

Chapter 11: Conic Sections

NCERT Exemplar · Class 11 · Maths

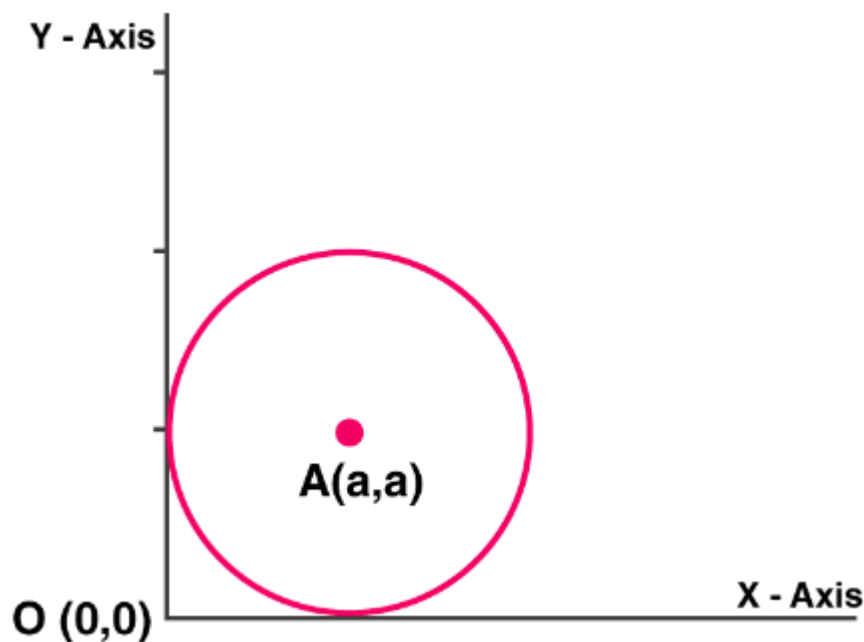
Exercise · Page 202 · Textbook page 202

Question 1

Find the equation of the circle which touches the both axes in first quadrant and whose radius is a .

Solution

The circle touches both the x and y axes in the first quadrant and the radius is a .



For a circle of radius a , the centre is (a, a) .

The equation of a circle having centre (h, k) , having radius as r units, is

$$(x - h)^2 + (y - k)^2 = r^2$$

Therefore, the equation of the circle becomes $(x - a)^2 + (y - a)^2 = a^2$

$$x^2 - 2ax + a^2 + y^2 - 2ay + a^2 - a^2 = 0$$

$$x^2 - 2ax + y^2 - 2ay + a^2 = 0$$

2. Show that the point (x, y) given by $x = \frac{2at}{1+t^2}$ and $y = \frac{a(1-t^2)}{1+t^2}$

lies on a circle for all real values of t such that $-1 \leq t \leq 1$ where a is any given real numbers.

Given

$$x = \frac{2at}{1+t^2}, y = \frac{a(1-t^2)}{1+t^2}$$

Squaring both the equations,

$$x^2 = \left[\frac{2at}{1+t^2} \right]^2 \text{ \& } y^2 = \left[\frac{a(1-t^2)}{1+t^2} \right]^2$$

Adding both the equations,

$$x^2 + y^2 = \frac{4a^2t^2}{(1+t^2)^2} + \frac{a^2(1-t^2)^2}{(1+t^2)^2}$$

Taking LCM and simplifying we get

$$\begin{aligned} &= \frac{a^2(4t^2 + (1-t^2)^2)}{(1+t^2)^2} \\ &= \frac{a^2(4t^2 + 1 + t^4 - 2t^2)}{(1+t^2)^2} \\ &= \frac{a^2(2t^2 + 1 + t^4)}{(1+t^2)^2} \\ &= \frac{a^2(1+t^2)^2}{(1+t^2)^2} \\ &= a^2 \end{aligned}$$

Therefore $x^2 + y^2 = a^2$

The equation of a circle having centre (h, k) , having radius as r units, is

$$(x-h)^2 + (y-k)^2 = r^2$$

Centre = $(0, 0)$ Radius = a units

Hence proved.

Question 2

Show that the point (x, y) given by $x = \frac{2at}{1+t^2}$ and $y = \frac{a(1-t^2)}{1+t^2}$ lies on a circle for all real values of t such that $-1 \leq t \leq 1$, where a is any given real number.

Solution

Substitute and compute $x^2 + y^2 = \frac{4a^2t^2 + a^2(1 - t^2)^2}{(1 + t^2)^2} = a^2$. Hence the point lies on the circle $x^2 + y^2 = a^2$.

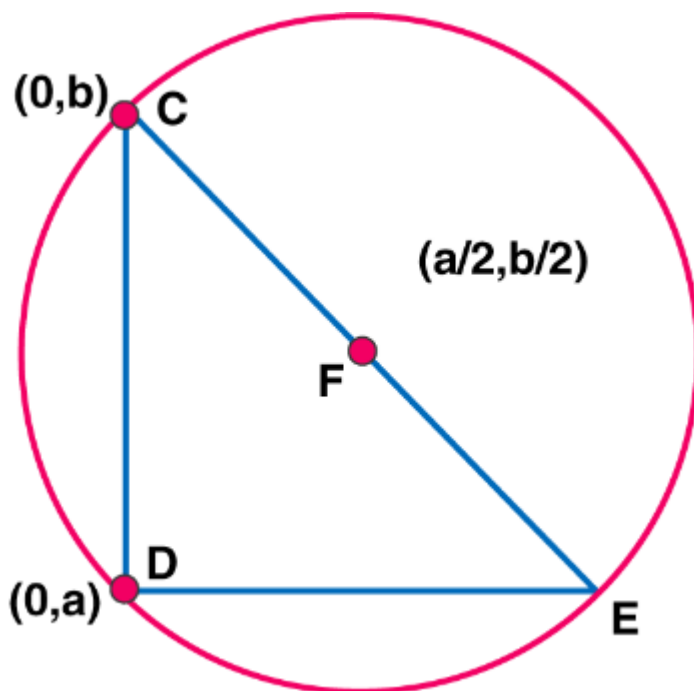
Question 3

If a circle passes through the point $(0, 0)$ $(a, 0)$, $(0, b)$ then find the coordinates of its centre.

Solution

The equation of a circle having centre (h, k) , having radius as r units, is

$$(x - h)^2 + (y - k)^2 = r^2$$



Putting the values of given co-ordinates in the above expression,

$$(0, 0)$$

$$(0 - h)^2 + (0 - k)^2 = r^2$$

$$h^2 + k^2 = r^2$$

$$(a, 0)$$

$$(a - h)^2 + (0 - k)^2 = r^2$$

$$a^2 + h^2 - 2ah + k^2 = r^2 \text{ — (1)}$$

$$(0, b)$$

$$(0 - h)^2 + (b - k)^2 = r^2$$

$$h^2 + b^2 + k^2 - 2bk = r^2 \text{ — (2)}$$

On solving equations (1) & (2), respectively,

$$a(a - 2h) = 0$$

$$b(b - 2k) = 0$$

$$\text{So, } a = 0 \text{ or } 2h$$

$$b = 0 \text{ or } 2k \text{ respectively.}$$

Since the circle passes through the centre (0, 0), so the co-ordinates are

$$a = 2h, h = a/2$$

$$b = 2k, k = b/2$$

The co-ordinates of the centre are (a/2, b/2)

Question 4

Find the equation of the circle which touches x-axis and whose centre is (1, 2).

Solution

Since the circle has a centre (1, 2) and also touches x-axis.

Radius of the circle is, $r = 2$

The equation of a circle having centre (h, k), having radius as r units, is

$$(x - h)^2 + (y - k)^2 = r^2$$

So, the equation of the required circle is:

$$(x - 1)^2 + (y - 2)^2 = 2^2$$

$$\Rightarrow x^2 - 2x + 1 + y^2 - 4y + 4 = 4$$

$$\Rightarrow x^2 + y^2 - 2x - 4y + 1 = 0$$

The equation of the circle is $x^2 + y^2 - 2x - 4y + 1 = 0$.

Question 5

If the lines $3x - 4y + 4 = 0$ and $6x - 8y - 7 = 0$ are tangents to a circle, then find the radius of the circle.

Solution

Given both lines are parallel & tangent to the circle.

Then distance between these two lines must be the diameter of the circle.

Lines:

$$3x - 4y + 4 = 0$$

$$3x - 4y - 3.5 = 0 \text{ (Equating the co-efficient of the equation)}$$

Distance (between 2 || lines) = $\left(\frac{|c-p|}{\sqrt{a^2+b^2}} \right)$ for the two given equations,
 $ax + by + c = 0$ & $ax + by + p = 0$

$$\text{Distance} = \left(\frac{|4 - (-3.5)|}{\sqrt{3^2 + (-4)^2}} \right) = \left(\frac{|7.5|}{\sqrt{9+16}} \right) = \frac{7.5}{5} = 1.5 \text{ units}$$

$$\text{Radius} = \frac{\text{Diameter}}{2} = \frac{1.5}{2}$$

$$= 0.75 \text{ units.}$$

Hence, the radius of the circle is $0.75 = \frac{3}{4}$ units.

Question 6

Find the equation of a circle which touches both the axes and the line $3x - 4y + 8 = 0$ and lies in the third quadrant.

Solution

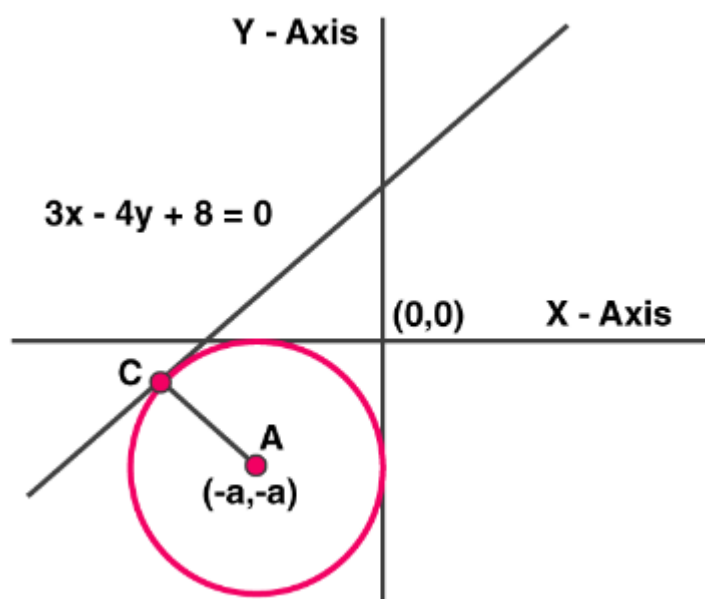
As the circle lies in third quadrant, then the centre is $(-a, -a)$.

$$\text{Perpendicular Distance (Between a point and line)} = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

Whereas the point is (x_1, y_1) and the line is expressed as $ax + by + c = 0$

The line which touches the circle is $3x - 4y + 8 = 0$, which is a tangent to the circle.

The perpendicular distance = a units (radius of the circle)



$$a = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

$$a = \frac{|3(-a) - 4(-a) + 8|}{\sqrt{3^2 + 4^2}}$$

$$a = \frac{|-3a + 4a + 8|}{\sqrt{9 + 16}}$$

$$a = \frac{|a + 8|}{\sqrt{25}}$$

$$a = \frac{a + 8}{5}$$

$$5a = a + 8$$

$$a = 2$$

Co-ordinates of the centre of the circle = $(-2, -2)$

Since, the equation of a circle having centre (h, k) , having radius as "r" units, is

$$(x - h)^2 + (y - k)^2 = r^2$$

$$(x - (-2))^2 + (y - (-2))^2 = 2^2$$

$$(x + 2)^2 + (y + 2)^2 = 4$$

$$x^2 + 4x + 4 + y^2 + 4y + 4 - 4 = 0$$

$$x^2 + y^2 + 4x + 4y + 4 = 0$$

The equation of the given circle is $x^2 + y^2 + 4x + 4y + 4 = 0$.

The equation of the given circle is $x^2 + y^2 + 4x + 4y + 4 = 0$.

Question 7

If one end of a diameter of the circle $x^2 + y^2 - 4x - 6y + 11 = 0$ is (3, 4), then find the coordinate of the other end of the diameter.

Solution

Given equation of the circle,

$$x^2 - 4x + y^2 - 6y + 11 = 0$$

$$x^2 - 4x + 4 + y^2 - 6y + 9 + 11 - 13 = 0$$

the above equation can be written as

$$x^2 - 2(2)x + 2^2 + y^2 - 2(3)y + 3^2 + 11 - 13 = 0$$

on simplifying we get

$$(x - 2)^2 + (y - 3)^2 = 2$$

$$(x - 2)^2 + (y - 3)^2 = (\sqrt{2})^2$$

Since, the equation of a circle having centre (h, k), having radius as r units, is

$$(x - h)^2 + (y - k)^2 = r^2$$

We have centre = (2, 3)

The centre point is the mid-point of the two ends of the diameter of a circle.

Let the points be (p, q)

$$(p + 3)/2 = 2 \text{ and } (q + 4)/2 = 3$$

$$p + 3 = 4 \text{ \& } q + 4 = 6$$

$$p = 1 \text{ \& } q = 2$$

Hence, the other ends of the diameter are (1, 2).

Question 8

Find the equation of the circle having (1, -2) as its centre and passing through $3x + y = 14$, $2x + 5y = 18$

Solution

Solving the given equations,

$$3x + y = 14 \dots\dots\dots 1$$

$$2x + 5y = 18 \dots\dots\dots 2$$

Multiplying the first equation by 5, we get

$$15x + 5y = 70 \dots\dots\dots 3$$

$$2x + 5y = 18 \dots\dots\dots 4$$

Subtract equation 4 from 3 we get

$$13x = 52,$$

$$\text{Therefore } x = 4$$

Substituting $x = 4$, in equation 1, we get

$$3(4) + y = 14$$

$$y = 14 - 12 = 2$$

So, the point of intersection is (4, 2)

Since, the equation of a circle having centre (h, k), having radius as r units, is

$$(x - h)^2 + (y - k)^2 = r^2$$

Putting the values of (4, 2) and centre co-ordinates (1,-2) in the above expression, we get

$$(4 - 1)^2 + (2 - (-2))^2 = r^2$$

$$3^2 + 4^2 = r^2$$

$$r^2 = 9 + 16 = 25$$

$$r = 5 \text{ units}$$

So, the expression is

$$(x - 1)^2 + (y - (-2))^2 = 5^2$$

Expanding the above equation we get

$$x^2 - 2x + 1 + (y + 2)^2 = 25$$

$$x^2 - 2x + 1 + y^2 + 4y + 4 = 25$$

$$x^2 - 2x + y^2 + 4y - 20 = 0$$

Hence the required expression is $x^2 - 2x + y^2 + 4y - 20 = 0$.

Question 9

If the line $y = \sqrt{3}x + k$ touches the circle $x^2 + y^2 = 16$, then find the value of k .

Solution

Since, the equation of a circle having centre (h, k) , having radius as r units, is

$$(x - h)^2 + (y - k)^2 = r^2$$

$$(x - 0)^2 + (y - 0)^2 = 4^2$$

Perpendicular Distance between a point $(0, 0)$ & the line

$$y = \sqrt{3}x + k \text{ or } \sqrt{3}x - y + k = 0$$

Perpendicular Distance (Between a point and line) =

Whereas the point is (x_1, y_1) and the line is expressed as $ax + by + c = 0$

$$D = \frac{\{|\sqrt{3}(0) + (0)(-1) + k|\}}{\sqrt{(\sqrt{3})^2 + 1^2}} = \frac{\{|k|\}}{\sqrt{3 + 1}} = \frac{\{k\}}{\sqrt{4}} = \frac{k}{2} \quad \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

$$\frac{k}{2} = 4 \quad (\text{Radius} = 4, \text{ Given})$$

$$k = 8$$

Hence, the required value of k is 8.

Hence, the required value of k is 8.

Question 10

Find the equation of a circle concentric with the circle $x^2 + y^2 - 6x + 12y + 15 = 0$ and has double of its area.

Solution

Given equation of the circle is

$$x^2 - 6x + y^2 + 12y + 15 = 0$$

The above equation can be written as

$$x^2 - 2(3)x + 3^2 + y^2 + 2(6)y + 6^2 + 15 - 9 + 36 = 0$$

$$(x - 3)^2 + (y - (-6))^2 - 30 = 0$$

$$(x - 3)^2 + (y - (-6))^2 = (\sqrt{30})^2$$

Since, the equation of a circle having centre (h, k), having radius as r units, is

$$(x - h)^2 + (y - k)^2 = r^2$$

$$\text{Centre} = (3, -6)$$

$$\text{Area of inner circle} = \pi r^2 = 22/7 \times 30 = 30\pi \text{ units square}$$

$$\text{Area of outer circle} = 2 \times 30\pi = 60\pi \text{ units square}$$

$$\text{So, } \pi r^2 = 60\pi$$

$$r^2 = 60$$

Equation of outer circle is,

$$(x - 3)^2 + (y - (-6))^2 = (\sqrt{60})^2$$

$$x^2 - 6x + 9 + y^2 - 12y + 36 = 60$$

$$x^2 - 6x + y^2 + 12y + 45 - 60 = 0$$

$$x^2 - 6x + y^2 + 12y - 15 = 0$$

Hence, the required equation of the circle is $x^2 - 6x + y^2 + 12y - 15 = 0$.

Question 11

If the latus rectum of an ellipse is equal to half of minor axis, then find its eccentricity.

Solution

$$\text{Equation of an ellipse} = \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Whereas

$$\text{Length of latus rectum} = \frac{2b^2}{a}$$

$$\text{Length of minor axis} = 2b$$

So,

$$\frac{2b}{2} = \frac{2b^2}{a} \quad (\text{Given})$$

$$Ab = 2b^2$$

$$2b^2 - ab = 0$$

$$b(2b - a) = 0$$

$$\text{So, } b = 0 \text{ or } a = 2b$$

$$b^2 = a^2(1 - e^2)$$

$$b^2 = (2b)^2(1 - e^2)$$

$$b^2 = 4b^2(1 - e^2)$$

$$1 - e^2 = \frac{1}{4}$$

$$e^2 = \frac{3}{4}$$

$$e = \frac{\sqrt{3}}{2}$$

$$\text{Hence, the eccentricity is } \frac{\sqrt{3}}{2}$$

$$Ab = 2b^2$$

$$2b^2 - ab = 0$$

$$b(2b - a) = 0$$

$$\text{So, } b = 0 \text{ or } a = 2b$$

$$b^2 = a^2(1 - e^2)$$

$$b^2 = (2b)^2(1 - e^2)$$

$$b^2 = 4b^2(1 - e^2)$$

On rearranging we get

$$1 - e^2 = \frac{1}{4}$$

$$e^2 = \frac{3}{4}$$

$$e = \frac{\sqrt{3}}{2}$$

Hence, the eccentricity is $\frac{\sqrt{3}}{2}$

Question 12

Given the ellipse with equation $9x^2 + 25y^2 = 225$, find the eccentricity and foci.

Solution

We know that equation of an ellipse is

$$= \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

Also we have,

$$\text{Length of latus rectum} = \frac{2b^2}{a}$$

$$\text{Length of minor axis} = 2b$$

$$9x^2 + 25y^2 = 225$$

Dividing the above equation by 225,

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

The above equation can be written as

$$\frac{x^2}{5^2} + \frac{y^2}{3^2} = 1$$

From the above equation we have,

$$a = 5, b = 3$$

$$b^2 = a^2 (1 - e^2)$$

$$3^2 = 5^2 (1 - e^2)$$

On rearranging we get

$$\frac{9}{25} = 1 - e^2$$

$$e^2 = 1 - \frac{9}{25}$$

Taking LCM and simplifying we get

$$e^2 = \frac{16}{25}$$

$$e = \frac{4}{5}$$

$$3^2 = 5^2 (1 - e^2)$$

On rearranging we get

$$\frac{9}{25} = 1 - e^2$$

$$e^2 = 1 - \frac{9}{25}$$

Taking LCM and simplifying we get

$$e^2 = \frac{16}{25}$$

$$e = \frac{4}{5}$$

We know foci = $(\pm a e, 0)$

$$= \left(\pm 5 \times \frac{4}{5}, 0 \right) = (\pm 4, 0)$$

Hence, the eccentricity is $4/5$ and foci is $(\pm 4, 0)$.

Question 13

If the eccentricity of an ellipse is $5/8$ and the distance between its foci is 10, then find latus rectum of the ellipse.

Solution

We know that equation of an ellipse

$$= \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

Also we have

$$\text{Length of latus rectum} = \frac{2b^2}{a}$$

$$\text{Length of minor axis} = 2b$$

$$e = \frac{5}{8}$$

We know that foci = $(\pm a e, 0)$

Given that distance between foci = 10

$$2ae = 10$$

$$2 \times a \times \frac{5}{8} = 10$$

$$a = 8$$

$$b^2 = a^2 (1 - e^2)$$

$$b^2 = 8^2 \left(1 - \left(\frac{5}{8} \right)^2 \right)$$

$$b^2 = 8^2 \left(1 - \left(\frac{5}{8} \right)^2 \right)$$

On simplifying we get

$$b^2 = 64 \left(1 - \frac{25}{64} \right)$$

$$b^2 = 64 \times \frac{39}{64} = 39$$

$$\text{Length of Latus Rectum} = \frac{2b^2}{a} = 2 \times \frac{39}{8} = 9.75$$

Hence, the length of latus rectum is 9.75 units.

Question 14

Find the equation of ellipse whose eccentricity is $\frac{2}{3}$, latus rectum is 5 and the centre is (0, 0).

Solution

We know that equation of an ellipse

$$= \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Also we have,

$$\text{Length of latus rectum} = \frac{2b^2}{a}$$

$$\text{Length of minor axis} = 2b$$

Given

$$e = \frac{2}{3}$$

$$b^2 = a^2 (1 - e^2) \dots\dots\dots 1$$

$$\text{Length of Latus Rectum} = \frac{2b^2}{a} = 5$$

$$b^2 = 2.5 a \dots\dots\dots 2$$

Now by substituting the values in equation 1 we get

$$\therefore 2.5 a = a^2 \left(1 - \left(\frac{2}{3} \right)^2 \right)$$

On simplifying and rearranging we get

$$2.5 a = a^2 \left(1 - \frac{4}{9} \right)$$

$$2.5 a = a^2 \left(\frac{5}{9} \right)$$

$$22.5a = 5a^2$$

$$5a^2 - 22.5a = 0$$

$$5a(a - 4.5) = 0$$

$a = 0$ {Not Possible, as the length of latus rectum is 5 units} or $4.5 = 9/2$

Now by substituting the values in equation 2 we get

$$b^2 = 2.5 \times 4.5 = 11.25 = \frac{45}{4}$$

Again we have equation of an ellipse

$$= \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

By substituting this values we get

$$\frac{x^2}{\left(\frac{9}{2}\right)^2} + \frac{y^2}{\frac{45}{4}} = 1$$

$$\frac{4x^2}{81} + \frac{4y^2}{45} = 1$$

∴ Equation of an ellipse is $\frac{4x^2}{81} + \frac{4y^2}{45} = 1$

Question 15

Find the distance between the directrices of the ellipse $x^2/36 + y^2/20 = 1$

Solution

We know that equation of an ellipse

$$= \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

Also we have Length of latus rectum = $\frac{2b^2}{a}$

Length of minor axis = $2b$

$$b^2 = a^2 (1 - e^2) \dots\dots\dots 1$$

Now by substituting we get

$$\frac{x^2}{(\sqrt{36})^2} + \frac{y^2}{(\sqrt{20})^2} = 1$$

$$\frac{x^2}{(6)^2} + \frac{y^2}{(2\sqrt{5})^2} = 1$$

From equation 1, we have

$$20 = 36 (1 - e^2)$$

On rearranging we get

$$\frac{20}{36} = 1 - e^2$$

$$\frac{20}{36} = 1 - e^2$$

$$e^2 = 1 - \frac{20}{36}$$

$$e^2 = \frac{16}{36} = \frac{4}{9}$$

$$e = \frac{2}{3}$$

$$\text{Directrices} = \pm \frac{a}{e}$$

$$\text{Distance between directrices} = \frac{2a}{e} = \frac{2 \times 6}{\frac{2}{3}} = 18$$

Hence the distance between directrices is 18 units.

Question 16

Find the coordinates of a point on the parabola $y^2 = 8x$ whose focal distance is 4.

Solution

We know that equation of an ellipse is $y^2 = 4ax$,

Also we have length of latus rectum = $4a$

Now by comparing the above two equations,

$$4a = 8$$

Therefore

$$a = 2$$

$$y^2 = 8 \times 2 = 16$$

$$y = \pm 4 \text{ and } x = 2$$

Hence the co – ordinates are (2, 4) and (2, -4).

Question 17

Find the length of the line-segment joining the vertex of the parabola $y^2 = 4ax$ and a point on the parabola where the line-segment makes an angle θ to the x-axis.

Solution

We know that equation of an ellipse is $y^2 = 4ax$,

Also we have

Length of latus rectum = $4a$

Let the point on parabola be (i, j)

From the figure, slope of OP = $\tan\theta = \frac{j}{i}$

$$j = i \tan\theta$$

Squaring the above equation on both sides we get

$$j^2 = i^2 \tan^2\theta \dots\dots 1$$

$$j^2 = 4ai \dots\dots 2$$

$$OP = \sqrt{i^2 + j^2}$$

By substituting the value of j then we get

Squaring the above equation on both sides we get

$$j^2 = i^2 \tan^2 \theta \dots\dots 1$$

$$j^2 = 4ai \dots\dots 2$$

$$OP = \sqrt{i^2 + j^2}$$

By substituting the value of j then we get

$$OP = \sqrt{i^2 + (i \tan \theta)^2}$$

$$= \sqrt{i^2 + i^2 \tan^2 \theta}$$

Taking i^2 common we get

$$= \sqrt{i^2 (1 + \tan^2 \theta)}$$

We know that

$$1 + \tan^2 \theta = \sec^2 \theta$$

$$= \sqrt{i^2 \sec^2 \theta}$$

The above equation can be written as

$$= i \sec \theta$$

Now by equating the equations 1 and 2 we get

$$4ai = i^2 \tan^2 \theta$$

On rearranging we get

$$i^2 \tan^2 \theta - 4ai = 0$$

Taking i common

$$i(i \tan^2 \theta - 4a) = 0$$

$i = 0$ which is not Possible

Or

$$i \tan^2 \theta - 4a = 0$$

$$i = \frac{4a}{\tan^2 \theta} \qquad \qquad \qquad 4a \qquad \qquad \qquad 4a \times \cos^2 \theta \times 1$$

$$i = \frac{4a}{\tan^2 \theta}$$

$$\text{Now } OP = i \sec \theta = \frac{4a}{\tan^2 \theta} \times \sec \theta = \frac{4a \times \cos^2 \theta \times 1}{\sin^2 \theta \times \cos \theta} = 4a \cot \theta \operatorname{cosec} \theta$$

Hence, the length of line segment is $4a \cot \theta \operatorname{cosec} \theta$ units.

Question 18

If the points (0, 4) and (0, 2) are respectively the vertex and focus of a parabola, then find the equation of the parabola.

Solution

Given

Vertex = (0, 4) Focus = (0, 2)

So, the directrix of the parabola is $y = 6$,

Since, distance of (x, y) from (0, 2) and perpendicular distance from (x, y) to directrix are always equal.

Using Distance Formula & Perpendicular Distance Formula,

Perpendicular Distance (Between a point and line) is

$$= \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}},$$

Where the point is (x_1, y_1) and the line is expressed as $ax + by + c = 0$ i.e.,

$x(0) + y - 6 = 0$ & (x, y)

Distance between the point of intersection & centre

We know distance formula = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Now by substituting the values we get

$$\sqrt{((x - 0)^2 + (y - 2)^2)} = \frac{|x(0) + y(1) - 6|}{\sqrt{0^2 + (-1)^2}} = \frac{y - 6}{1} = y - 6$$

Squaring both the sides,

$$\left[\sqrt{((x - 0)^2 + (y - 2)^2)} \right]^2 = (y - 6)^2$$

$$x^2 + y^2 - 4y + 4 = y^2 - 12y + 36$$

$$x^2 + 8y - 32 = 0$$

Hence, the required equation is $x^2 + 8y - 32 = 0$.

Question 19

If the line $y = mx + 1$ is tangent to the parabola $y^2 = 4x$ then find the value of m.

Solution

Given equations are,

$$y = mx + 1 \text{ \& } y^2 = 4x$$

By solving given equations we get

$$(mx + 1)^2 = 4x$$

Expanding the above equation we get

$$m^2x^2 + 2mx + 1 = 4x$$

On rearranging we get

$$m^2x^2 + 2mx - 4x + 1 = 0$$

$$m^2x^2 + x(2m - 4) + 1 = 0$$

As the line touches the parabola, above equation must have equal roots,

$$\text{Discriminant (D)} = 0$$

$$(2m - 4)^2 - 4(m^2)(1) = 0$$

$$4m^2 - 16m + 16 - 4m^2 = 0$$

$$-16m + 16 = 0$$

$$-m + 1 = 0$$

$$m = 1$$

Hence, the required value of m is 1.

Question 20

If the distance between the foci of a hyperbola is 16 and its eccentricity is $\sqrt{2}$, then obtain the equation of the hyperbola.

Solution

We know that equation of Hyperbola

$$= \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Also we have foci = $(\pm a e, 0)$

Given distance between foci is $2ae = 16$

$$e = \sqrt{2}$$

$$2 \times a \times \sqrt{2} = 16$$

$$a = \frac{16}{2 \times \sqrt{2}} = \frac{8}{\sqrt{2}} = 4\sqrt{2}$$

$$\because b^2 = a^2 (e^2 - 1)$$

$$b^2 = (4\sqrt{2})^2 ((\sqrt{2})^2 - 1)$$

$$= 32(2 - 1) = 32$$

$$\therefore \text{Equation is } \frac{x^2}{32} - \frac{y^2}{32} = 1$$

Question 21

Find the eccentricity of the hyperbola $9y^2 - 4x^2 = 36$.

Solution

Given

$$9y^2 - 4x^2 = 36$$

Dividing the above equation by 36, we get

$$\frac{y^2}{4} - \frac{x^2}{9} = 1$$

The above equation can be written as

$$\frac{y^2}{2^2} - \frac{x^2}{3^2} = 1$$

We know that equation of Hyperbola = $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

Now by comparing the equations, we get

$$a = 2 \text{ and } b = 3$$

$$b^2 = a^2 (e^2 - 1)$$

$$3^2 = (2)^2 [(e)^2 - 1]$$

$$9 = 4 (e^2 - 1)$$

$$e^2 - 1 = \frac{9}{4}$$

$$\frac{y^2}{2^2} - \frac{x^2}{3^2} = 1$$

We know that equation of Hyperbola = $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

Now by comparing the equations, we get

$$a = 2 \text{ and } b = 3$$

$$b^2 = a^2 (e^2 - 1)$$

$$3^2 = (2)^2 [(e)^2 - 1]$$

$$9 = 4 (e^2 - 1)$$

$$e^2 - 1 = \frac{9}{4}$$

$$e^2 = 1 + \frac{9}{4} = \frac{13}{4}$$

$$e = \frac{\sqrt{13}}{2}$$

Hence, the eccentricity of given hyperbola is $\frac{\sqrt{13}}{2}$

Question 22

Find the equation of the hyperbola with eccentricity $3/2$ and foci at $(\pm 2, 0)$.

Solution

Given

$$e = \frac{3}{2}$$

We have foci = $(\pm a e, 0) = (\pm 2, 0)$

Therefore the hyperbola lies on x – axis,

$$\text{Equation is } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Given $a e = 2$

$$a \times \frac{3}{2} = 2$$

$$a = \frac{4}{3}$$

$$\therefore b^2 = a^2 (e^2 - 1)$$

$$\begin{aligned} b^2 &= \left(\frac{4}{3}\right)^2 \left(\left(\frac{3}{2}\right)^2 - 1\right) \\ &= \frac{16}{9} \left(\frac{9}{4} - 1\right) = \frac{16}{9} \times \frac{5}{4} = \frac{20}{9} \end{aligned}$$

$$\text{Equation is } \frac{x^2}{\left(\frac{4}{3}\right)^2} - \frac{y^2}{\frac{20}{9}} = 1$$

$$\frac{9x^2}{16} - \frac{9y^2}{20} = 1$$

$$\text{Hence, the required equation is } \frac{9x^2}{16} - \frac{9y^2}{20} = 1$$

$$\text{Equation is } \frac{x^2}{\left(\frac{4}{3}\right)^2} - \frac{y^2}{\frac{20}{9}} = 1$$

$$\frac{9x^2}{16} - \frac{9y^2}{20} = 1$$

$$\text{Hence, the required equation is } \frac{9x^2}{16} - \frac{9y^2}{20} = 1$$

Long Answer type:

Question 23

If the lines $2x - 3y = 5$ and $3x - 4y = 7$ are the diameters of a circle of area 154square units, then obtain the equation of the circle.

Solution

Since, diameters of a circle intersect at the centre of a circle,

$$2x - 3y = 5 \dots\dots\dots 1$$

$$3x - 4y = 7 \dots\dots\dots 2$$

Solving the above equations,

Multiplying equation 1 by 3 we get

$$6x - 9y = 15$$

Multiplying equation 2 by 2 we get

$$6x - 8y = 14$$

$$y = 1$$

$$y = -1$$

Putting $y = -1$, in equation 1, we get

$$2x - 3(-1) = 5$$

$$2x + 3 = 5$$

$$2x = 2$$

$$x = 1$$

Coordinates of centre = (1,-1)

Given area = 154

$$\text{Area} = \pi r^2 = 154$$

$$22/7 \times r^2 = 154$$

$$r^2 = 154 \times 7/22$$

$$r = 7 \text{ units}$$

Since, the equation of a circle having centre (h, k), having radius as r units, is

$$(x - h)^2 + (y - k)^2 = r^2$$

$$(x - 1)^2 + (y - (-1))^2 = 7^2$$

$$x^2 - 2x + 1 + (y + 1)^2 = 49$$

$$x^2 - 2x + 1 + y^2 + 2y + 1 - 49 = 0$$

$$x^2 - 2x + y^2 + 2y - 47 = 0$$

Hence the required equation of the given circle is $x^2 - 2x + y^2 + 2y - 47 = 0$.

Question 24

Find the equation of the circle which passes through the points (2, 3) and (4, 5) and the centre lies on the straight line $y - 4x + 3 = 0$.

Solution

Since, the equation of a circle having centre (h, k), having radius as r units, is

$$(x - h)^2 + (y - k)^2 = r^2 \dots\dots\dots 1$$

Substituting (2, 3) & (4, 5) in the above equation, we get

$$(2 - h)^2 + (3 - k)^2 = r^2$$

$$4 - 4h + h^2 + 9 + k^2 - 6k = r^2$$

$$h^2 - 4h + k^2 - 6k + 13 = r^2 \dots\dots\dots 2$$

$$(4 - h)^2 + (5 - k)^2 = r^2$$

$$16 - 8h + h^2 + 25 + k^2 - 10k = r^2$$

$$h^2 - 8h + k^2 - 10k + 41 = r^2 \dots\dots\dots 3$$

Equating both the equations 2 & 3, as their RHS are equal, we get

$$h^2 - 4h + k^2 - 6k + 13 = h^2 - 8h + k^2 - 10k + 41$$

On simplifying we get

$$8h - 4h + 10k - 6k = 41 - 13$$

$$4h + 4k = 28$$

$$h + k = 7 \dots\dots\dots 4$$

As centre lies on the given line, so it satisfies the values too,

$$k - 4h + 3 = 0 \dots\dots\dots 5$$

Solving equations 3 and 4 simultaneously,

$$h + k = 7$$

$$-4h + k = -3$$

Subtracting both the equations, we get

$$5h = 10$$

$$h = 2$$

$$2 + k = 7$$

$$k = 5$$

Putting $h = 2$ & $k = 5$ in equation 2,

$$h^2 - 4h + k^2 - 6k + 13 = r^2$$

$$2^2 - 4(2) + 5^2 - 6(5) + 13 = r^2$$

$$4 - 8 + 25 - 30 + 13 = r^2$$

$$r^2 = 4$$

$$r = 2 \text{ units}$$

Putting the values of $h = 2$, $k = 5$ & $r = 2$, respectively in equation 1,

$$(x - h)^2 + (y - k)^2 = r^2$$

$$(x - 2)^2 + (y - 5)^2 = 2^2$$

$$x^2 - 4x + 4 + y^2 - 10y + 25 = 4$$

$$x^2 - 4x + y^2 - 10y + 25 = 0$$

Hence, the required equation is $x^2 - 4x + y^2 - 10y + 25 = 0$.

Question 25

Find the equation of a circle whose centre is $(3, -1)$ and which cuts off a chord of length 6 units on the line $2x - 5y + 18 = 0$.

Solution

Given equation of the chord is,

$$2x - 5y + 18 = 0 \dots\dots\dots 1$$

$$5y = 2x + 18$$

$$y = \frac{2}{5}x + \frac{18}{5}$$

As we have, $y = mx + C$

Where, m is the slope of the line,

$$m = \frac{2}{5}$$

Slope of the line perpendicular to the chord,

$$m' = -\frac{5}{2}$$

As the product of slope of perpendicular lines = -1,

$$y - y_1 = m' (x - x_1)$$

$$y - (-1) = -\frac{5}{2}(x - 3)$$

$$2y + 2 = -5x + 15$$

$$5x + 2y = 13 \dots\dots\dots 2 \text{ [Equation of line passing from centre and cutting the chord]}$$

Solving both the equations,

$$2x - 5y = -18 \text{ \& } 5x + 2y = 13$$

Multiplying the equation 1 & equation 2 by 2 & 5 respectively, we get

$$4x - 10y = -36$$

$$25x + 10y = 65$$

$$29x = 29$$

$$x = 1$$

$$4x - 10y = -36$$

$$25x + 10y = 65$$

$$29x = 29$$

$$x = 1$$

$$2(1) - 5y = -18$$

$$2 - 5y + 18 = 0$$

$$5y = 20$$

$$y = 4$$

Point of intersection at chord and radius = (1, 4)

Distance between the point of intersection & centre

$$\text{Distance formula} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(1 - 3)^2 + (4 - (-1))^2}$$

$$= \sqrt{(-2)^2 + (4 + 1)^2}$$

$$= \sqrt{4 + (5)^2}$$

$$= \sqrt{4 + 25}$$

$$= \sqrt{29} \text{ units}$$

Using Pythagoras Theorem,

$$(\text{Hypotenuse})^2 = (\text{Base})^2 + (\text{Perpendicular})^2$$

$$= (3)^2 + (\sqrt{29})^2 = 29 + 9$$

$$= \sqrt{38}$$

$$\text{Hypotenuse} = \sqrt{38} \text{ units (radius)}$$

Since, the radius bisects the chord into two equal halves,

Since, the equation of a circle having centre (h, k), having radius as r units, is

$$(x - h)^2 + (y - k)^2 = r^2$$

$$(x - 3)^2 + (y - (-1))^2 = (\sqrt{38})^2$$

$$x^2 - 6x + 9 + (y + 1)^2 = 38$$

$$x^2 - 6x + y^2 + 2y + 1 + 9 - 38 = 0$$

$$x^2 - 6x + y^2 + 2y - 28 = 0$$

Hence, the required equation of the circle is $x^2 - 6x + y^2 + 2y - 28 = 0$.

Question 26

Find the equation of a circle of radius 5 which is touching another circle $x^2 + y^2 - 2x - 4y - 20 = 0$ at (5, 5).

Solution

$$\text{Given } x^2 - 2x + y^2 - 4y - 20 = 0$$

$$x^2 - 2x + 1 + y^2 - 4y + 4 - 20 - 5 = 0$$

$$(x - 1)^2 + (y - 2)^2 = 25$$

$$(x - 1)^2 + (y - 2)^2 = 5^2$$

Since, the equation of a circle having centre (h, k), having radius as r units, is

$$(x - h)^2 + (y - k)^2 = r^2$$

$$\text{Centre} = (1, 2)$$

$$\text{Point of Intersection} = (5, 5)$$

It intersects the line into 1: 1, as the radius of both the circles is 5 units.

Using Ratio Formula,

$$\frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}, \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2}$$

$$\text{Ratio} = m_1 : m_2$$

Assuming the co-ordinates of the centre of the circle be (p, q)

$$5 = \frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}$$

$$5 = \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2}$$

Now by substituting the values we get

$$\frac{(1)p + (1)1}{1 + 1} = \frac{p + 1}{2} = 5$$

$$\frac{1(q) + 1(2)}{1 + 1} = \frac{q + 2}{2} = 5$$

$$p + 1 = 10, q + 2 = 10$$

$$p = 9 \text{ \& } q = 8$$

$$\text{Co-ordinates} = (9, 8)$$

Therefore the equation is,

$$(x - h)^2 + (y - k)^2 = r^2$$

$$(x - 9)^2 + (y - 8)^2 = 5^2$$

$$x^2 - 18x + 81 + y^2 - 16y + 64 = 25$$

$$x^2 - 18x + y^2 - 16y + 145 - 25 = 0$$

$$x^2 - 18x + y^2 - 16y + 120 = 0$$

Hence, the required equation is $x^2 - 18x + y^2 - 16y + 120 = 0$.

Question 27

Find the equation of a circle passing through the point (7, 3) having radius 3 units and whose centre lies on the line $y = x - 1$.

Solution

Since, the equation of a circle having centre (h, k), having radius as r units, is

$$(x - h)^2 + (y - k)^2 = r^2 \text{ Centre lies on the line i.e., } y = x - 1,$$

$$\text{Co - Ordinates are } (h, k) = (h, h - 1)$$

$$(x - h)^2 + (y - k)^2 = r^2$$

$$(7 - h)^2 + (3 - (h - 1))^2 = 3^2$$

$$49 + h^2 - 14h + (3 - h + 1)^2 = 9$$

On rearranging we get

$$h^2 - 14h + 49 + 16 + h^2 - 8h - 9 = 0$$

$$2h^2 - 22h + 56 = 0$$

$$h^2 - 11h + 28 = 0$$

$$h^2 - 4h - 7h + 28 = 0$$

$$h(h - 4) - 7(h - 4) = 0$$

$$(h - 7)(h - 4) = 0$$

$$h = 7 \text{ or } 4$$

$$\text{Centre} = (7, 6) \text{ or } (4, 3)$$

$$(x - h)^2 + (y - k)^2 = r^2$$

Equation, having centre (7, 6)

$$(x - 7)^2 + (y - 6)^2 = 3^2$$

$$x^2 - 14x + 49 + y^2 - 12y + 36 - 9 = 0$$

$$x^2 - 14x + y^2 - 12y + 76 = 0$$

Equation, having centre (4, 3)

$$(x - 4)^2 + (y - 3)^2 = 3^2$$

$$x^2 - 8x + 16 + y^2 - 6y + 9 - 9 = 0$$

$$x^2 - 8x + y^2 - 6y + 16 = 0$$

Hence, the required equation is $x^2 - 14x + y^2 - 12y + 76 = 0$ or $x^2 - 8x + y^2 - 6y + 16 = 0$.

Question 28

Find the equation of each of the following parabolas

(a) Directrix $x = 0$, focus at (6, 0)

(b) Vertex at (0, 4), focus at (0, 2)

(c) Focus at (-1, -2), directrix $x - 2y + 3 = 0$

Solution

(a) The distance of any point on the parabola from its focus and its directrix is same.

Given that, directrix, $x = 0$ and focus = (6, 0)

If a parabola has a vertical axis, the standard form of the equation of the parabola is $(x - h)^2 = 4p(y - k)$, where $p \neq 0$.

The vertex of this parabola is at (h, k) .

The focus is at $(h, k + p)$ & the directrix is the line $y = k - p$.

As the focus lies on x - axis,

Therefore the equation is $y^2 = 4ax$ or $y^2 = -4ax$

So, for any point $P(x, y)$ on the parabola

Distance of point from directrix = Distance of point from focus

$$x^2 = (x - 6)^2 + y^2$$

$$x^2 = x^2 - 12x + 36 + y^2$$

$$y^2 - 12x + 36 = 0$$

Hence the required equation is $y^2 - 12x + 36 = 0$.

(b) Given Vertex = (0, 4) & Focus = (0, 2)

We know the distance between the vertex and directrix is same as the distance between the vertex and focus.

Directrix is $y - 6 = 0$

For any point of P(x, y) on the parabola

Distance of P from directrix = Distance of P from focus

Perpendicular Distance (Between a point and line) = $\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$

Where the point is (x_1, y_1) and the line is expressed as $ax + by + c = 0$ i.e.,

$x(0) + y - 6 = 0$ & point = (x, y)

Distance between the point of intersection & centre

Distance formula = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ (between (x, y) & (0, 2))

$$\frac{|0(x) + (1)y - 6|}{\sqrt{0^2 + 1^2}} = \sqrt{((x - 0)^2 + (y - 2)^2)}$$

$$\frac{y - 6}{1} = \sqrt{x^2 + y^2 - 4y + 4}$$

Squaring both the sides,

$$x^2 + y^2 - 4y + 4 = (y - 6)^2$$

$$x^2 + y^2 - 4y + 4 = y^2 - 12y + 36$$

$$x^2 + 8y - 32 = 0$$

Hence, the required equation is $x^2 + 8y - 32 = 0$.

(c) Focus = $(-1, -2)$, directrix is $x - 2y + 3 = 0$

For any point (x, y) on parabola, the distance from focus to that point is always equal to the perpendicular distance from that point to the directrix,

Perpendicular Distance (Between a point and line) = $\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$

Where the point is (x_1, y_1) and the line is expressed as $ax + by + c = 0$ i.e.,

$x - 2y + 3 = 0$ and point = (x, y)

Distance between the point of intersection & centre

Distance formula = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$$\frac{|1(x) + (-2)y + 3|}{\sqrt{(-2)^2 + 1^2}} = \sqrt{((x - (-1))^2 + (y - (-2))^2)}$$

$$\frac{|x - 2y + 3|}{\sqrt{4 + 1}} = \sqrt{((x + 1)^2 + (y + 2)^2)}$$

$$\frac{x - 2y + 3}{\sqrt{5}} = \sqrt{((x + 1)^2 + (y + 2)^2)}$$

Squaring both the sides,

$$\left[\frac{x - 2y + 3}{\sqrt{5}} \right]^2 = \left[\sqrt{((x + 1)^2 + (y + 2)^2)} \right]^2$$

$$\frac{(x - 2y + 3)^2}{5} = (x + 1)^2 + (y + 2)^2$$

$$x^2 + 4y^2 + 9 - 4xy + 6x - 12y = 5 [x^2 + 2x + 1 + y^2 + 4y + 4]$$

$$x^2 + 4y^2 + 9 - 4xy + 6x - 12y = 5x^2 + 10x + 5y^2 + 20y + 25$$

$$4x^2 + y^2 + 4xy + 4x + 32y + 16 = 0$$

Hence the required equation is $4x^2 + y^2 + 4xy + 4x + 32y + 16 = 0$