$

C. $2 - \sqrt{3}$

D. 1

Solution

A. $2\sqrt{3}$

Explanation:

According to the question,

We have,

$$\tan 75^\circ - \cot 75^\circ$$

$$\begin{aligned} &= \frac{\sin 75^\circ}{\cos 75^\circ} - \frac{\cos 75^\circ}{\sin 75^\circ} \\ &= \frac{\sin^2 75^\circ - \cos^2 75^\circ}{\cos 75^\circ \sin 75^\circ} \\ &= \frac{2(\sin^2 75^\circ - \cos^2 75^\circ)}{2\cos 75^\circ \sin 75^\circ} \\ &= \frac{-2\cos 150^\circ}{\sin 150^\circ} \end{aligned}$$

$$= -2\cot 150^\circ$$

$$= -2 \cot (180^\circ - 30^\circ)$$

$$= 2\cot 30^\circ$$

$$= 2\sqrt{3}$$

Thus, option (A) $2\sqrt{3}$ is the correct answer.