

## Chapter 6: Applications of Derivatives

NCERT · Class 12 · Maths

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### Exercise 6.1

#### Question 1

Find the rate of change to the area of a circle with respect to its radius when

a)  $r = 3\text{cm}$

b)  $r = 4\text{cm}$

#### Solution

Ans: The area of a circle ( $A$ ) with radius ( $r$ ) is  $A = \pi r^2$ .

Now, for a given radius, the change in the area of the circle is given by,

$$\frac{dA}{dr} = \frac{d}{dr}(\pi r^2) = 2\pi r$$

1. When  $r = 3\text{cm}$

$$\frac{dA}{dr} = 2\pi(3)$$

$$= 6\pi$$

As a result, when the radius of the circle is 3cm, the area of the circle changes at a rate of  $6\pi$  cm.

2. When  $r = 4\text{cm}$

$$\frac{dA}{dr} = 2\pi(4)$$

$$= 8\pi$$

As a result, when the radius of the circle is 4cm, the area of the circle changes at a rate of  $8\pi$ cm.

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#### Question 2

The volume of a cube is increasing at the rate of  $8\text{cm}^3/\text{s}$ . How fast is the surface area increasing when the length of an edge is 12 cm?

#### Solution

Ans:

Let  $x$  be the length of a side,  $v$  be the volume, and  $s$  be the surface area of the cube.

When  $x$  is a function of time  $t$ , we have

$$V = x^3 \text{ and } S = 6x^2$$

It is given that  $\frac{dV}{dt} = 8\text{cm}^3/\text{s}$

Then, by chain rule we have

$$\begin{aligned} \therefore 8 &= \frac{dv}{dt} = (x^3) \\ &= \frac{d}{dx}(x^3) \cdot \frac{dx}{dt} = 3x^2 \cdot \frac{dx}{dt} \\ \Rightarrow \frac{dx}{dt} &= \frac{8}{3x^2} \end{aligned} \quad \dots\dots (1)$$

$$\begin{aligned} \text{Now, } \Rightarrow \frac{ds}{dt} &= \frac{d}{dt}(6x^2) = \frac{d}{dx}(6x^2) \cdot \frac{dx}{dt} \quad (\text{By chain rule}) \\ &= 12x \cdot \frac{dx}{dt} = \frac{32}{x} \end{aligned}$$

Thus, when  $x = 12\text{cm}$ ,

$$\begin{aligned} \frac{dS}{dt} &= \frac{32}{12}\text{cm}^2/\text{s} \\ &= \frac{8}{3}\text{cm}^2/\text{s}. \end{aligned}$$

As a result, if the cube's edge length is  $12\text{cm}$ , the surface area is rising at a rate of  $\frac{8}{3}\text{cm}^2/\text{s}$ .

### Question 3

The radius of a circle is increasing uniformly at the rate of  $3\text{cm/s}$ . Find the rate at which the area of the circle is increasing when the radius is  $10\text{ cm}$ .

### Solution

Ans: For the radius ( $r$ ), the area of a circle ( $A$ ) is  $A = \pi r^2$ .

Now, for time ( $t$ ), the rate of change of area ( $A$ ) is given by,

$$\begin{aligned} \frac{dA}{dt} &= \frac{d}{dt}(\pi r^2) \\ &= 2\pi r \frac{dr}{dt} \quad (\text{By chain rule}) \end{aligned}$$

It is given that the increase in radius of circle is,

$$\begin{aligned} \frac{dr}{dt} &= 3\text{cm/s} \\ \frac{dA}{dt} &= 2\pi r (3) = 6\pi r \end{aligned}$$

Thus, when  $r = 10\text{cm}$

$$\frac{dA}{dt} = 6\pi (10) = 60\pi\text{cm}^2/\text{s}$$

As a result, when the radius of the circle is  $10\text{cm}$ , the rate at which the area of the circle increases is  $60\pi\text{cm}^2/\text{s}$ .

#### Question 4

An edge of a variable cube is increasing at the rate of  $3\text{cm/s}$ . How fast is the volume of the cube increasing when the edge is  $10\text{cm}$  long?

#### Solution

Ans: Let  $x$  be the length of a side and  $v$  be the volume of the cube. Then,  $V = x^3$

$$\therefore \frac{dV}{dt} = 3x^2 \cdot \frac{dx}{dt} \quad (\text{By chain rule})$$

It is given that,

$$\frac{dx}{dt} = 3\text{cm/s}$$

$$\therefore \frac{dV}{dt} = 3x^2(3) = 9x^2$$

Then, when  $x = 10\text{cm}$ ,

$$\frac{dV}{dt} = 9(10)^2 = 900\text{cm}^3/\text{s}$$

As a result, when the edge is  $10\text{cm}$  long, the volume of the cube increases at the rate of  $900\text{cm}^3/\text{s}$ .

#### Question 5

A stone is dropped into a quiet lake and waves move in circles at the speed of  $5\text{ cm/s}$ . At the instant when the radius of the circular wave is  $8\text{ cm}$ , how fast is the enclosed area increasing?

#### Solution

Ans: The area of a circle (A) with radius ( $r$ ) is given by,

$$A = \pi r^2$$

Now, for time ( $t$ ), the rate of change of area (A) is given by,

$$\begin{aligned} \therefore \frac{dA}{dt} &= \frac{d}{dt}(\pi r^2) \\ &= \frac{d}{dr}(\pi r^2) \frac{dr}{dt} = 2\pi r \frac{dr}{dt} \quad (\text{By chain rule}) \end{aligned}$$

It is given that  $\frac{dr}{dt} = 5\text{cm/s}$ .

Thus, when  $r = 8\text{cm}$

$$\frac{dA}{dt} = 80\pi$$

As a result, when the circular wave's radius is 8cm, the enclosed area grows at a pace of  $80\pi\text{cm}^2/s$ .

### Question 6

The radius of a circle is increasing at the rate of 0.7cm/s. What is the rate of increase of its circumference?

### Solution

Ans. The circumference of a circle (C) with the radius ( $r$ ) is given by

$$C = 2\pi r.$$

Now, the rate of change of circumference (C) for time ( $t$ ) is given by,

$$\frac{dC}{dt} = \frac{dC}{dr} \cdot \frac{dr}{dt} \quad (\text{By chain rule})$$

$$= \frac{d}{dr}(2\pi r) \frac{dr}{dt}$$

$$= 2\pi \cdot \frac{dr}{dt}$$

It is given that  $\frac{dr}{dt} = 0.7\text{cm/s}$ .

Hence, the rate of increase of the circumference is  $1.4\pi\text{cm/s}$ .

### Question 7

The length  $x$  of a rectangle is decreasing at the rate of 5 cm/minute and the width  $y$  is increasing at the rate of 4 cm/minute. When  $x = 8\text{cm}$  and  $y = 6\text{cm}$ , find the rates of change of (a) the perimeter, and (b) the area of the rectangle.

### Solution

Ans: It is given that the length  $x$  is decreasing at the rate of  $5\text{cm/min}$  and the width  $y$  is increasing at the rate of  $4\text{cm/min}$ , i.e.,

$$\frac{dx}{dt} = -5\text{cm/min},$$

$$\frac{dy}{dt} = 4\text{cm/min}$$

(a) The perimeter (P) of a rectangle is given by

$$P = 2(x + y)$$

$$\therefore \frac{dP}{dt} = 2 \left( \frac{dx}{dt} + \frac{dy}{dt} \right)$$

$$= -2\text{cm/min}$$

Hence, the rate of decrease in the perimeter of rectangle is  $2\text{cm}/\text{min}$ .

(b) The area ( $A$ ) of a rectangle is given by,

$$A = x \times y$$

$$\frac{dA}{dt} = \frac{dx}{dt} \cdot y + x \cdot \frac{dy}{dt}$$

$$= -5y + 4x$$

When  $x = 8\text{cm}$  and  $y = 6\text{cm}$ ,

$$\frac{dA}{dt} = (-5 \times 6 + 4 \times 8) \text{cm}^2/\text{min}$$

$$= 2\text{cm}^2/\text{min}$$

Hence, the rate of increase in the area of the rectangle is  $2\text{cm}^2/\text{min}$ .

### Question 8

A balloon, which always remains spherical on inflation, is being inflated by pumping in 900 cubic centimetres of gas per second. Find the rate at which the radius of the balloon increases when the radius is 15 cm.

### Solution

Ans: The volume of a sphere ( $V$ ) with radius ( $r$ ) is given by,

$$V = \frac{4}{3}\pi r^3$$

For time ( $t$ ), the rate of change of volume ( $V$ ) is given by,

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} \quad (\text{By chain rule})$$

$$= \frac{d}{dr} \left( \frac{4}{3}\pi r^3 \right) \cdot \frac{dr}{dt}$$

$$= 4\pi r^2 \cdot \frac{dr}{dt}$$

It is given that

$$\frac{dv}{dt} = 900\text{cm}^3/\text{s}$$

$$\therefore 900 = 4\pi r^2 \cdot \frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \frac{900}{4\pi r^2}$$

$$= \frac{225}{\pi r^2}$$

Therefore, when the radius is 15cm,

$$\frac{dr}{dt} = \frac{225}{\pi(15)^2} = \frac{1}{\pi}$$

As a result, the rate at which the balloon's radius increases when the radius is  $15\text{cm}$  is  $\frac{1}{\pi}\text{cm/sec}$ .

### Question 9

A balloon, which always remains spherical has a variable radius. Find the rate at which its volume is increasing with the radius when the latter is  $10\text{ cm}$ .

### Solution

Ans: The volume of a sphere ( $V$ ) with radius ( $r$ ) is given by,

$$V = \frac{4}{3}\pi r^3$$

Rate of change of volume ( $V$ ) for time ( $t$ ) is given by,

$$\frac{dV}{dr} = \frac{d}{dr} \left( \frac{4}{3}\pi r^3 \right)$$

$$= 4\pi r^2$$

Therefore, when the radius is  $10\text{cm}$ ,

$$\frac{dV}{dr} = 4\pi(10)^2$$

$$= 400\pi$$

Hence, the rate of increase in the volume of the balloon is  $400\pi\text{cm}^3/\text{sec}$ .

### Question 10

A ladder  $5\text{ m}$  long is leaning against a wall. The bottom of the ladder is pulled along the ground, away from the wall, at the rate of  $2\text{ cm/s}$ . How fast is its height on the wall decreasing when the foot of the ladder is  $4\text{ m}$  away from the wall?

### Solution

Ans: Let the height of the wall be  $y\text{m}$  and the foot of the ladder be  $x\text{m}$  away from the wall.

Then, by Pythagoras Theorem, we have:

$$x^2 + y^2 = 25 \text{ (Length of the ladder is } 5\text{m)}$$

$$\Rightarrow y = \sqrt{25 - x^2}$$

Then, the rate of change of height  $y$  for time  $t$  is given by,

$$\frac{dy}{dt} = \frac{-x}{\sqrt{25-x^2}} \cdot \frac{dx}{dt}$$

It is given that  $\frac{dx}{dt} = 2\text{cm/s}$

$$\therefore \frac{dy}{dt} = \frac{-2x}{\sqrt{25-x^2}}$$

Now, when  $x = 4\text{m}$  we have:

$$\frac{dy}{dt} = \frac{8}{3}$$

Hence, the rate of decrease in the height of the ladder on the wall is  $\frac{8}{2}\text{cm/sec}$ .

### Question 11

A particle moving along the curve  $6y = x^3 + 2$ . Find the points on the curve at which the  $y$ -coordinate is changing 8 times as fast as the  $x$ -coordinate.

### Solution

Ans: Given the equation of the curve as  $6y = x^3 + 2$ .

The rate of change of the position of the particle for time

(t) is given by,

$$6\frac{dy}{dt} = 3x^2\frac{dx}{dt} + 0$$

$$\Rightarrow 2\frac{dy}{dt} = x^2\frac{dx}{dt}$$

When the particle's  $y$ -coordinate changes 8 times as fast as its  $x$ -coordinate i.e.,  $\left(\frac{dy}{dt} = 8\frac{dx}{dt}\right)$ , we have:

$$2\left(8\frac{dx}{dt}\right) = x^2\frac{dx}{dt}$$

$$\Rightarrow 16\frac{dx}{dt} = x^2\frac{dx}{dt}$$

$$\Rightarrow (x^2 - 16)\frac{dx}{dt} = 0$$

$$\Rightarrow x^2 = 16$$

$$\Rightarrow x = \pm 4$$

When  $x = -4$ ,

$$y = \frac{-31}{3}$$

and when  $x = 4$ ,

$$y = 11$$

Hence, the points required on the curve are  $(4, 11)$  and  $\left(-4, \frac{-31}{3}\right)$ .

### Question 12

The radius of an air bubble is increasing at the rate of  $1/2$  cm/s. At what rate is the volume of the bubbles of the bubble increasing when the radius is 1cm?

### Solution

Ans: The air bubble is in the shape of a sphere.

The volume of an air bubble ( $V$ ) with radius ( $r$ ) is given by,

$$V = \frac{4}{3}\pi r^3$$

Rate of change of volume ( $V$ ) for time ( $t$ ) is given by,

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} \quad (\text{By chain rule})$$

$$= \frac{d}{dr} \left( \frac{4}{3}\pi r^3 \right) \cdot \frac{dr}{dt}$$

$$= 4\pi r^2 \cdot \frac{dr}{dt}$$

It is given that  $\frac{dr}{dt} = \frac{1}{2} \text{ cm/s}$

Therefore, when  $r$  is 1cm,

$$\frac{dV}{dt} = 4\pi(1)^2 \left( \frac{1}{2} \right)$$

$$= 2\pi \text{ cm}^3/\text{s}$$

Hence, the volume of the bubble increases at the rate of  $2\pi \text{ cm}^3/\text{s}$ .

### Question 13

A balloon, which always remains spherical, has a variable diameter  $\frac{3}{2}(2x + 1)$ . Find the rate of change of its volume with respect to  $x$ .

### Solution

Ans: The volume of a sphere ( $V$ ) with radius

$$(r)$$

is given by,

$$V = \frac{4}{3}\pi r^3.$$

Diameter is  $\frac{3}{2}(2x + 1)$

$$r = \frac{3}{4}(2x + 1)$$

$$V = \frac{4}{3}\pi r^3$$

$$= \frac{9}{16}\pi(2x + 1)^3$$

Hence, the rate of change of volume with respect to  $x$  is as

$$\frac{dV}{dt} = \frac{9}{16}\pi \frac{d}{dt}(2x + 1)^3$$

$$= \frac{27}{8}\pi(2x + 1)^3.$$

### Question 14

Sand is pouring from a pipe at the rate of  $12\text{cm}^3/\text{sec}$ . The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand cone increasing when the height is 4 cm?

### Solution

Ans: The volume of a cone ( $V$ ) with radius and height ( $h$ ) is given by.

$$V = \frac{1}{3}\pi r^2 h$$

It is provided that,

$$h = \frac{1}{6}r$$

$$\Rightarrow r = 6h$$

$$\therefore V = \frac{1}{3}\pi(6h)^2 h$$

$$= 12\pi h^3$$

The rate of change of volume for time ( $t$ ) is given by

$$\frac{dV}{dt} = 12\pi \frac{d}{dh}(h^3) \cdot \frac{dh}{dt} \quad (\text{by chain rule})$$

$$= 12\pi(3h^2) \frac{dh}{dt}$$

$$= 36\pi h^2 \frac{dh}{dt}$$

It is provided that  $\frac{dV}{dt} = 12\text{cm}^2/\text{s}$ .

Therefore, when  $h = 4\text{cm}$  we have:

$$12 = 36\pi(4)^2 \frac{dh}{dt}$$

$$\Rightarrow \frac{dh}{dt} = \frac{12}{36\pi(16)}$$

$$\frac{dh}{dt} = \frac{1}{48\pi}$$

Hence, when the height of the sand cone is 4 cm, its height is increasing at the rate of  $\frac{1}{48} \text{ cm/sec}$ .

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### Question 15

The total cost  $C(x)$  in Rupees associated with the production of  $x$  units of an item is given by

$$C(x) = 0.007x^3 - 0.003x^2 + 15x + 4000$$

.

Find the marginal cost when 17 units are produced.

### Solution

Ans: The rate of change in total cost for output is known as marginal cost.

$$\begin{aligned} MC &= \frac{dC}{dx} = 0.007(3x^2) - 0.003(2x) + 15 \\ &= 0.021x^2 - 0.006x + 15 \end{aligned}$$

When

$$\begin{aligned} x = 17, MC &= 0.021(17)^2 - 0.006(17) + 15 \\ &= 0.21(289) - 0.006(17) + 15 \\ &= 6.069 - 0.102 + 15 \\ &= 20.967 \end{aligned}$$

Hence, the marginal cost when 17 units are produced is Rs.20.967.

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### Question 16

The total revenue in Rupees received from the sale of  $x$  units of a product is given by  $R(x) = 13x^2 + 26x + 15$ .

Find the marginal revenue when  $x = 7$ .

### Solution

Ans: The rate of change in total cost for the number of units sold is known as marginal cost. Let the number of units sold be  $x$ .

$$MR = \frac{dR}{dx} = 13(2x) + 26$$

$$= 26x + 26$$

When  $x = 7$ ,

$$MR = 208$$

Hence, the required marginal revenue is Rs.208.

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### Question 17

The rate of change of the area of a circle with respect to its radius  $r$  at  $r = 6\text{cm}$  is (A)  $10\pi$  (B)  $12\pi$  (C)  $8\pi$  (D)  $11\pi$ .

### Solution

Ans: The area of a circle ( $A$ ) with radius ( $r$ ) is given by,  $A = \pi r^2$

As a result, the area's rate of change in relation to its radius  $r$  is  $\frac{dA}{dt} = \frac{d}{dr}(\pi r^2) = 2\pi r$

Therefore, when  $r = 6\text{cm}$ ,

$$\frac{dA}{dr} = 2\pi \times 6$$

$$= 12\pi\text{cm}^2/\text{s}$$

Hence, the required rate of change of the area of a circle is  $12\pi\text{cm}^2/\text{s}$ .

Therefore, option B is correct.

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### Question 18

The total revenue in Rupees received from the sale of  $x$  units of a product is given by  $R(x) = 3x^2 + 36x + 5$ . The marginal revenue, when  $x = 15$  is (A) 116 (B) 96 (C) 90 (D) 126.

### Solution

Ans: The rate of change in total cost for the number of units sold is known as marginal cost.

$$\therefore MR = \frac{dR}{dx} = 6x + 36$$

Therefore, when  $x = 15$ ,

$$MR = 90 + 36 = 126$$

Hence, the required marginal revenue is Rs.126.

The correct option is D.

## Exercise 6.2

### Question 1

Show, that the function given by  $f(x) = 3x + 17$  is strictly increasing on  $R$ .

#### Solution

Ans: Let the two numbers in  $R$  be  $x_1$  and  $x_2$ .

$$x_1 < x_2 \Rightarrow 3x_1 < 3x_2$$

$$\Rightarrow 3x_1 + 17 < 3x_2 + 17$$

$$= f(x_1) < f(x_2)$$

Thus, the function is strictly increasing on  $R$ .

### Question 2

Show that the function given  $f(x) = e^{2x}$  is strictly increasing on  $R$ .

#### Solution

Ans: Let the two numbers in  $R$  be  $x_1$  and  $x_2$ .

Then, we have:

$$x_1 < x_2 \Rightarrow 2x_1 < 2x_2$$

$$\Rightarrow e^{2x_1} < e^{2x_2}$$

$$\Rightarrow f(x_1) < f(x_2)$$

Hence,  $f$  is strictly increasing on  $R$ .

### Question 3

Show that the function given by  $f(x) = \sin x$  is

A) Strictly increasing in  $\left(0, \frac{\pi}{2}\right)$

B) Strictly decreasing in  $\left(\frac{\pi}{2}, \pi\right)$

C) Neither increasing nor decreasing in  $(0, \pi)$

**Solution**

Ans: The given function is  $f(x) = \sin x$ .

$$\therefore f'(x) = \cos x$$

A) Since for each  $x \in \left(0, \frac{\pi}{2}\right)$ ,  $\cos x > 0$ , we have  $f'(x) > 0$ .

Hence,  $f$  is strictly increasing in  $\left(0, \frac{\pi}{2}\right)$ .

B) Since for each  $x \in \left(\frac{\pi}{2}, \pi\right)$ ,  $\cos x < 0$ , we have  $f'(x) < 0$ .

Hence,  $f$  is strictly increasing in  $\left(\frac{\pi}{2}, \pi\right)$

C) It is clear from the results obtained in (A) and (B) that  $f$  is neither increasing nor decreasing in  $(0, \pi)$ .

**Question 4**

Find the intervals in which the function  $f$  given by  $f(x) = 2x^2 - 3x$  is

(A) Strictly increasing

(B) Strictly decreasing

**Solution**

Ans: The function provided is  $f(x) = 2x^2 - 3x$ .

$$f'(x) = 4x - 3$$

$$\therefore f'(x) = 0$$

$$\Rightarrow x = \frac{3}{4}$$

Now, the point  $\frac{3}{4}$  divides the real line into two disjoint intervals i.e.,

$$\left(-\infty, \frac{3}{4}\right) \text{ and } \left(\frac{3}{4}, \infty\right).$$

In interval,  $\left(-\infty, \frac{3}{4}\right)$ ,  $f'(x) = 4x - 3 < 0$

Hence, the given function ( $f$ ) is strictly decreasing in interval  $\left(-\infty, \frac{3}{4}\right)$

In interval,  $\left(\frac{3}{4}, \infty\right)$ ,  $f'(x) = 4x - 3 > 0$

Hence, the given function ( $f$ ) is strictly increasing in interval  $\left(\frac{3}{4}, \infty\right)$

**Question 5**

Find the intervals in which the function  $f$  is given as  $f(x) = 2x^3 - 3x^2 - 36x + 7$

(A) strictly increasing

(B) strictly decreasing

**Solution**

Ans: The given function is

$$f(x) = 2x^3 - 3x^2 - 36x + 7$$

$$f'(x) = 6x^2 - 6x - 36$$

$$= 6(x^2 - x - 6)$$

$$= 6(x + 2)(x - 3)$$

$$\therefore f'(x) = 0$$

$$\Rightarrow x = -2, 3$$

The points  $x = -2, 3$  divide the real line into three disjoint intervals

i.e.,  $(-\infty, -2)$ ,  $(-2, 3)$ , and  $(3, \infty)$ .

In intervals  $(-\infty, -2)$  and  $(3, \infty)$ ,  $f'(x)$  is positive while in the interval  $(-2, 3)$ ,  $f'(x)$  is negative.

Hence, the given function ( $f$ ) is strictly increasing in intervals  $(-\infty, -2) \cup (3, \infty)$  while the function ( $f$ ) is strictly decreasing in interval  $(-2, 3)$ .

**Question 6**

Find the intervals in which the following functions are strictly

increasing or decreasing:

(a)  $x^2 + 2x - 5$

(b)  $10 - 6x - 2x^2$

(c)  $-2x^3 - 9x^2 - 12x + 1$

(d)  $6 - 9x - x^2$

(e)  $(x + 1)^3(x - 3)^3$

**Solution**

Ans:

a) Given that

$$f(x) = x^2 + 2x - 5$$

$$\therefore f'(x) = 2x + 2$$

Now,

$$f'(x) = 0$$

$$\Rightarrow x = -1$$

The real line is divided into two disjoint intervals  $(-\infty, -1)$  and  $(-1, \infty)$  by the point  $x = -1$ .

In interval  $(-\infty, -1)$ ,

$$\therefore f'(x) = 2x + 2 < 0$$

Therefore,  $f$  is strictly decreasing in the interval  $(-\infty, -1)$ .

Thus  $f$  is strictly decreasing for  $x < -1$

In interval  $(-1, \infty)$ ,

$$\therefore f'(x) = 2x + 2 > 0$$

Therefore,  $f$  is strictly increasing in interval  $(-1, \infty)$

Thus  $f$  is strictly increasing for  $x > -1$ .

b) Given that,

$$f(x) = 10 - 6x - 2x^2$$

$$\therefore f'(x) = -6 - 4x$$

Now,

$$f'(x) = 0$$

$$\Rightarrow x = -\frac{3}{2}$$

The real line is divided into two disjoint intervals  $(-\infty, -\frac{3}{2})$  and  $(-\frac{3}{2}, \infty)$  by the point  $x = -\frac{3}{2}$ .

In interval  $(-\infty, -\frac{3}{2})$  i.e., when  $x < -\frac{3}{2}$ ,  $f'(x) = -6 - 4x < 0$ .

$\therefore f$  is strictly increasing for  $x < -\frac{3}{2}$

In interval  $(-\frac{3}{2}, \infty)$  i.e., when  $x > -\frac{3}{2}$ ,  $f'(x) = -6 - 4x < 0$ .

$\therefore f$  is strictly increasing for  $x < -\frac{3}{2}$

c) Given that,

$$f(x) = -2x^3 - 9x^2 - 12x + 1$$

$$\begin{aligned}\therefore f'(x) &= -6x^2 - 18x - 12 \\ &= -6(x^2 + 3x + 2) \\ &= -6(x + 1)(x + 2)\end{aligned}$$

Now,

$$f'(x) = 0$$

$$\Rightarrow x = -1$$

and  $x = -2$

The real line is divided into three disjoint interval  $(-\infty, -2)$ ,  $(-2, -1)$  and  $(-1, \infty)$  by the points  $x = -1, -2$  when  $x < -2$  and  $x > -1$ ,

$$f'(x) = -6(x + 1)(x + 2) < 0$$

$\therefore f$  is strictly increasing for  $x < -2$  and  $x > -1$

Now, in interval  $(-2, -1)$  i.e., when  $-2 < x < -1$ ,

$$f'(x) = -6(x + 1)(x + 2) > 0$$

$\therefore f$  is strictly increasing for  $-2 < x < -1$

d) Given that,

$$f(x) = 6 - 9x - x^2$$

$$\therefore f'(x) = -9 - 2x$$

$$\text{Now, } f'(x) = 0 \text{ gives } x = -\frac{9}{2}$$

The real line is divided into two disjoint intervals  $(-\infty, -\frac{9}{2})$  and  $(-\frac{9}{2}, \infty)$  by the point  $x = -\frac{9}{2}$ .

In interval  $(-\infty, -\frac{9}{2})$  i.e., for  $x < -\frac{9}{2}$ ,

$\therefore f$  is strictly increasing for  $x < -\frac{9}{2}$

In interval  $(-\frac{9}{2}, \infty)$  i.e., for  $x > -\frac{9}{2}$ ,  $f'(x) = -9 - 2x > 0$

$\therefore f$  is strictly increasing for  $x > -\frac{9}{2}$

In interval  $(-\frac{9}{2}, \infty)$  i.e., for  $x > -\frac{9}{2}$ ,  $f'(x) = -9 - 2x < 0$

$\therefore f$  is strictly increasing for  $x > -\frac{9}{2}$

e) Given that,

$$f(x) = (x + 1)^3(x - 3)^3$$

$$\begin{aligned}
 f'(x) &= 3x(x+1)^2(x-3)^3 + 3(x-3)^2(x+1)^3 \\
 &= 3(x+1)^2(x-3)^2[x-3+x+1] \\
 &= 6(x+1)^2(x-3)^2(x-1)
 \end{aligned}$$

Now,

$$f'(x) = 0$$

$$\Rightarrow x = -1, 3, 1$$

The points  $x = -1, 1, 3$  divided the real line into four disjoint intervals. i.e.,

$$(-\infty, -1), (-1, 1), (1, 3) \text{ and } (3, \infty)$$

$$\text{In intervals } (-\infty, -1) \text{ and } (-1, 1), f'(x) = 6(x+1)^2(x-3)^2(x-1) < 0$$

$\therefore f$  is strictly decreasing in interval  $(-\infty, -1)$  and  $(-1, 1)$

$$\text{In intervals } (1, 3) \text{ and } (3, \infty), f'(x) = 6(x+1)^2(x-3)^2(x-1) > 0$$

$\therefore f$  is strictly decreasing in the interval  $(1, 3)$  and  $(3, \infty)$ .

### Question 7

Show that  $y = \log(1+x) - \frac{2x}{2+x}, x > -1$ , is an increasing function of

$x$  throughout its domain.

### Solution

Ans: We are given that,

$$y = \log(1+x) - \frac{2x}{2+x}$$

$$\therefore \frac{dy}{dx} = \frac{1}{1+x} - \frac{(2+x)(2) - 2x(1)}{(2+x)^2}$$

$$= \frac{x^2}{(2+x)^2}$$

Now,

$$\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{x^2}{(2+x)^2} = 0$$

$$\Rightarrow x^2 = 0$$

$$\Rightarrow x = 0$$

Since  $x > -1$  point  $x = 0$  divides the domain  $(-1, \infty)$  in two disjoint

intervals i.e.,  $-1 < x < 0$  and  $x > 0$ .

When  $-1 < x < 0$  we have:

$$x < 0 \Rightarrow x^2 > 0$$

$$x > -1 \Rightarrow (2 + x) > 0$$

$$\Rightarrow (2 + x)^2 > 0$$

$$\therefore y = \frac{x^2}{(2 + x)^2} > 0$$

Also, when  $x > 0$

$$x < 0 \Rightarrow x^2 > 0,$$

$$(2 + x)^2 > 0$$

$$\therefore y = \frac{x^2}{(2+x)^2} > 0$$

Hence, the function  $f$  is increasing throughout this domain.

---

### Question 8

Find the values of  $x$  for which  $y = [x(x - 2)]^2$  is an increasing function.

### Solution

Ans: We have,

$$y = [x(x - 2)]^2$$

$$= [x^2 - 2x]^2$$

$$\therefore \frac{dy}{dx} = 2(x^2 - 2x)(2x - 2)$$

$$= 4x(x - 2)(x - 1)$$

$$\therefore \frac{dy}{dx} = 0$$

$$\Rightarrow x = 0, 1, 2$$

The points  $x = 0, 1, 2$  divide the real line into four disjoint intervals i.e.,

$(-\infty, 0)$ ,  $(0, 1)$ ,  $(1, 2)$  and  $(2, \infty)$ .

In intervals  $(-\infty, 0)$  and  $(1, 2)$ ,  $\frac{dy}{dx} < 0$

$\therefore y$  is strictly decreasing in intervals  $(-\infty, 0)$  and  $(1, 2)$

However, in intervals  $(0, 1)$  and  $(2, \infty)$ ,  $\frac{dy}{dx} > 0$

$\therefore y$  is strictly decreasing in intervals  $(0, 1)$  and  $(2, \infty)$

$\therefore y$  is strictly decreasing in intervals  $0 < x < 1$  and  $x > 2$ .

### Question 9

Prove that  $y = \frac{4 \sin \theta}{(2 + \cos \theta)} - \theta$  is an increasing function of  $\theta$  in  $\left[0, \frac{\pi}{2}\right]$ .

### Solution

Ans: We have,

$$y = \frac{4 \sin \theta}{(2 + \cos \theta)} - \theta$$

$$\therefore \frac{dy}{d\theta} = \frac{(2 + \cos \theta)(4 \cos \theta) - 4 \sin \theta(-\sin \theta)}{(2 + \cos \theta)^2} - 1$$

$$= \frac{8 \cos \theta + 4 \cos^2 \theta + 4 \sin^2 \theta}{(2 + \cos \theta)^2} - 1$$

$$= \frac{8 \cos \theta + 4}{(2 + \cos \theta)^2} - 1$$

Now,

$$\frac{dy}{d\theta} = 0$$

$$\Rightarrow \frac{8 \cos \theta + 4}{(2 + \cos \theta)^2} = 1$$

$$\Rightarrow 8 \cos \theta + 4 = 4 + \cos^2 \theta + 4 \cos \theta$$

$$\Rightarrow \cos^2 \theta - 4 \cos \theta = 0$$

$$\Rightarrow \cos \theta = 0, 4$$

Since,

$$\cos \theta \neq 4,$$

$$\cos \theta = 0$$

$$\Rightarrow \theta = \frac{\pi}{2}$$

$$\text{Now, } \frac{dy}{d\theta} = \frac{\cos(4 - \cos \theta)}{(2 + \cos \theta)^2}$$

In the interval  $\left[0, \frac{\pi}{2}\right]$ , we have  $\cos \theta > 0$ ,

$$\text{Also, } 4 > \cos \theta \Rightarrow 4 - \cos \theta > 0$$

$$\therefore \cos \theta (4 - \cos \theta) > 0 \text{ and also } (2 + \cos \theta)^2 > 0$$

$$\Rightarrow \frac{\cos \theta (4 - \cos \theta)}{(2 + \cos \theta)^2} > 0$$

$$\Rightarrow \frac{dy}{dx} > 0$$

Therefore  $y$  is strictly increasing in the interval  $\left(0, \frac{\pi}{2}\right)$

Hence,  $y$  is increasing in the interval  $\left(0, \frac{\pi}{2}\right)$ .

---

### Question 10

Prove that the logarithmic function is strictly increasing on  $(0, \infty)$ .

#### Solution

Ans: The given function is  $f(x) \log x$ .

$$f'(x) = \frac{1}{x}$$

It is clear that for  $x > 0$ ,  $f'(x) = \frac{1}{x} > 0$

Hence,  $f(x) \log x$  is strictly increasing in the interval  $(0, \infty)$ .

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### Question 11

Prove that the function  $f$  is given by  $f(x) = x^2 - x + 1$  is neither strictly increasing nor strictly decreasing on  $(-1, 1)$ .

#### Solution

Ans: The given function is  $f(x) = x^2 - x + 1$

$$\therefore f'(x) = 2x - 1$$

Now,

$$f'(x) = 0$$

$$\Rightarrow x = \frac{1}{2}.$$

The point  $\frac{1}{2}$  divides the interval  $(-1, 1)$  into two disjoint intervals i.e.,

$$\left(-1, \frac{1}{2}\right) \text{ and } \left(\frac{1}{2}, 1\right).$$

Now, in the interval  $\left(-1, \frac{1}{2}\right)$ ,  $f'(x) = 2x - 1 < 0$

Therefore,  $f$  is strictly decreasing in the interval  $\left(-1, \frac{1}{2}\right)$ ,

However, in the interval  $\left(\frac{1}{2}, 1\right)$ ,  $f'(x) = 2x - 1 > 0$

Therefore,  $f$  is strictly decreasing in the interval  $\left(\frac{1}{2}, 1\right)$ ,

Hence,  $f$  is neither strictly increasing nor decreasing in interval  $(-1, 1)$ .

### Question 12

Which of the following functions are strictly decreasing on  $\left(0, \frac{\pi}{2}\right)$ ?

(A)

**COSX**

(B)  $\cos 2x$  (C)  $\cos 3x$  (D)  $\tan x$

### Solution

Ans.

A) Let  $f_1(x) = \cos x$

$$\therefore f_1'(x) = -\sin x$$

In interval,  $\left(0, \frac{\pi}{2}\right)$ ,  $f_1'(x) = -\sin x < 0$

$\therefore f_1(x) = \cos x$  is strictly decreasing in the interval  $\left(0, \frac{\pi}{2}\right)$ .

B) Let  $f_2(x) = \cos 2x$

$$\therefore f_2'(x) = -2 \sin 2x$$

Now,

$$0 < x < \frac{\pi}{2} \Rightarrow 0 < 2x < \pi$$

$$\Rightarrow \sin 2x > 0 \Rightarrow -2 \sin 2x < 0$$

$$\therefore f_2'(x) = -2 \sin 2x < 0 \text{ on } \left(0, \frac{\pi}{2}\right)$$

$\therefore f_2(x) = \cos 2x$  is strictly decreasing in interval  $\left(0, \frac{\pi}{2}\right)$  \$\$

C) Let  $f_3(x) = \cos 3x$

$$\therefore f_3'(x) = -3 \sin 3x$$

Now,

$$f_3'(x) = 0$$

$$\Rightarrow \sin 3x = 0$$

$$\Rightarrow 3x = \pi, x \in \left(0, \frac{\pi}{2}\right)$$

$$\Rightarrow x = \frac{\pi}{3}$$

The point  $x = \frac{\pi}{3}$  divides the interval  $\left(0, \frac{\pi}{2}\right)$  into two disjoint intervals i.e.,  $\left(0, \frac{\pi}{3}\right)$  and  $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$ .

Now, in the interval  $\left(0, \frac{\pi}{3}\right)$ ,  $f_3(x) = -3 \sin 3x < 0 \left[0 < x < \frac{\pi}{3} \Rightarrow 0 < 3x < \pi\right]$

$\therefore f_3$  is strictly decreasing in interval  $\left(0, \frac{\pi}{3}\right)$

However, in the interval  $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$ ,  $f_3(x) = -3 \sin 3x > 0 \left[\frac{\pi}{3} < x < \frac{\pi}{2} \Rightarrow \pi < 3x < \frac{3\pi}{2}\right]$

$\therefore f_3$  is strictly increasing in the interval  $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$

Hence,  $f_3$  is neither increasing nor decreasing in interval  $\left(0, \frac{\pi}{2}\right)$

D) Let  $f_4(x) = \tan x$

$$\therefore f_4'(x) = \sec^2 x$$

In interval  $\left(0, \frac{\pi}{2}\right)$ ,  $f_4'(x) = \sec^2 x > 0$

$$\therefore f_4$$

is strictly increasing in the interval  $\left(0, \frac{\pi}{2}\right)$ .

Therefore, functions  $\cos x$  and  $\cos 2x$  are strictly decreasing in  $\left(0, \frac{\pi}{2}\right)$ .

Hence, the correct options are A and B.

### Question 13

On which of the following intervals is the function  $f$  is given by

$f(x) = x^{100} + \sin x - 1$  strictly decreasing?

A.

$(0, 1)$

B.

$$\left(\frac{\pi}{2}, \pi\right)$$

C.

$$\left(0, \frac{\pi}{2}\right)$$

D.

None of these

### Solution

Ans: We have,

$$f(x) = x^{100} + \sin x - 1$$

$$\therefore f'(x) = 100x^{90} + \cos x$$

In interval  $(0, 1)$ ,  $\cos x > 0$  and  $100x^{90} > 0$

$$\therefore f'(x) > 0$$

Thus, the function  $f$  is strictly increasing in the interval  $(0, 1)$ .

In interval  $\left(\frac{\pi}{2}, \pi\right)$ ,  $\cos x < 0$  and

$$100x^{90} > 0$$

.

Also,  $100x^{90} > \cos x$

$$\therefore f'(x) > 0 \text{ in } \left(\frac{\pi}{2}, \pi\right)$$

Thus, the function  $f$  is strictly increasing in the interval  $\left(\frac{\pi}{2}, \pi\right)$ .

In the interval  $\left(0, \frac{\pi}{2}\right)$ ,  $\cos x > 0$  and  $100x^{90} > 0$

$$\therefore 100x^{90} + \cos x > 0$$

$$\Rightarrow f'(x) > 0 \text{ on } \left(0, \frac{\pi}{2}\right)$$

$\therefore f$  is strictly increasing in the interval  $\left(0, \frac{\pi}{2}\right)$

Hence, the function  $f$  is strictly decreasing in none of the intervals. The correct option is D.

### Question 14

Find the least value of  $a$  such that the function  $f$  given

$f(x) = x^2 + ax + 1$  is strictly increasing on  $(1, 2)$ .

### Solution

Ans: We have,

$$f(x) = x^2 + ax + 1$$

$$\therefore f'(x) = 2x + a$$

Now, the function  $f$  will be increasing in  $(1, 2)$  if  $f'(x) > 0$  in  $(1, 2)$ .

$$\Rightarrow 2x + a > 0$$

$$\Rightarrow 2x > -a$$

$$\Rightarrow x > \frac{-a}{2}$$

As a result, we must determine the smallest value of  $a$  such that

$$x > \frac{-a}{2}, x \in (1, 2)$$

$$\Rightarrow x > \frac{-a}{2} (1 < x < 2)$$

Thus, the least value is given by

$$\frac{-a}{2} = 1$$

$$\Rightarrow a = -2$$

Hence, the required value of  $a$  is  $-2$ .

### Question 15

Let  $I$  be any interval disjoint from  $(-1, 1)$ , prove that the function  $f$  given by  $f(x) = x + \frac{1}{x}$  is strictly increasing on  $I$ .

### Solution

Ans: We have,

$$f(x) = x + \frac{1}{x}$$

$$\therefore f'(x) = 1 - \frac{1}{x^2}$$

Now,

$$f'(x) = 0$$

$$\Rightarrow x = \pm 1$$

The real line is divided into three disjoint intervals i.e.,

$(-\infty, -1), (-1, 1), (1, \infty)$  by the points  $x = 1, -1$ .

In the interval  $(-1, 1)$  we observe,

$$-1 < x < 1$$

$$\Rightarrow x^2 < 1$$

$$\Rightarrow 1 < \frac{1}{x^2}, x \neq 0$$

$$\therefore f'(x) = 1 - \frac{1}{x^2} < 0 \text{ on } (-1, 1) \sim \{0\}.$$

$\therefore f$  is strictly decreasing on  $(-1, 1) \sim \{0\}$ .

In the interval,  $(-\infty, -1)$  and  $(1, \infty)$ , it is observed that:

$$x < -1$$

$$\Rightarrow x^2 > 1$$

$$\Rightarrow 1 > \frac{1}{x^2}$$

$$\Rightarrow 1 - \frac{1}{x^2} > 0$$

$$\therefore f'(x) = 1 - \frac{1}{x^2} > 0 \text{ on } (-\infty, -1) \text{ and } (1, \infty).$$

$\therefore f$  is strictly increasing on  $(-\infty, -1)$  and  $(1, \infty)$ .

Hence, the function  $f$  is strictly increasing in the interval disjoint from  $(-1, 1)$ .

Hence, the given result is proved.

### Question 16

Prove that the function  $f$  given by  $f(x) = \log \sin x$  is strictly increasing on  $\left(0, \frac{\pi}{2}\right)$  and strictly decreasing on  $\left(\frac{\pi}{2}, \pi\right)$ .

### Solution

Ans: We have,

$$f(x) = \log \sin x$$

$$\therefore f'(x) = \frac{1}{\sin x} \cos x = \cot x$$

$$\text{In interval, } \left(0, \frac{\pi}{2}\right), f'(x) = \cot x > 0$$

$\therefore f$  is strictly increasing in  $\left(0, \frac{\pi}{2}\right)$

In interval,  $\left(\frac{\pi}{2}, \pi\right)$ ,  $f'(x) = \cot x < 0$

$\therefore f$  is strictly increasing in  $\left(\frac{\pi}{2}, \pi\right)$ .

---

### Question 17

Prove that the function  $f$  is given by  $f(x) = \log \cos x$  is strictly decreasing on  $\left(0, \frac{\pi}{2}\right)$  and strictly increasing on  $\left(\frac{\pi}{2}, \pi\right)$ .

### Solution

Ans: We have,

$$f(x) = \log \cos x$$

$$\therefore f'(x) = \frac{1}{\cos x}(-\sin x) = -\tan x$$

In interval  $\left(0, \frac{\pi}{2}\right)$ ,

$$\tan x > 0 \Rightarrow -\tan x < 0$$

$$\therefore f'(x) < 0 \text{ on } \left(0, \frac{\pi}{2}\right)$$

$\therefore f$  is strictly decreasing on  $\left(0, \frac{\pi}{2}\right)$

In interval  $\left(\frac{\pi}{2}, \pi\right)$ ,

$$\tan x < 0 \Rightarrow -\tan x > 0$$

$$\therefore f'(x) > 0 \text{ on } \left(\frac{\pi}{2}, \pi\right)$$

$\therefore f$  is strictly increasing on  $\left(\frac{\pi}{2}, \pi\right)$ .

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### Question 18

Prove that the function given by  $f(x) = x^3 - 3x^2 + 3x = 100$  is increasing in  $\mathbb{R}$ .

### Solution

Ans: We have,

$$f(x) = x^3 - 3x^2 + 3x = 100$$

$$f'(x) = 3x^2 - 6x + 3$$

$$= 3(x^2 - 2x + 1)$$

$$= 3(x - 1)^2$$

For any  $x \in \mathbb{R}$

$$(x - 1)^2 > 0$$

Thus,  $f'(x)$  is always positive in  $\mathbb{R}$ .

Hence, the given function  $f$  is increasing in  $\mathbb{R}$ .

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### Question 19

The interval in which  $y = x^2 e^{-x}$  is increasing is

A.

$$(-\infty, \infty)$$

B.

$$(-2, 0)$$

C.

$$(2, \infty)$$

D.

$$(0, 2)$$

### Solution

Ans: We have,

$$y = x^2 e^{-x}$$

$$\therefore \frac{dy}{dx} = 2x e^{-x} - x^2 e^{-x} = x e^{-x} (2 - x)$$

Now,

$$\frac{dy}{dx} = 0$$

$$\Rightarrow x = 0, 2$$

The points  $x = 0, 2$  divided the real line into the three disjoint intervals

i.e.,  $(-\infty, 0)$ ,  $(0, 2)$ ,  $(2, \infty)$ .

In intervals  $(-\infty, 0)$  and  $(2, \infty)$ ,  $f'(x) < 0$  as  $e^{-x}$  is always positive.

$\therefore f$  is decreasing on  $(-\infty, 0)$  and  $(2, \infty)$ .

In interval  $(0, 2)$ ,  $f'(x) > 0$

$\therefore f$  is strictly increasing on  $(0, 2)$ .

Hence  $f$  is strictly increasing in the interval  $(0, 2)$

Thus, D is the correct option.

## Exercise 6.3

### Question 1

Find the maximum and minimum values, if any, of the following given by

1.  $f(x) = (2x - 1)^2 + 3$

2.  $f(x) = 9x^2 + 12x + 2$

3.  $f(x) = -(x - 1)^2 + 10$

4.  $g(x) = x^3 + 1$

### Solution

Ans:

i) The given function is  $f(x) = (2x - 1)^2 + 3$ .

It can be observed that  $(2x - 1)^2 \geq 0$  for every  $x \in R$ .

Therefore,  $f(x) = (2x - 1)^2 + 3 \geq 3$  for every  $x \in R$ .

The minimum value of  $f$  is attained when  $2x - 1 = 0$

$$2x - 1 = 0, x = \frac{1}{2}$$

$$\text{Minimum value of } f\left(\frac{1}{2}\right) = \left(2 \cdot \frac{1}{2} - 1\right)^2 + 3 = 3.$$

Hence, function  $f$  does not have a maximum value.

ii) The given function is  $f(x) = 9x^2 + 12x + 2 = (3x^2 + 2)^2 - 2$ .

It can be observed that  $(3x^2 + 2)^2 \geq 0$  for every  $x \in R$ .

Therefore,  $f(x) = (3x^2 + 2)^2 - 2 \geq -2$  for every  $x \in R$ .

The minimum value of  $f$  is attained when  $3x + 2 = 0$

$$3x + 2 = 0, x = -\frac{2}{3}$$

$$\text{Minimum value of } f\left(-\frac{2}{3}\right) = \left(3\left(-\frac{2}{3}\right) + 2\right)^2 - 2 = -2.$$

Hence, function  $f$  does not have a maximum value.

iii) The given function is  $f(x) = -(x-1)^2 + 10$ .

It can be observed that  $(x-1)^2 \geq 0$  for every  $x \in R$ .

Therefore,  $f(x) = -(x-1)^2 + 10 \leq 10$  for every  $x \in R$ .

The minimum value of  $f$  is attained when  $(x-1) = 0$

$$x-1=0, x=1$$

$$\text{Minimum value of } f = f(1) = -(1-1)^2 + 10 = 10$$

Hence, function  $f$  does not have a maximum value.

vi) The given function is  $g(x) = x^3 + 1$ .

Hence, function  $g$  neither has a maximum value nor a minimum value.

---

## Question 2

Find the maximum and minimum values, if any, of the following functions given by

i)  $f(x) = |x+2| - 1$

ii)  $g(x) = -|x+1| + 3$

iii)  $h(x) = \sin 2x + 5$

vi)  $f(x) = |\sin 4x + 3|$

v)  $h(x) = x + 4, x \in (-1, 1)$

## Solution

Ans: We know that  $|x+2| \geq 0$  for every  $x \in R$

Therefore,  $f(x) = |x+2| - 1 \geq -1$  for every  $x \in R$

The minimum value of  $f$  is attained when  $|x+2| = 0$

$$\Rightarrow x = -2$$

$$\text{Minimum value of } f = f(-2) = |-2+2| - 1 = -1$$

Hence, function  $f$  does not have a maximum value.

Ans: We know that  $-|x+1| \leq 0$  for every  $x \in R$

Therefore,  $g(x) = -|x+1| + 3 \leq 3$  for every  $x \in R$

The minimum value of  $g$  is attained when  $|x+1| = 0$

$$\Rightarrow x = -1$$

$$\text{Minimum value of } g = g(-1) = -|-1 + 1| + 3 = 3$$

Hence, function  $g$  does not have a maximum value.

$$\text{Ans: We know that } -1 \leq \sin 2x \leq 1$$

$$-1 + 5 \leq \sin 2x + 5 \leq 1 + 5$$

$$4 \leq \sin 2x + 5 \leq 6$$

Hence, the maximum and minimum values of  $h$  are 6 & 4 respectively.

$$\text{Ans: We know that } -1 \leq \sin 4x \leq 1$$

$$2 \leq \sin 4x + 3 \leq 4$$

$$2 \leq |\sin 4x + 3| \leq 4$$

Hence, the maximum and minimum values of  $f$  are 4 & 2 respectively.

$$\text{Ans: Here, if a point } x_0 \text{ is closest to } -1, \text{ then we find } \frac{x_0}{2} + 1 < x_0 + 1 \text{ for all } x_0 \in (-1, 1).$$

$$\text{Also, if } x_1 \text{ is closet to } -1, \text{ then we find } x_1 + 1 < \frac{x_1 + 1}{2} + 1 \text{ for all } x_0 \in (-1, 1).$$

Hence, function  $h(x)$  has neither maximum nor minimum value at  $(-1, 1)$ .

### Question 3

Find the local maxima and local minima, if any, of the following functions. Find also the local maximum and the local minimum values, as the case may be:

i)  $f(x) = x^2$

ii)  $g(x) = x^3 - 3x$

iii)  $h(x) = \sin x + \cos x, 0 < x < \frac{\pi}{2}$

iv)  $f(x) = \sin x - \cos x, 0 < x < 2\pi$

v)  $f(x) = x^3 - 6x^2 + 9x + 15$

vi)  $g(x) = \frac{x}{2} + \frac{2}{x}, x > 0$

vii)  $g(x) = \frac{1}{x^2 + 2}$

viii)  $f(x) = x\sqrt{1-x}, x > 0$

### Solution

$$\text{Ans: } f(x) = x^2$$

$$\therefore f'(x) = 2x$$

$$\text{Now, } f'(x) = 0 \Rightarrow x = 0$$

Thus,  $x = 0$  is the only critical point that could be the point of local maxima or local minima of  $f$ .

We have  $f''(0) = 2$ , which is positive.

Therefore, by second derivative test,  $x = 0$  is a point of local minima and local minimum value of  $f$  at  $x = 0$  is  $f(0) = 0$ .

$$\text{Ans: } g(x) = x^3 - 3x$$

$$\therefore g'(x) = 3x^2 - 3$$

Now,

$$g'(x) = 0 \Rightarrow 3x^2 = 3 \Rightarrow x = \pm 1$$

$$g''(x) = 6x$$

$$g''(1) = 6 > 0$$

$$g''(-1) = -6 < 0$$

By second derivative test,  $x = 1$  is a point of local minima and local minimum value of  $g$  at  $x = 1$  is  $g(1) = 1^3 - 3 = -2$ .

$$\text{Ans: } h(x) = \sin x + \cos x, 0 < x < \frac{\pi}{2}$$

$$\therefore h'(x) = \cos x - \sin x$$

$$h'(x) = 0 \Rightarrow \cos x = \sin x \Rightarrow \tan x = 1$$

$$\Rightarrow x = \frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right)$$

$$h''(x) = -\sin x - \cos x$$

$$h''\left(\frac{\pi}{4}\right) = -\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)$$

$$= -\frac{1}{\sqrt{2}} = -\sqrt{2} < 0$$

Therefore, by second derivative test,  $x = \frac{\pi}{4}$  is a point of local maxima and the local maximum value of  $h$  at  $x = \frac{\pi}{4}$

$$h\left(\frac{\pi}{4}\right) = \sin \frac{\pi}{4} + \cos \frac{\pi}{4}$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}.$$

$$\text{Ans: } f(x) = \sin x - \cos x, 0 < x < 2\pi$$

$$\therefore f'(x) = \cos x + \sin x$$

$$f'(x) = 0 \Rightarrow \cos x = -\sin x$$

$$\tan x = -1 \Rightarrow x = \frac{3\pi}{4}, \frac{7\pi}{4} \in (0, 2\pi)$$

$$f''(x) = -\sin x + \cos x$$

$$f''\left(\frac{3\pi}{4}\right) = -\sin \frac{3\pi}{4} + \cos \frac{3\pi}{4} = -\sqrt{2} < 0$$

$$f''\left(\frac{7\pi}{4}\right) = -\sin \frac{7\pi}{4} + \cos \frac{7\pi}{4} = \sqrt{2} > 0$$

Therefore, by second derivative test,  $x = \frac{3\pi}{4}$  is a point of local maxima and the local maximum value of  $f$  at  $x = \frac{3\pi}{4}$  is

$f\left(\frac{3\pi}{4}\right) = \sin \frac{3\pi}{4} \cos \frac{3\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$ . However,  $x = \frac{7\pi}{4}$  is a point of local minima and the local minimum value of  $f$  at  $x = \frac{7\pi}{4}$

$$\text{is } f\left(\frac{7\pi}{4}\right) = \sin \frac{7\pi}{4} - \cos \frac{7\pi}{4} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2}.$$

$$\text{Ans: } f(x) = x^3 - 6x + 9x + 15$$

$$\therefore f'(x) = 3x^2 - 12x + 9$$

$$f(x) = 0 \Rightarrow 3(x^2 - 4x + 3) = 0$$

$$\Rightarrow 3(x - 1)(x - 3) = 0$$

$$\Rightarrow x = 1, 3$$

Now,

$$f''(x) = 6x - 12 = 6(x - 2)$$

$$f''(1) = 6(1 - 2) = -6 < 0$$

$$f''(3) = 6(3 - 2) = 6 > 0$$

Therefore, by second derivative test,  $x = 1$  is a point of local maxima and the local maximum value of  $f$  at  $x = 1$  is  $f(1) = 1 - 6 + 9 + 15 = 19$ . However,  $x = 3$  is a point of local minima and the local minimum value of  $f$  at  $x = 3$  is  $f(3) = 27 - 54 + 27 + 15 = 15$ .

$$\text{Ans: } g(x) = \frac{x}{2} + \frac{2}{x}, x > 0$$

$$\therefore g'(x) = \frac{1}{2} - \frac{2}{x^2}$$

Now,

$$g'(x) = 0 \text{ gives } \frac{2}{x^2} = \frac{1}{2}$$

$$\Rightarrow x^2 = 4 \Rightarrow x = \pm 2$$

Since  $x > 0$  we take  $x = 2$ .

Now,

$$g''(x) = \frac{4}{x^3}$$

$$g''(2) = \frac{4}{2^3} = \frac{1}{2} > 0$$

Therefore, by second derivative test,  $x = 2$  is a point of local minima and the local minimum value of  $g$  at  $x = 2$  is  $g(2) = \frac{2}{2} + \frac{2}{2} = 1 + 1 = 2$ .

$$\text{Ans: } g(x) = \frac{1}{x^2+2}$$

$$\therefore g'(x) = \frac{-(2x)}{(x^2+2)^2}$$

$$g'(x) = 0 \Rightarrow \frac{-(2x)}{(x^2+2)^2} = 0 \Rightarrow x = 0$$

Now, for values close to  $x = 0$  and to the left of 0,  $g'(x) > 0$ . Also, for values close to  $x = 0$  and to the right of  $g'(x) < 0$ .

Therefore, by first derivative test  $x = 0$  is a point of local maxima and the local maximum value of  $g(0)$  is  $\frac{1}{0+2} = \frac{1}{2}$ .

$$\text{Ans: } f(x) = x\sqrt{1-x}, x > 0$$

$$\therefore f'(x) = x\sqrt{1-x} + x \cdot \frac{1}{2\sqrt{1-x}}(-1) = \sqrt{1-x} - \frac{x}{2\sqrt{1-x}}$$

$$= \frac{2(1-x)-x}{2\sqrt{1-x}} = \frac{2-3x}{2\sqrt{1-x}}$$

$$f'(x) = 0 \Rightarrow \frac{2-3x}{2\sqrt{1-x}} = 0$$

$$\Rightarrow 2 - 3x = 0 \Rightarrow x = \frac{2}{3}$$

$$f''(x) = \frac{1}{2} \left[ \frac{\sqrt{1-x}(-3) - (2-3x)\left(\frac{-1}{2\sqrt{1-x}}\right)}{1-x} \right]$$

$$= \frac{\sqrt{1-x}(-3) + 2(2-3x)\left(\frac{1}{2\sqrt{1-x}}\right)}{2(1-x)}$$

$$= \frac{3x-4}{4(1-x)^{\frac{3}{2}}}$$

$$f''\left(\frac{2}{3}\right) = \frac{2-4}{4\left(\frac{1}{3}\right)^{\frac{3}{2}}} = \frac{-1}{2\left(\frac{1}{3}\right)^{\frac{3}{2}}} < 0$$

Therefore, by second derivative test,  $x = \frac{2}{3}$  is a point of local

maxima and the local maximum value of  $f$  at  $x = \frac{2}{3}$  is  $f\left(\frac{2}{3}\right) = \frac{2-4}{4\left(\frac{1}{3}\right)^{\frac{3}{2}}} = \frac{-1}{2\left(\frac{1}{3}\right)^{\frac{3}{2}}} < 0$ .

**Question 4**

Prove that the following functions do not have maxima or minima:

i)  $f(x) = e^x$

ii)  $g(x) = \log x$

iii)  $h(x) = x^3 + x^2 + x + 1$

**Solution**

Ans:  $f(x) = e^x$

$$\therefore f'(x) = e^x$$

Now, if  $f'(x) = 0$  then  $e^x = 0$ . But the exponential function can never assume 0 for any value of  $x$ .

Therefore, there does not exist  $c \in R$  such that  $f'(c) = 0$ .

Hence, function  $f$  does not have maxima or minima.

Ans: We have,

$$g(x) = \log x$$

$$\therefore g'(x) = \frac{1}{x}$$

Since  $\log x$  is defined for a positive number  $x$ ,  $g'(x) > 0$  for any  $x$

Therefore, there does not exist  $c \in R$  such that  $g'(c) = 0$ .

Hence, function  $g$  does not have maxima or minima.

Ans: We have,

$$h(x) = x^3 + x^2 + x + 1$$

$$\therefore h'(x) = 3x^2 + 2x + 1$$

Now,

Therefore, there does not exist  $c \in R$  such that  $h'(c) = 0$ .

Hence, function  $h$  does not have maxima or minima.

**Question 5**

Find the absolute maximum value and the absolute minimum value of the following functions in the given intervals:

i)  $f(x) = x^3, x \in [-2, 2]$

ii)  $f(x) = \sin x + \cos x, x \in [0, \pi]$

$$\text{iii) } f(x) = 4x - \frac{1}{2}x^2, x \in \left[-2, \frac{9}{2}\right]$$

$$\text{iv) } f(x) = (x-1)^2 + 3, x \in [-3, 1]$$

### Solution

Ans: The given function is  $f(x) = x^3$ .

$$\therefore f'(x) = 3x^2$$

Now,

$$f'(x) = 0 \Rightarrow x = 0$$

Then, we evaluate the value of  $f$  at critical point  $x = 0$  and at endpoints of the interval  $[-2, 2]$ .

$$f(0) = 0$$

$$f(-2) = (-2)^3 = -8$$

$$f(2) = (2)^3 = 8$$

Hence, we can conclude that the absolute maximum value of  $f$  on  $[-2, 2]$  is 8 occurring at  $x = 2$ . Also, the absolute minimum value of  $f$  on  $[-2, 2]$  is  $-8$  occurring at  $x = -2$ .

Ans: The given function is  $f(x) = \sin x + \cos x$

$$\therefore f'(x) = \cos x - \sin x$$

Now,

$$f'(x) = 0 \Rightarrow \sin x = \cos x$$

$$\Rightarrow \tan x = 1 \Rightarrow x = \frac{\pi}{4}$$

Then, we evaluate the value of  $f$  at critical point  $x = \frac{\pi}{4}$  and at the endpoints of the interval  $[0, \pi]$ .

$$f\left(\frac{\pi}{4}\right) = \sin \frac{\pi}{4} + \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$$

$$f(0) = \sin 0 + \cos 0 = 1$$

$$f(\pi) = \sin \pi + \cos \pi = -1$$

Hence, we can conclude that the absolute maximum value of  $f$  on  $[0, \pi]$  is  $\sqrt{2}$  occurring at  $x = \frac{\pi}{4}$ . Also, the absolute minimum value of  $f$  on  $[0, \pi]$  is  $-1$  occurring at  $x = \pi$ .

Ans: The given function is  $f(x) = 4x - \frac{1}{2}x^2$

$$\therefore f'(x) = 4 - x$$

Now,

$$f'(x) = 0 \Rightarrow x = 4$$

Then, we evaluate the value of  $f$  at critical point  $x = 4$  and at the endpoints of the interval  $\left[-2, \frac{9}{2}\right]$ .

$$f(4) = 16 - \frac{1}{2}(16) = 8$$

$$f(-2) = -8 - \frac{1}{2}(4) = -10$$

$$f\left(\frac{9}{2}\right) = 4\left(\frac{9}{2}\right) - \frac{1}{2}\left(\frac{9}{2}\right)^2 = 18 - \frac{81}{8} = 7.875$$

Hence, we can conclude that the absolute maximum value of  $f$  on  $\left[-2, \frac{9}{2}\right]$  is 8 occurring at  $x = 4$ . Also, the absolute minimum value of  $f$  on  $\left[-2, \frac{9}{2}\right]$  is  $-10$  occurring at  $x = -2$ .

Ans: The given function is  $f(x) = (x - 1)^2 + 3$

$$\therefore f'(x) = 2(x - 1)$$

Now,

$$f'(x) = 0 \Rightarrow 2(x - 1) = 0, x = 1$$

Then, we evaluate the value of  $x$  at critical point  $x = 1$  and at the endpoints of the interval  $[-3, 1]$ .

$$f(1) = (1 - 1)^2 + 3 = 3$$

$$f(-3) = (-3 - 1)^2 + 3 = 19$$

Hence, we can conclude that the absolute maximum value of  $f$  on  $[-3, 1]$  is 19 taking place at  $x = -3$ . Also, the absolute minimum value of  $f$  on  $[-3, 1]$  is 3 taking place at  $x = 1$ .

---

### Question 6

Find the maximum profit that a company can make if the profit function is given by  $p(x) = 41 - 24x - 18x^2$ .

### Solution

Ans: The profit function is given as  $p(x) = 41 - 24x - 18x^2$ .

$$\therefore p'(x) = -24 - 36x$$

$$p''(x) = -36$$

Now,

$$p'(x) = 0$$

$$\Rightarrow x = \frac{-24}{-36} = -\frac{2}{3}$$

Also,

$$p''\left(-\frac{2}{3}\right) = -36 < 0$$

By second derivatives test,  $x = -\frac{2}{3}$  is the point of local maximum of  $p$

$$\text{Therefore, Maximum profit} = p\left(-\frac{2}{3}\right)$$

$$= 41 - 24\left(-\frac{2}{3}\right) - 18\left(-\frac{2}{3}\right)^2$$

$$= 41 + 16 - 8$$

$$= 49$$

Hence, the maximum profit that the company can make is 49 units.

---

### Question 7

Find both the maximum value and the minimum value of  $3x^4 - 8x^3 + 12x^2 - 48x + 25$  on the interval  $[0,3]$ .

### Solution

Ans: Let's take  $f(x) = 3x^4 - 8x^3 + 12x^2 - 48x + 25$  on  $[0,3]$

$$f'(x) = 12x^3 - 24x^2 + 24x - 48$$

$$\text{Now } f'(x) = 0$$

$$12x^3 - 24x^2 + 24x - 48 = 0$$

$$x^3 - 2x^2 + 2x - 4 = 0$$

$$(x - 2)(x^2 + 2) = 0$$

$$x = 2 \text{ or } x = \pm\sqrt{2}$$

As  $x = \pm\sqrt{2}$  is imaginary, therefore this value is not considered.

$$\text{At } x=2, f(2) = 3(16) - 8(8) + 12(4) - 48(2) + 25 = -39$$

$$\text{At } x=0, f(0) = 3(0) - 8(0) + 12(0) - 48(0) + 25 = 25$$

$$\text{At } x=3, f(3) = 3(81) - 8(27) + 12(9) - 48(3) + 25 = 16$$

Therefore, the minimum value of the given function is -39 and maximum value is 25.

**Question 8**

At what points in the interval  $[0, 2\pi]$  does the function  $\sin 2x$  attain, its maximum value?

**Solution**

Ans: Let

$$f(x) = \sin 2x$$

$$\therefore f'(x) = 2 \cos 2x$$

Now,

$$f'(x) = 0 \Rightarrow \cos 2x = 0$$

$$\Rightarrow 2x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$$

$$\Rightarrow x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

Then, we evaluate the values of  $f$  at critical points

$$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \text{ and at the endpoints of}$$

the interval  $[0, 2\pi]$ .

$$f\left(\frac{\pi}{4}\right) = \sin \frac{\pi}{2} = 1,$$

$$f\left(\frac{3\pi}{4}\right) = \sin \frac{3\pi}{2} = -1,$$

$$f\left(\frac{5\pi}{4}\right) = \sin \frac{5\pi}{2} = 1,$$

$$f\left(\frac{7\pi}{4}\right) = \sin \frac{7\pi}{2} = -1,$$

$$f(0) = \sin 0 = 0,$$

$$f(2\pi) = \sin 2\pi = 0$$

Hence, we can conclude that the absolute maximum value of

$$f[0, 2\pi] \text{ is occurring at } x = \frac{\pi}{4} \text{ and } x = \frac{5\pi}{4}.$$

**Question 9**

What is the maximum value of the function  $\sin x + \cos x$ ?

**Solution**

$$\text{Ans: Let } f(x) = \sin x + \cos x$$

$$\therefore f'(x) = \cos x - \sin x$$

$$f'(x) = 0 \Rightarrow \sin x = \cos x$$

$$\Rightarrow \tan x = 1 \Rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4}, \dots$$

$$f'(x) = -\sin x - \cos x = -(\sin x + \cos x)$$

Now, when  $(\sin x + \cos x)$  is positive, i.e.,

when  $\sin x$  and  $\cos x$  are both positive,  $f'(x)$  will be negative.

We also know that in the first quadrant, both  $\sin x$  and  $\cos x$  are positive. Then,  $f''(x)$  will be negative when  $x \in \left(0, \frac{\pi}{2}\right)$ .

As a result, we consider  $x = \frac{\pi}{4}$

$$f''\left(\frac{\pi}{4}\right) = -\left(\sin \frac{\pi}{4} + \cos \frac{\pi}{4}\right) = -\left(\frac{2}{\sqrt{2}}\right) = -\sqrt{2} < 0$$

By second derivative test,  $f$  will be the maximum at  $x = \frac{\pi}{4}$  and the

$$\text{maximum value of } f \text{ is } f\left(\frac{\pi}{4}\right) = \sin \frac{\pi}{4} + \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}.$$

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### Question 10

Find the maximum value of  $2x^3 - 24x + 107$  in the interval  $[1, 3]$

Find the maximum value of the same function in  $[-3, -1]$ .

### Solution

Ans: Let  $f(x) = 2x^3 - 24x - 107$

$$\therefore f'(x) = 6x^2 - 24 = 6(x^2 - 4)$$

Now,

$$f'(x) = 0 \Rightarrow 6(x^2 - 4) = 0$$

$$\Rightarrow x^2 = 4 \Rightarrow x = \pm 2$$

We first consider the interval  $[1, 3]$ .

Then, we evaluate the value of  $f$  at the critical point  $x \in [1, 3]$  and at the endpoints of the interval  $[1, 3]$ .

$$f(2) = 2(8) - 24(2) + 107 = 75$$

$$f(1) = 2(1) - 24(1) + 107 = 85$$

$$f(3) = 2(27) - 24(3) + 107 = 89$$

Hence, the absolute maximum value of  $f(x)$  in the interval  $[1, 3]$  is

89 occurring at  $x = 3$ .

Next, we consider the interval  $[-3, -1]$ .

Evaluate the value of  $f$  at the critical point  $x \in [1, 3]$  and at

the endpoints of the interval  $[-3, -1]$ .

$$f(-3) = 2(-27) - 24(-3) + 107 = 125$$

$$f(-1) = 2(-1) - 24(-1) + 107 = 129$$

$$f(-2) = 2(-8) - 24(-2) + 107 = 139$$

Hence, the absolute maximum value of  $f(x)$  in the interval  $[-3, -1]$  is

139 occurring at  $x = -2$ .

---

### Question 11

It is given that at  $x = 1$  the function  $x^4 - 62x^2 + ax + 9$  attains its maximum value, on the interval  $[0, 2]$ . Find the value of  $a$ .

### Solution

Ans: Let  $f(x) = x^4 - 62x^2 + ax + 9$

$$\therefore f'(x) = 4x^3 - 124x + a$$

It is given that function  $f$  attains its maximum value on the interval

$[0, 2]$  at  $x = 1$ .

$$\therefore f'(1) = 0$$

$$\Rightarrow 4 - 124 + a = 0$$

$$\Rightarrow a = 120$$

Hence, the value of  $a$  is 120.

---

### Question 12

Find the maximum and minimum values of  $x + \sin 2x$  on  $[0, 2\pi]$ .

### Solution

Ans: Let  $f(x) = x + \sin 2x$

$$\therefore f'(x) = 1 + 2 \cos 2x$$

$$\text{Now, } f'(x) = 0 \Rightarrow \cos 2x = -\frac{1}{2} = \cos \frac{2\pi}{3}$$

$$2x = 2\pi \pm \frac{2\pi}{3}, n \in \mathbb{Z}$$

$$\Rightarrow x = \pi \pm \frac{\pi}{3}, n \in \mathbb{Z}$$

$$\Rightarrow x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3} \in [0, 2\pi]$$

Then, we evaluate the value of  $f$  at critical points  $\Rightarrow x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$  and the end points of the interval  $[0, 2\pi]$ .

$$f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} + \sin \frac{2\pi}{3} = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$$

$$f\left(\frac{2\pi}{3}\right) = \frac{2\pi}{3} + \sin \frac{4\pi}{3} = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$$

$$f\left(\frac{4\pi}{3}\right) = \frac{4\pi}{3} + \sin \frac{8\pi}{3} = \frac{4\pi}{3} + \frac{\sqrt{3}}{2}$$

$$f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} + \sin \frac{10\pi}{3} = \frac{5\pi}{3} - \frac{\sqrt{3}}{2}$$

$$f(0) = 0 + \sin 0 = 0$$

$$f(2\pi) = 2\pi + \sin 4\pi = 2\pi$$

Hence, we can conclude that the absolute maximum value of  $f(x)$  in

the interval  $[0, 2\pi]$  is  $2\pi$  occurring at  $x = 2\pi$  and the absolute maximum value of  $f(x)$  in the interval  $[0, 2\pi]$  is 0 occurring at  $x = 0$ .

### Question 13

Find two numbers whose sum is 24 and whose product is as large as possible.

#### Solution

Ans: Let one number be  $x$ . Then, the other number is  $(24 - x)$ .

Let  $p(x)$  denote the product of the two numbers. Thus, we have:

$$P(x) = x(24 - x) = 24x - x^2$$

$$\therefore P'(x) = 24 - 2x$$

$$P''(x) = -2$$

Now,

$$P'(x) = 0 \Rightarrow x = 12$$

Also,

$$P''(12) = -2 < 0$$

By second derivative test,  $x = 12$  is the point of local maxima of  $P$ .

Hence, the product of the numbers is the maximum when the numbers are 12 and  $24 - 12 = 12$ .

---

### Question 14

Find two positive numbers  $x$  and  $y$  such that  $x + y = 60$  and  $xy^3$  is maximum.

### Solution

Ans: It is assumed that the two numbers are  $x$  and  $y$  such that  $x + y = 60$ .

$$y = 60 - x$$

$$\text{Let } f(x) = xy^3$$

$$\Rightarrow f(x) = x(60 - x)^3$$

$$\therefore f'(x) = (60 - x)^3 - 3x(60 - x)^2$$

$$= (60 - x)^2 [60 - 4x]$$

And,

$$f''(x) = -2(60 - x)(60 - 4x) - 4(60 - x)^2$$

$$= -2(60 - x)(180 - 6x)$$

$$= -12(60 - x)(30 - x)$$

$$\text{Now, } f'(x) = 0$$

$$\Rightarrow x = 60, 15$$

$$\text{When } x = 60, f''(x) = 0$$

$$\text{When } x = 15, f''(x) = -12(60 - 15)(30 - 15) < 0.$$

By second derivative test,  $x = 15$  is a point of local maxima of  $f$ .

Thus, function  $xy^3$  maximum when  $x = 15$  and  $y = 60 - 15 = 45$ .

Hence, the required numbers are 15 and 45.

---

### Question 15

Find two positive numbers  $x$  and  $y$  such that their sum is 35 and the product  $x^2y^5$  is a maximum.

### Solution

Ans: Let one number be  $x$ . Then, the other number is  $y = (35 - x)$

Let  $p(x) = x^2y^5$ . Then, we have:

$$P(x) = x^2(35 - x)^5$$

$$\therefore P'(x) = 2x(35 - x)^5 - 5x^2(35 - x)^4$$

$$= x(35 - x)^4 [2(35 - x) - 5x]$$

$$= 7x(35 - x)^4 (10 - x)$$

And,

$$P''(x) = 7(35 - x)^4 (10 - x) + 7x \left[ -(35 - x)^4 - 4(35 - x)^3 (10 - x) \right]$$

$$= 7(35 - x)^4 (10 - x) - 7x(35 - x)^4 - 28x(35 - x)^3 (10 - x)$$

$$= 7(35 - x)^3 [350 - 45x + x^2 - 35x + x^2 - 40x + 4x^2]$$

$$= 7(35 - x)^3 (6x^2 - 120x + 350)$$

$$\text{Now, } P'(x) = 0 \Rightarrow x = 0, x = 35, x = 10$$

$$\text{When, } x = 35, f'(x) = f(x) = 0$$

$$y = 35 - 35 = 0.$$

This makes the product equal to 0.

When  $x = 0, y = 35 - 0 = 35$  and the product  $x^2 y^5$  will be 0.

$x = 0$  and  $x = 35$  cannot be the possible values of  $x$ .

When  $x = 10$  we have:

$$P''(x) = 7(35 - 10)^3 (-250) < 0$$

By second derivative test,  $P(x)$  will be the maximum when  $x = 10$  and

$$y = 35 - 10 = 25.$$

Hence, the required numbers are 10 and 25.

### Question 16

Find two positive numbers whose sum is 16 and the sum of whose cubes is minimum.

### Solution

Ans: Let one number be  $x$ . Then, the other number is  $(16 - x)$ .

Let the sum of the cubes of these numbers be denoted by  $S(x)$ . Then,

$$S(x) = x^3 + (16 - x)^3$$

$$\therefore S'(x) = 3x^2 - 3(16 - x)^2,$$

$$S''(x) = 6x + 6(16 - x)$$

$$\text{Now, } S'(x) = 0 \Rightarrow 3x^2 - 3(16 - x)^2 = 0$$

$$\Rightarrow x^2 - (16 - x)^2 = 0$$

$$\Rightarrow x = \frac{256}{32} = 8$$

Now,

$$S''(8) = 6(8) + 6(16 - 8) = 96 > 0$$

By second derivative test,  $x = 8$  is the point of local minima of  $S$ .

Hence, the sum of the cubes of the numbers is the minimum when the numbers are 8 and  $16 - 8 = 8$ .

### Question 17

A square piece of tin of side 18 cm is to be made into a box without a top, by cutting a square from each corner and folding up the flaps to form the box. What should be the side of the square to be cut off so that the volume of the box is the maximum possible?

### Solution

Ans: Let the side of the square be cut off by  $x$  cm.

Then, the length and the breadth of the box will be  $(18 - 2x)$  cm each.

The box has a height of  $x$  cm. As a result, the box's volume  $V(x)$  is given by,

$$V(x) = x(18 - 2x)^2$$

$$\therefore V'(x) = (18 - 2x)^2 - 4x(18 - 2x)$$

$$= (18 - 2x)[18 - 2x - 4x]$$

$$= 12(9 - x)(3 - x)$$

And,

$$V'(x) = 12[-(9 - x) - (3 - x)]$$

$$= -24(6 - x)$$

$$\text{Now, } V'(x) = 0 \Rightarrow x = 9, 3$$

If  $x = 9$ , then the length and the breadth will become 0.

$$\therefore x = 9$$

$$\Rightarrow x = 3$$

$$\text{Now, } V''(3) = -24(6 - 3) = -72 < 0$$

Therefore, by second derivative test,  $x = 3$  is the point of maxima of  $V$ .

As a result, the volume of the box obtained is the biggest achievable by removing a square of side 3 cm from each corner of the square tin and building a box from the remaining sheet.

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### Question 18

A rectangular sheet of tin 45 cm by 24 cm is to be made into a box without a top, by cutting off squares from each corner and folding up the flaps. What should be the side of the square to be cut off so that the volume of the box is the maximum possible?

### Solution

Ans: Let the side of the square be cut by  $x$  cm.

Then, the height of the box is  $x$ , the length is  $45 - 2x$ , and the breadth is  $24 - 2x$ .

$$V(x) = x(45 - 2x)(24 - 2x)$$

$$= 4x^3 - 138x^2 + 1080x$$

$$V'(x) = 12x^2 - 276 + 1080$$

$$= 12(x^2 - 23x + 90)$$

$$= 12(x - 18)(x - 5)$$

$$V''(x) = 24x - 276 = 12(2x - 23)$$

$$\text{Now, } V'(x) = 0 \Rightarrow x = 18, 5$$

It is not possible to cut off a square of side 18 cm from each corner of the rectangular sheet,

Thus,  $x$  cannot be equal to 18.

$$x = 5$$

$$\text{Now, } V''(5) = 12(10 - 23) = -156 < 0$$

Therefore, by second derivative test  $x = 5$  is the point of maxima.

Hence, the side of the square to be cut off to make the volume of the box maximum possible is 5cm.

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### Question 19

Show that of all the rectangles inscribed in a given fixed circle, the square has the maximum area.

**Solution**

Ans: Let a rectangle of length  $l$  and breadth  $b$  be inscribed in the given circle of radius  $a$ .

Then, the diagonal of length  $2a$  cm passes through the center.

By Pythagoras theorem, we have:

$$(2a)^2 = l^2 + b^2$$

$$\Rightarrow b^2 = 4a^2 - l^2$$

$$\Rightarrow b = \sqrt{4a^2 - l^2}$$

$$\text{Area of the rectangle, } A = l\sqrt{4a^2 - l^2}$$

$$\therefore \frac{dA}{dl} = \sqrt{4a^2 - l^2} + l \frac{1}{2\sqrt{4a^2 - l^2}}(-2l) = \sqrt{4a^2 - l^2} + \frac{l^2}{\sqrt{4a^2 - l^2}} = \frac{4a^2 - 2l^2}{\sqrt{4a^2 - l^2}}$$

$$\frac{d^2A}{dl^2} = \frac{\sqrt{4a^2 - l^2}(-4l) - (4a^2 - 2l^2) \frac{(-2l)}{2\sqrt{4a^2 - l^2}}}{(4a^2 - l^2)}$$

$$= \frac{(4a^2 - l^2)(-4l) + 1(4a^2 - 2l^2)}{(4a^2 - l^2)^{\frac{3}{2}}}$$

$$= \frac{-2l(6a^2 - l^2)}{(4a^2 - l^2)^{\frac{3}{2}}}$$

$$\text{Now, } \frac{dA}{dl} = 0 \text{ gives } 4a^2 = 2l^2 \Rightarrow l = \sqrt{2a}$$

$$\Rightarrow b = \sqrt{4a^2 - 2a^2} = \sqrt{2a^2} = \sqrt{2a}$$

$$\text{Now, when } l = \sqrt{2a}$$

$$\frac{d^2A}{dl^2} = \frac{-2(\sqrt{2a})(-6a^2 - 2a^2)}{2\sqrt{2a^3}} = \frac{-8\sqrt{2a^3}}{2\sqrt{2a^3}} = -4 < 0$$

By second derivative test, when  $l = \sqrt{2a}$  then the area of the rectangle is the maximum.

Since  $l = b = \sqrt{2a}$  the rectangle is a square.

Hence, it has been proved that the square has the maximum area of all the rectangles inscribed in the given fixed circle.

**Question 20**

Show that the right circular cylinder of given surface and maximum volume is such that its height is equal to the diameter of the base.

**Solution**

Ans: Let  $r$  and  $h$  denote the cylinder's radius and height respectively.

Then, the cylinder has the following Surface Area ( $S$ ):

$$S = 2\pi r^2 + 2\pi r h$$

$$\Rightarrow h = \frac{S - 2\pi r^2}{2\pi r}$$

$$= \frac{S}{2\pi} \left( \frac{1}{r} \right) - r$$

Let  $V$  be the volume of the cylinder. Then,

$$V = \pi r^2 h = \pi r^2 \left[ \frac{S}{2\pi} \left( \frac{1}{r} \right) - r \right] = \frac{Sr}{2}$$

$$= -\pi r^3 = \frac{S}{2\pi} \left( \frac{1}{r} \right) - r$$

$$\text{Then, } \frac{dV}{dr} = \frac{S}{2} - 3\pi r^2, \frac{d^2V}{dr^2} = -6\pi r$$

$$\text{Now, } \frac{dV}{dr} = 0 \Rightarrow \frac{S}{2} = 3\pi r^2 \Rightarrow r^2 = \frac{S}{6\pi}$$

$$\text{When } r^2 = \frac{S}{6\pi} \text{ then } h = \frac{6\pi r^2}{2\pi} \left( \frac{1}{r} \right) - r = 3r - r = 2r$$

Hence, the volume is the maximum when the height is twice the radius i.e. when the height is equal to the diameter.

**Question 21**

**Of all the closed cylindrical cans (right circular), of a given volume of 100 cubic centimeters, find the dimensions of the can which has the minimum surface area?**

**Solution**

Ans: Let  $r$  and  $h$  denote the cylinder's radius and height respectively.

Then, the cylinder has the following volume ( $V$ ),

$$V = \pi r^2 h = 100$$

$$\therefore h = \frac{100}{\pi r^2}$$

The cylinder has the following surface area ( $S$ ),

$$S = 2\pi r^2 + 2\pi r h = 2\pi r^2 + \frac{200}{r}$$

$$\frac{dS}{dr} = 0 \Rightarrow 4\pi r = \frac{200}{r^2}$$

$$\Rightarrow r^3 = \frac{200}{4\pi} = \frac{50}{\pi}$$

$$\Rightarrow r = \left(\frac{50}{\pi}\right)^{\frac{1}{3}}$$

Now, it is observed that when  $r = \left(\frac{50}{\pi}\right)^{\frac{1}{3}}$ ,  $\frac{d^2S}{dr^2} > 0$ .

By second derivative test, when the radius of the cylinder is  $\left(\frac{50}{\pi}\right)^{\frac{1}{3}}$  cm, the surface area is the minimum.

When  $r = \left(\frac{50}{\pi}\right)^{\frac{1}{3}}$ ,  $h = 2\left(\frac{50}{\pi}\right)^{\frac{1}{3}}$  cm.

Hence the required dimensions of the can which has the minimum surface area are given as  $radius = \left(\frac{50}{\pi}\right)^{\frac{1}{3}}$  cm and  $height = 2\left(\frac{50}{\pi}\right)^{\frac{1}{3}}$  cm.

### Question 22

A Wire of length 28 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the length of the two pieces so that the combined area of the circle is minimum?

### Solution

Ans: Let a piece of length  $l$  be cut from the given wire to make a square.

Then, the other piece of wire to be made into a circle is of length  $(28 - l)$  m.

Let  $r$  be the circle's radius. Then  $2\pi r = 28 - l \Rightarrow r = \frac{1}{2\pi}(28 - l)$ .

The combined areas of the square and the circle ( $A$ ) is given by,

$$A = \pi^2 + r^2 \text{ (side of the square)}$$

$$= \frac{l^2}{16} + \pi \left[ \frac{1}{2\pi}(28 - l) \right]^2$$

$$= \frac{l^2}{16} + \frac{1}{4\pi}(28 - l)^2$$

$$\therefore \frac{dA}{dl} = \frac{2l}{16} + \frac{2}{4\pi}(28 - l)(-1) = \frac{l}{8} - \frac{1}{2\pi}(28 - l)$$

$$\frac{d^2A}{dl^2} = \frac{l}{8} + \frac{1}{2\pi} > 0$$

$$\text{Now, } \frac{dA}{dl} = 0 \Rightarrow \frac{l}{8} - \frac{1}{2\pi}(28 - l) = 0$$

$$\Rightarrow (\pi + 4)l - 112 = 0$$

$$\Rightarrow l = \frac{112}{\pi + 4}$$

Thus, when  $l = \frac{112}{\pi + 4}$ ,  $\frac{d^2A}{dl^2} > 0$

By second derivative test, area (A) is the minimum when  $l = \frac{112}{\pi+4}$ . Hence, when the length of the wire is  $\frac{112}{\pi+4}$  cm in making the square the combined area is the minimum while the length of the wire in making the circle is  $28 - \frac{112}{\pi+4} = \frac{28\pi}{\pi+4}$  cm.

### Question 23

Prove that the volume of the largest cone that can be inscribed in a sphere of radius R is  $\frac{8}{27}$  of the volume of the sphere.

### Solution

Ans: Let r and h be the radius and height of the cone respectively inscribed in a sphere of radius R.

Let V be the volume of the cone.

$$\text{Then, } V = \frac{1}{3}\pi r^2 h$$

Height of the cone is given by,

$$h = R + AB = R + \sqrt{R^2 - r^2} \quad (\text{ABC is a right triangle})$$

$$V = \frac{1}{3}\pi r^2 \left( R + \sqrt{R^2 - r^2} \right)$$

$$= \frac{1}{3}\pi r^2 R + \frac{1}{3}\pi r^2 \sqrt{R^2 - r^2}$$

$$\frac{dV}{dr} = \frac{2}{3}\pi r R + \frac{2}{3}\pi r \sqrt{R^2 - r^2} + \frac{1}{3}\pi r^2 \cdot \frac{(-2r)}{2\sqrt{R^2 - r^2}}$$

$$= \frac{2}{3}\pi r R + \frac{2}{3}\pi r \sqrt{R^2 - r^2} - \frac{1}{3}\pi \cdot \frac{r^3}{\sqrt{R^2 - r^2}}$$

$$= \frac{2}{3}\pi r R + \frac{2\pi R^2 - 3\pi r^3}{3\sqrt{R^2 - r^2}}$$

$$\frac{d^2V}{dr^2} = \frac{2}{3}\pi r R + \frac{9(R^2 - r^2)(2\pi R^2 - 9\pi r^2) + 2\pi r R^2 - 3\pi r^4}{27(R^2 - r^2)^{\frac{3}{2}}}$$

$$\text{Now, } \frac{dV}{dr} = 0 \Rightarrow \pi \frac{2}{3} r R = \frac{3\pi r^3 - 2\pi R^2}{3\sqrt{R^2 - r^2}}$$

$$\Rightarrow 2R = 2R\sqrt{R^2 - r^2} = 3r^2 - 2R^2$$

$$\Rightarrow 4R^4 - 4R^2 r^2 = 9r^4 + 4R^2$$

$$\Rightarrow 9r^4 = 8R^2 r^2$$

$$\Rightarrow r^2 = \frac{8}{9}R^2$$

$$\text{When } r^2 = \frac{8}{9}R^2 \text{ then } \frac{d^2V}{dr^2} < 0$$

By second derivative test, the volume of the cone is the maximum when

$$r^2 = \frac{8}{9}R^2.$$

$$\text{When } r^2 = \frac{8}{9}R^2, h = R + \sqrt{R^2 - \frac{8}{9}R^2} = R + \frac{R}{3} = \frac{4}{3}R$$

Therefore,

$$= \frac{1}{3}\pi \left(\frac{8}{9}R^2\right) \left(\frac{4}{3}R\right)$$

$$= \frac{8}{27} \left(\frac{4}{3}\pi R^3\right)$$

$$= \frac{8}{27} \times (\text{Volume})$$

Hence, the volume of the largest cone that can be inscribed in the sphere is  $\frac{8}{27}$  the volume of the sphere.

### Question 24

Show that the right circular cone of the least curved surface and given volume has an altitude equal to  $\sqrt{2}$  times the radius of the base.

### Solution

Ans: Let  $r$  and  $h$  be the radius and the altitude of the cone respectively.

Then, the cone has the following volume ( $V$ ):

$$V = \frac{1}{3}\pi r^2 h \Rightarrow h = \frac{3V}{r^2}$$

The cone has the following surface area ( $S$ ),

$$S = \pi r l \quad (l = \text{slant height})$$

$$= \pi r \sqrt{r^2 + h^2}$$

$$= \pi r \sqrt{r^2 + \frac{9V^2}{r^4}} = \pi \frac{r \sqrt{9r^2 + V^2}}{\pi r^2}$$

$$= \frac{1}{r} \sqrt{\pi^2 r^6 + 9V^2}$$

$$\frac{dS}{dr} = \frac{r \cdot \frac{6\pi^2 r^5}{2\sqrt{\pi^2 r^6 + 9V^2}} - \sqrt{\pi^2 r^6 + 9V^2}}{r^2}$$

$$= \frac{2\pi^2 r^6 - 9V^2}{r^2 \sqrt{\pi^2 r^6 + 9V^2}}$$

$$\text{Now, } \frac{dS}{dr} = 0 \Rightarrow 2\pi^2 r^6 = 9V^2 \Rightarrow r^6 = \frac{9V^2}{2\pi^2}$$

Thus, it can be easily verified that when  $r^6 = \frac{9V^2}{2\pi^2}$ ,  $\frac{d^2S}{dr^2} < 0$

By second derivative test, the surface area of the cone is the least when  $r^6 = \frac{9V^2}{2\pi^2}$

$$\text{When, } r^6 = \frac{9V^2}{2\pi^2}, h = \frac{3V}{\pi r^2} \left( \frac{2\pi^2 r^6}{9} \right)^{\frac{1}{2}} = \frac{3}{\pi r^2} \frac{\sqrt{2\pi r^3}}{3} = \sqrt{2}r$$

Hence, for a given volume, the right circular cone of the least curved surface has an altitude equal to  $\sqrt{2}$  times the radius of the base.

### Question 25

Show that the semi-vertical angle of the cone of the maximum volume and of the given slant height is  $\tan^{-1}\sqrt{2}$ .

### Solution

Ans: Let  $\theta$  be the semi-vertical angle of the cone.

It is clear that  $\theta \in \left[0, \frac{\pi}{2}\right]$

Let  $r$ ,  $h$ , and  $l$  be the radius, height, and the slant height of the cone respectively.

The slant height of the cone is given as constant.

Now  $r = l \sin \theta$  and  $h = l \cos \theta$

The volume ( $V$ ) of the cone is given by,

$$\begin{aligned} V &= \frac{1}{3} \pi r^2 h \\ &= \frac{1}{3} \pi (l^2 \sin^2 \theta) (l \cos \theta) \\ &= \frac{1}{3} \pi l^3 \sin^2 \theta \cos \theta \\ \therefore \frac{dV}{d\theta} &= \frac{l^3 \pi}{3} [\sin^2 \theta (-\sin \theta) + \cos \theta (2 \sin \theta \cos \theta)] \\ &= \frac{l^3 \pi}{3} [-\sin^3 \theta + 2 \sin \theta \cos^2 \theta] \\ \frac{d^2 V}{d\theta^2} &= \frac{l^3 \pi}{3} [-3 \sin^2 \theta \cos \theta + 2 \cos^3 \theta - 4 \sin^2 \theta \cos \theta] \\ &= \frac{l^3 \pi}{3} [2 \cos^3 \theta - 7 \sin^2 \theta \cos \theta] \end{aligned}$$

$$\text{Now, } \frac{dV}{d\theta} = 0$$

$$\Rightarrow \sin^3 \theta = 2 \sin \theta \cos^2 \theta$$

$$\Rightarrow \tan \theta = 2$$

$$\Rightarrow \theta = \tan^{-1} \sqrt{2}$$

Now, when  $\theta = \tan^{-1} \sqrt{2}$ , then  $\tan^2 \theta = 2$  or  $\sin^2 \theta = 2 \cos^2 \theta$

Then,

$$\frac{d^2V}{d\theta^2} = \frac{l^3\pi}{3} [2\cos^3\theta - 14\cos^3\theta] = -4\pi l^3 \cos^3\theta < 0 \quad (\text{For } \theta \in [0, \frac{\pi}{2}])$$

By second derivative test, the volume (V) is the maximum when  $\theta = \tan^{-1}\sqrt{2}$ . Hence, the semi-vertical angle of the cone of the maximum volume is  $\tan^{-1}\sqrt{2}$  for a given slant height.

### Question 26

Show that semi-vertical angle of right circular cone of given surface area and maximum volume is  $\sin^{-1}\left(\frac{1}{3}\right)$

### Solution

Ans: Let's assume x be the radius and y be the height of the cone and semi-vertical angle be  $\Theta$ .

And, Total Surface area of cone (S) =  $\pi x \sqrt{x^2 + y^2} + \pi x^2$

$$x \sqrt{x^2 + y^2} + x^2 = \frac{S}{\pi}$$

Let's say  $\frac{S}{\pi} = k$

$$x \sqrt{x^2 + y^2} = k - x^2$$

Squaring on both sides

$$x^2 (x^2 + y^2) = (k - x^2)^2$$

$$x^2 y^2 = k^2 - 2kx^2$$

$$x^2 = \frac{k^2}{y^2 + 2k} \quad \dots\dots(1)$$

$$\text{Volume of cone (V)} = \frac{1}{3} \pi x^2 y$$

$$= \frac{1}{3} \pi \left( \frac{k^2}{y^2 + 2k} \right) y$$

$$= \frac{1}{3} \pi k^2 \left( \frac{y}{y^2 + 2k} \right)$$

Derive on both sides

$$\frac{dV}{dy} = \frac{1}{3} \pi k^2 \frac{d}{dy} \frac{y}{y^2 + 2k}$$

$$= \frac{1}{3} \pi k^2 \left[ \frac{(y^2 + 2k) \cdot 1 - y \cdot 2y}{(y^2 + 2k)^2} \right] \quad [\text{Using quotient rule}]$$

$$\frac{dV}{dy} = \frac{1}{3} \pi k^2 \frac{(2k - y^2)}{(y^2 + 2k)^2} \quad \dots\dots(2)$$

$$\text{Now } \frac{dV}{dy} = 0$$

$$\frac{1}{3} \pi k^2 \frac{(2k-y^2)}{(y^2+2k)^2} = 0$$

$$2k - y^2 = 0$$

$$y^2 = 2k$$

$$y = \pm \sqrt{2k}$$

Height can't be negative so we neglect  $y = -\sqrt{2k}$

$$\text{So, } y = \sqrt{2k}$$

As,  $\frac{dV}{dy} > 0$ , therefore, Volume is maximum at  $y = \sqrt{2k}$

From equation (1), We get,

$$x^2 = \frac{k^2}{2k+2k} = \frac{k^2}{4k} = \frac{k}{4}$$

$$x = \frac{\sqrt{k}}{2}$$

Now, Semi-vertical angle of the cone  $\sin \theta = \frac{x}{\sqrt{x^2+y^2}}$

$$= \frac{\frac{\sqrt{k}}{2}}{\sqrt{\frac{k}{4}+2k}} = \frac{\sqrt{k}}{2} \times \sqrt{\frac{4}{9k}} = \frac{1}{3}$$

Which implies

$$\theta = \sin^{-1} \frac{1}{3}$$

### Question 27

The point on the curve  $x^2 = 2y$  which is nearest to the point (0, 5) is

A.

$$(2\sqrt{2}, 4)$$

B.

$$(2\sqrt{2}, 0)$$

C.

$$(0, 0)$$

D.

$$(2, 2)$$

**Solution**

Ans: The given curve is  $x^2 = 2y$

For each value of  $x$ ,  $y$  will be  $\frac{x^2}{2}$  the position of the point will be  $\left(x, \frac{x^2}{2}\right)$

The distance  $d(x)$  between the points  $\left(x, \frac{x^2}{2}\right)$  and  $(0, 5)$  is given by

$$d(x) = \sqrt{(x-0)^2 + \left(\frac{x^2}{2} - 5\right)^2} = \sqrt{\frac{x^4}{4} - 4x^2 + 25}$$

$$\therefore d'(x) = \frac{(x^3 - 8x)}{\sqrt{x^4 - 16x^2 + 100}}$$

$$\text{Now, } d'(x) = 0 \Rightarrow x^3 - 8x = 0$$

$$\Rightarrow x = 0, \pm 2\sqrt{2}$$

$$\text{And } d''(x) = \frac{(x^4 - 16x^2 + 100)(3x^2 - 8) - 2(x^3 - 8x)^2}{(x^4 - 16x^2 + 100)^{\frac{3}{2}}}$$

$$\text{When } x = 0 \text{ then } d''(x) = \frac{36(-8)}{6^3} < 0$$

$$\text{When } x = \pm 2\sqrt{2} \text{ then } d(x) > 0$$

By second derivative test,  $d(x)$  is the minimum at  $x = \pm 2\sqrt{2}$ .

$$\text{When } x = \pm 2\sqrt{2}, y = 4.$$

Hence, the point on the curve  $x^2 = 2y$  which is nearest to the point  $(0, 5)$

is  $(\pm 2\sqrt{2}, 4)$ .

Therefore, the correct option is A.

**Question 28**

For all real values of  $x$ , the minimum value of  $\frac{1-x+x^2}{1+x+x^2}$  is

A.

0

B.

1

C.

3

D.

$$\frac{1}{3}.$$

**Solution**

Ans: Let  $f(x) = \frac{1-x+x^2}{1+x+x^2}$

$$f'(x) = \frac{(1+x+x^2)(-1-2x) - (1-x+x^2)(1+2x)}{(1+x+x^2)^2}$$

$$= \frac{2(x^2-1)}{(1+x+x^2)^2}$$

$$\therefore f'(x) = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$$

$$\text{Now, } f''(x) = 4 \frac{(1+3x-x^3)}{(1+x+x^2)^3}$$

$$\text{And, } f''(1) = \frac{4(1+3-1)}{(1+1+1)^3} = \frac{4}{9} > 0$$

$$\text{And, } f''(-1) = \frac{4(1-3+1)}{(1+1+1)^3} = \frac{-4}{27} < 0$$

By second derivative test, f is the minimum at  $x = 1$  and the minimum

value is given by  $f(1) = \frac{1}{3}$

The correct option is D.

**Question 29**

The maximum value of  $[x(x+1)+1]^{\frac{1}{3}}, 0 \leq x \leq 1$  is

A.

$$\left(\frac{1}{3}\right)^{\frac{1}{3}}$$

B.

$$\frac{1}{2}$$

C.

1

D.

0

**Solution**

Ans: Let  $f(x) = [x(x+1) + 1]^{\frac{1}{3}}$ .

$$\therefore f'(x) = \frac{2x+1}{3[x(x+1)+1]^{\frac{2}{3}}}$$

$$\text{Now, } f(x) = 0 \Rightarrow x = -\frac{1}{2}$$

But,  $x = -1/2$  is not part of the interval  $[0,1]$

Now, evaluate the value of  $f$  at the endpoints of the interval  $[0, 1]$  {i.e.,

at  $x = 0$  and  $x = 1$ }.

$$f(0) = [0(0-1) + 1]^{\frac{1}{3}} = 1$$

$$f(1) = [1(1-1) + 1]^{\frac{1}{3}} = 1$$

Hence, we can conclude that the maximum value of  $f$  is 1 in the interval  $[0, 1]$ .

The correct option is 13.

**Miscellaneous Exercise****Question 1**

Show that the function given by  $f(x) = \frac{\log x}{x}$  has maximum at  $x = e$ .

**Solution**

$$\text{Ans: } f(x) = \frac{\log x}{x}$$

$$f'(x) = \frac{x \left( \frac{1}{x} \right) - \log x}{x^2} = \frac{1 - \log x}{x^2}$$

$$f'(x) = 0$$

$$\Rightarrow 1 - \log x = 0$$

$$\Rightarrow \log x = 1$$

$$\Rightarrow \log x = \log e$$

$$\Rightarrow x = e$$

$$\begin{aligned} f''(x) &= \frac{x^2 \left( -\frac{1}{x} \right) - (1 - \log x)(2x)}{x^4} \\ &= \frac{-x - 2x(1 - \log x)}{x^4} \\ &= \frac{-3 + 2 \log x}{x^3} \\ f''(e) &= \frac{-3 + 2 \log e}{e^3} = \frac{-3 + 2}{e^3} = \frac{-1}{e^3} < 0 \end{aligned}$$

$f$  is the maximum at  $x = e$

## Question 2

The two equal sides of an isosceles triangle with fixed base  $b$  are decreasing at the rate of 3 cm per second. How fast is the area decreasing when the two equal sides are equal to the base?

### Solution

Ans: Let  $\triangle ABC$  be isosceles where

$BC$

is the base of fixed length  $b$ .

Let the length of the two equal sides of  $\triangle ABC$  be  $a$ .

Draw  $AD \perp BC$ .

$$AD = \sqrt{a^2 - \frac{b^2}{4}}$$

$$\text{Area of triangle} = \frac{1}{2}b\sqrt{a^2 - \frac{b^2}{4}}$$

$$\frac{dA}{dt} = \frac{1}{2}b \cdot \frac{2a}{2\sqrt{a^2 - \frac{b^2}{4}}} \frac{da}{dt} = \frac{ab}{\sqrt{4a^2 - b^2}} \frac{da}{dt}$$

$$\frac{da}{dt} = -3 \text{ cm/s}$$

$$\therefore \frac{da}{dt} = \frac{-3ab}{\sqrt{4a^2 - b^2}}$$

when  $a = b$

$$\frac{dA}{dt} = \frac{-3b^2}{\sqrt{4b^2 - b^2}} = \frac{-3b^2}{\sqrt{3b^2}} = -\sqrt{3b}$$

### Question 3

Find the intervals in which the function  $f$  given by  $f(x) = \frac{4 \sin x - 2x - x \cos x}{2 + \cos x}$  is

(i) increasing (ii) decreasing

### Solution

$$\text{Ans: } f(x) = \frac{4 \sin x - 2x - x \cos x}{2 + \cos x}$$

$$\begin{aligned} \therefore f'(x) &= \frac{(2 + \cos x)(4 \cos x - 2 - \cos x + x \sin x) - (4 \sin x - 2x - x \cos x)(-\sin x)}{(2 + \cos x)^2} \\ &= \frac{(2 + \cos x)(3 \cos x - 2 + x \sin x) + \sin x(4 \sin x - 2x - x \cos x)}{(2 + \cos x)^2} \\ &= \frac{6 \cos x - 4 + 2x \sin x + 3 \cos^2 x - 2 \cos x + x \sin x \cos x + 4 \sin^2 x - 2 \sin^2 x - 2x \sin x - x \sin x \cos x}{(2 + \cos x)^2} \\ &= \frac{4 \cos x - 4 + 3 \cos^2 x + 4 \sin^2 x}{(2 + \cos x)^2} \\ &= \frac{4 \cos x - \cos^2 x}{(2 + \cos x)^2} = \frac{\cos x(4 - \cos x)}{(2 + \cos x)^2} \end{aligned}$$

$$f'(x) = 0$$

$$\Rightarrow \cos x = 0 \quad \cos x = 4$$

$$\cos x \neq 4$$

$$\cos x = 0$$

$$\Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$\text{In } \left(0, \frac{\pi}{2}\right) \text{ and } \left(\frac{3\pi}{2}, 2\pi\right), f'(x) > 0$$

$f(x)$  is increasing for  $0 < x < \frac{x}{2}$  and  $\frac{3\pi}{2} < x < 2\pi$ .

$$\text{In } \left( \frac{\pi}{2}, \frac{3\pi}{2} \right), f'(x) < 0$$

$f(x)$  is decreasing for  $\frac{\pi}{2} < x < \frac{3\pi}{2}$ .

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#### Question 4

Find the intervals in which the function  $f$  given by  $f(x) = x^3 + \frac{1}{x^3}, x \neq 0$  is

(i) increasing

(ii) decreasing

#### Solution

Ans:  $f(x) = x^3 + \frac{1}{x^3}$

$$\therefore f'(x) = 3x^2 - \frac{3}{x^4} = \frac{3x^6 - 3}{x^4}$$

$$f'(x) = 0 \Rightarrow 3x^6 - 3 = 0 \Rightarrow x^6 = x = \pm 1$$

In  $(-\infty, 1)$  and  $(1, \infty)$  i.e., when  $x < -1$  and  $x > 1$ ,  $f'(x) > 0$ . when  $x < -1$  and  $x > 1$ ,  $f$  is increasing. In  $(-1, 1)$  i.e., when  $-1 < x < 1$ ,  $f'(x) < 0$ .

Thus, when  $-1 < x < 1$ ,  $f$  is decreasing.

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#### Question 5

Find the maximum area of an isosceles triangle inscribed in the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  with its vertex at one end of the major axis.

#### Solution

Ans:

Ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Let  $ABC$ , be the triangle inscribed in the ellipse where vertex  $C$  is at  $(a, 0)$ . Since the ellipse is symmetrical with  $x$ -axis and  $y$ -axis  $y_1 = \pm \frac{b}{a} \sqrt{a^2 - x_1^2}$ .

Coordinates of  $A$  are  $\left(-x_1, \frac{b}{a} \sqrt{a^2 - x_1^2}\right)$  and coordinates of  $B$  are  $\left(x_1, -\frac{b}{a} \sqrt{a^2 - x_1^2}\right)$ . As the point  $(-x_1, y_1)$  lies on the ellipse, the area of triangle  $ABC$  is

$$A = \frac{1}{2} \left| a \left( \frac{2b}{a} \sqrt{a^2 - x_1^2} \right) + (-x_1) \left( -\frac{b}{a} \sqrt{a^2 - x_1^2} \right) + (-x_1) \left( -\frac{b}{a} \sqrt{a^2 - x_1^2} \right) \right|$$

$$\Rightarrow A = ba\sqrt{a^2 - x_1^2} + x_1 \frac{b}{a} \sqrt{a^2 - x_1^2}$$

$$\therefore \frac{dA}{dx_1} = \frac{-2xb}{2\sqrt{a^2 - x_1^2}} + \frac{b}{a} \sqrt{a^2 - x_1^2} - \frac{2bx_1^2}{a^2\sqrt{a^2 - x_1^2}}$$

$$= \frac{b}{2\sqrt{a^2 - x_1^2}} [-x_1a + (a^2 - x_1^2) - x_1^2]$$

$$= \frac{b(-2x_1^2 - x_1a + a^2)}{a\sqrt{a^2 - x_1^2}}$$

$$\frac{dA}{dx_1} = 0$$

$$\Rightarrow -2x_1^2 - x_1a + a^2 = 0$$

$$\Rightarrow x_1 = \frac{a \pm \sqrt{a^2 - 4(-2)(a^2)}}{2(-2)}$$

$$= \frac{a \pm \sqrt{9a^2}}{-4}$$

$$= \frac{a \pm 3a}{-4}$$

$$\Rightarrow x_1 = -a, \frac{a}{2}$$

$$x_1 \text{ cannot be equal to } a. \therefore x_1 = \frac{a}{2} \Rightarrow y_1 = \frac{b}{a} \sqrt{a^2 - \frac{a^2}{4}} = \frac{ba}{2a} \sqrt{3} = \frac{\sqrt{3}b}{2}$$

$$\text{Now, } \frac{d^2A}{dx_1^2} = \frac{b}{a} \left\{ \frac{\sqrt{a^2 - x_1^2}(-4x_1 - a) - (-2x_1^2 - x_1a + a^2) \frac{(-2x_1)}{2\sqrt{a^2 - x_1^2}}}{a^2 - x_1^2} \right\}$$

$$= \frac{b}{a} \left\{ \frac{(a^2 - x_1^2)(-4x_1 - a) + x_1(-2x_1^2 - x_1a + a^2)}{(a^2 - x_1^2)^{\frac{2}{3}}} \right\}$$

$$= \frac{b}{a} \left\{ \frac{2x^3 - 3a^2x - a^3}{(a^2 - x_1^2)^{\frac{3}{2}}} \right\}$$

when  $x_1 = \frac{a}{2}$ ,

$$\frac{d^2A}{dx_1^2} = \frac{b}{a} \left\{ \frac{2\frac{a^3}{8} - 3\frac{a^3}{2} - a^3}{\left(\frac{3a^2}{4}\right)^{\frac{3}{2}}} \right\} = \frac{b}{a} \left\{ \frac{\frac{a^3}{4} - \frac{3}{2}a^3 - a^3}{\left(\frac{3a^2}{4}\right)^{\frac{3}{2}}} \right\}$$

$$= \frac{b}{a} \left\{ \frac{\frac{9}{4}a^3}{\left(\frac{3a^2}{4}\right)^{\frac{3}{2}}} \right\} < 0$$

Area is the maximum when  $x_1 = \frac{a}{2}$ . Maximum area of the triangle is  $A = b\sqrt{a^2 - \frac{a^2}{4}} + \left(\frac{a}{2}\right) \frac{b}{a} \sqrt{a^2 - \frac{a^2}{4}}$

$$= ab\frac{\sqrt{3}}{2} + \left(\frac{a}{2}\right) \frac{b}{a} \times \frac{a\sqrt{3}}{2}$$

$$= \frac{ab\sqrt{3}}{2} + \frac{ab\sqrt{3}}{4} = \frac{3\sqrt{3}}{4}ab$$

### Question 6

A tank with a rectangular base and rectangular sides, open at the top is to be constructed so that its depth is

$$2m$$

and volume is  $8m^3$ . If the building of the tank costs Rs 70 per sq meter for the base and Rs 45 per sq meter for the sides. What is the cost of the least expensive tank?

## Solution

Ans: Let

$$l, b$$

and  $h$  represent the length, breadth, and height of the tank respectively.

$$\text{height } (h) = 2m$$

$$\text{Volume of the tank} = 8 \text{ m}^3 \quad \text{Volume of the tank} = l \times b \times h \quad 8 = l \times b \times 2$$

$$\Rightarrow lb = 4 \Rightarrow b = \frac{4}{l}$$

$$\text{Area of the base} = lb = 4$$

$$\text{Area of 4 walls } (A) = 2h(l + b)$$

$$\therefore A = 4 \left( l + \frac{4}{l} \right)$$

$$\Rightarrow \frac{dA}{dl} = 4 \left( 1 - \frac{4}{l^2} \right)$$

$$\text{Now, } \frac{dA}{dl} = 0$$

$$\Rightarrow l - \frac{4}{l^2} = 0$$

$$\Rightarrow l^2 = 4$$

$$\Rightarrow l = \pm 2$$

Therefore, we have  $l = 4$ .

$$\therefore b = \frac{4}{l} = \frac{4}{2} = 2$$

$$\frac{d^2A}{dl^2} = \frac{32}{l^3}$$

$$l = 2, \frac{d^2A}{dl^2} = \frac{32}{8} = 4 > 0.$$

Area is the minimum when  $l = 2$ .

We have  $l = b = h = 2$ .

Cost of building base

=

$$\text{Rs } 70 \times (lb) = \text{Rs } 70(4) = \text{Rs } 280$$

$$\text{Cost of building walls} = Rs 2h(l + h) \times 45 = Rs 90(2)(2 + 2) =$$

$$Rs 8(90) = Rs 720$$

$$\text{Required total cost} = Rs(280 + 720) = Rs 1000$$

### Question 7

The sum of the perimeter of a circle and square is  $k$ , where  $k$  is some constant. Prove that the sum of their area is least when the side of the square is double the radius of the circle.

### Solution

$$\text{Ans: } 2\pi r + 4a = k \text{ (where } k \text{ is constant)} \Rightarrow a = \frac{k - 2\pi r}{4}$$

the sum of the areas of the circle and the square ( $A$ ) is given by,

$$A = \pi r^2 + a^2 = \pi r^2 + \frac{(k - 2\pi r)^2}{16}$$

$$\therefore \frac{dA}{dr} = 2\pi r + \frac{2(k - 2\pi r)(-2\pi)}{16} = 2\pi r$$

$$= -\frac{\pi(k - 2\pi r)}{4}$$

$$\text{Now, } \frac{dA}{dr} = 0$$

$$\Rightarrow 2\pi r = \frac{\pi(k - 2\pi r)}{4}$$

$$8r = k - 2\pi r$$

$$\Rightarrow (8 + 2\pi)r = k$$

$$\Rightarrow r = \frac{k}{8 + 2\pi} = \frac{k}{2(4 + \pi)}$$

$$\text{Now, } \frac{d^2 A}{dr^2} = 2\pi + \frac{\pi^2}{2} > 0$$

$$\therefore \text{where } r = \frac{k}{2(4 + \pi)}, \frac{d^2 A}{dr^2} > 0.$$

$$\text{area is least when } r = \frac{k}{2(4 + \pi)} \text{ where } r = \frac{k}{2(4 + \pi)},$$

$$a = \frac{k - 2\pi \left[ \frac{k}{2(4 + \pi)} \right]}{4} = \frac{8k + 2\pi k - 2\pi k}{2(4 + \pi) \times 4} = \frac{k}{4 + \pi} = 2r$$

### Question 8

A window is in the form of a rectangle surmounted by a semicircular opening. The total perimeter of the window is 10 m. Find the dimensions of the window to admit maximum light through the whole opening.

### Solution

Ans:  $x$  and  $y$  are the length and breadth of the rectangular window.

$$\text{Radius of semicircular opening} = \frac{x}{2}$$

$$\therefore x + 2y + \frac{\pi x}{2} = 10$$

$$\Rightarrow x \left( 1 + \frac{\pi}{2} \right) + 2y = 10$$

$$\Rightarrow 2y = 10 - x \left( 1 + \frac{\pi}{2} \right)$$

$$\Rightarrow y = 5 - x \left( \frac{1}{2} + \frac{\pi}{4} \right)$$

$$A = xy + \frac{\pi}{2} \left( \frac{x}{2} \right)^2$$

$$= x \left[ 5 - x \left( \frac{1}{2} + \frac{\pi}{4} \right) \right] + \frac{\pi}{8} x^2$$

$$= 5x - x^2 \left( \frac{1}{2} + \frac{\pi}{4} \right) + \frac{\pi}{8} x^2$$

$$\therefore \frac{dA}{dx} = 5 - 2x \left( \frac{1}{2} + \frac{\pi}{4} \right) + \frac{\pi}{4} x$$

$$\frac{d^2 A}{dx^2} = - \left( 1 + \frac{\pi}{2} \right) + \frac{\pi}{4} = -1 - \frac{\pi}{4}$$

$$\frac{dA}{dx} = 0$$

$$\Rightarrow 5 - x \left( 1 + \frac{\pi}{2} \right) + \frac{\pi}{4} x = 0$$

$$\Rightarrow 5 - x - \frac{\pi}{4} x = 0$$

$$\Rightarrow x \left( 1 + \frac{\pi}{4} \right) = 5$$

$$\Rightarrow x = \frac{5}{\left(1 + \frac{\pi}{4}\right)} = \frac{20}{\pi + 4}$$

$$x = \frac{20}{\pi + 4}, \frac{d^2 A}{dx^2} < 0.$$

area is maximum when length  $x = \frac{20}{\pi + 4}m$ .

$$\text{Now, } y = 5 - \frac{20}{\pi + 4} \left( \frac{2 + \pi}{4} \right) = 5 - \frac{5(2 + \pi)}{\pi + 4} = \frac{10}{\pi + 4}m$$

the required dimensions length =  $\frac{20}{\pi + 4}m$  and breadth =  $\frac{10}{\pi + 4}m$ .

### Question 9

A point of the hypotenuse of a triangle is at distance  $a$  and  $b$  from the sides of the triangle. Show that the minimum

length of the hypotenuse is  $\left( a^{\frac{2}{3}} + b^{\frac{2}{3}} \right)^{\frac{3}{2}}$

### Solution

Ans:  $\triangle ABC$  right-angled at  $B$ .  $AB = x$  and  $BC = y$ .

$P$  be a point on hypotenuse such that  $P$  is at a distance of  $a$  and  $b$  from the sides

$AB$

and

$BC$

respectively.

$$\angle C = \theta$$

$$AC = \sqrt{x^2 + y^2}$$

$$PC = b \operatorname{cosec} \theta$$

$$AP = a \sec \theta$$

$$AC = AP + PC$$

$$AC = b \operatorname{cosec} \theta + a \sec \theta \dots (1)$$

$$\therefore \frac{d(AC)}{d\theta} = -b \operatorname{cosec} \theta \cot \theta + a \sec \theta \tan \theta$$

$$\therefore \frac{d(AC)}{d\theta} = 0$$

$$\Rightarrow a \sec \theta \tan \theta = b \operatorname{cosec} \theta \cot \theta$$

$$\Rightarrow \frac{a}{\cos \theta} \cdot \frac{\sin \theta}{\cos \theta} = \frac{b}{\sin \theta} \frac{\cos \theta}{\sin \theta}$$

$$\Rightarrow a \sin^3 \theta = b \cos^3 \theta$$

$$\Rightarrow (a)^{\frac{1}{3}} \sin \theta = (b)^{\frac{1}{3}} \cos \theta$$

$$\Rightarrow \tan \theta = \left( \frac{b}{a} \right)^{\frac{1}{3}}$$

$$\therefore \sin \theta = \frac{(b)^{\frac{1}{3}}}{\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}} \text{ and } \cos \theta = \frac{(a)^{\frac{1}{3}}}{\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}} \dots (2)$$

$$\text{clearly } \frac{d^2(AC)}{d\theta^2} < 0 \text{ when } \tan \theta = \left( \frac{b}{a} \right)^{\frac{1}{3}}.$$

$$\text{the length of the hypotenuse is the maximum when } \tan \theta = \left( \frac{b}{a} \right)^{\frac{1}{3}}.$$

$$\text{Now, when } \tan \theta = \left( \frac{b}{a} \right)^{\frac{1}{3}}$$

$$\tan \theta = \left( \frac{b}{a} \right)^{\frac{1}{3}}$$

$$AC = \frac{b \sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}}{\frac{1}{b^{\frac{1}{3}}}} + \frac{a \sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}}{\frac{1}{a^{\frac{1}{3}}}}$$

$$= a \sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}} \left( b^{\frac{2}{3}} + a^{\frac{2}{3}} \right)$$

$$= \left( a^{\frac{2}{3}} + b^{\frac{2}{3}} \right)^{\frac{3}{2}}$$

$$\text{maximum length of the hypotenuse is} = \left( a^{\frac{2}{3}} + b^{\frac{2}{3}} \right)^{\frac{3}{2}}.$$

### Question 10

Find the points at which the function  $f$  given by  $f(x) = (x - 2)^4(x + 1)^3$  has

(i) local maxima

(ii) local minima

(iii) point of inflexion

### Solution

$$\text{Ans: } f(x) = (x - 2)^4(x + 1)^3$$

$$\therefore f'(x) = 4(x - 2)^3(x + 1)^3 + 3(x + 1)^2(x - 2)^4$$

$$= (x - 2)^3(x + 1)^2[4(x + 1) + 3(x - 2)]$$

$$= (x - 2)^3(x + 1)^2(7x - 2)$$

$$f'(x) = 0 \Rightarrow x = -1 \text{ and } x = \frac{2}{7} \text{ or } x = 2$$

for  $x$  close to  $\frac{2}{7}$  and to left of  $\frac{2}{7}$ ,  $f'(x) > 0$ .

for  $x$  close to  $\frac{2}{7}$  and to right of  $\frac{2}{7}$ ,  $f'(x) < 0$ .  $x = \frac{2}{7}$  is point of local minima.

as the value of  $x$  varies  $f'(x)$  does not changes its sign.

$x = -1$  is point of inflexion.

### Question 11

Find the absolute maximum and minimum values of the function  $f$  given by

$$f(x) = \cos^2 x + \sin x, x \in [0, \pi]$$

**Solution**

Ans:  $f(x) = \cos^2 x + \sin x$

$$f'(x) = 2 \cos x (-\sin x) + \cos x$$

$$= -2 \sin x \cos x + \cos x$$

$$f'(x) = 0$$

$$\Rightarrow 2 \sin x \cos x = \cos x \Rightarrow \cos x (2 \sin x - 1) = 0$$

$$\Rightarrow \sin x = \frac{1}{2} \text{ or } \cos x = 0$$

$$\Rightarrow x = \frac{\pi}{6}, \text{ or } \frac{\pi}{2} \text{ as } x \in [0, \pi]$$

$$f\left(\frac{\pi}{6}\right) = \cos^2 \frac{\pi}{6} + \sin \frac{\pi}{6} = \left(\frac{\sqrt{3}}{2}\right)^2 + \frac{1}{2} = \frac{5}{4}$$

$$f(0) = \cos^2 0 + \sin 0 = 1 + 0 = 1$$

$$f(\pi) = \cos^2 \pi + \sin \pi = (-1)^2 + 0 = 1$$

$$f\left(\frac{\pi}{2}\right) = \cos^2 \frac{\pi}{2} + \sin \frac{\pi}{2} = 0 + 1 = 1$$

absolute maximum value of  $f$  is  $\frac{5}{4}$  at  $x = \frac{\pi}{6}$

The absolute minimum value of  $f$  is 1 at  $x = 0$ ,  $x = \frac{\pi}{2}$ , and  $\pi$ .

**Question 12**

Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius  $r$  is  $\frac{4r}{3}$ .

**Solution**

Ans:  $V = \frac{1}{3} \pi R^2 h$

$$BC = \sqrt{r^2 - R^2}$$

$$h = r + \sqrt{r^2 - R^2}$$

$$\therefore V = \frac{1}{3} \pi R^2 \left( r + \sqrt{r^2 - R^2} \right) = \frac{1}{3} \pi R^2 r + \frac{1}{3} \pi R^2 \sqrt{r^2 - R^2}$$

$$\therefore \frac{dV}{dR} = \frac{2}{3}\pi Rr + \frac{2\pi}{3}\pi R\sqrt{r^2 - R^2} + \frac{R^2}{3} \cdot \frac{(-2R)}{2\sqrt{r^2 - R^2}}$$

$$= \frac{2}{3}\pi Rr + \frac{2\pi}{3}\pi R\sqrt{r^2 - R^2} - \frac{R^3}{3\sqrt{r^2 - R^2}}$$

$$= \frac{2}{3}\pi Rr + \frac{2\pi Rr(r^2 - R^2) - \pi R^3}{3\sqrt{r^2 - R^2}}$$

$$= \frac{2}{3}\pi Rr + \frac{2\pi Rr^2 - 3\pi Rr^3}{3\sqrt{r^2 - R^2}}$$

$$\frac{dV}{dR^2} = 0$$

$$\Rightarrow \frac{2\pi r R}{3} = \frac{3\pi R^2 - 2\pi Rr^2}{3\sqrt{r^2 - R^2}}$$

$$\Rightarrow 2r\sqrt{r^2 - R^2} = 3R^2 - 2r^2$$

$$\Rightarrow 4r^2(r^2 - R^2) = (3R^2 - 2r^2)^2$$

$$\Rightarrow 14r^4 - 4r^2R^2 = 9R^4 + 4r^4 - 12R^2r^2$$

$$\Rightarrow 9R^4 - 8r^2R^2 = 0$$

$$\Rightarrow 9R^2 = 8r^2$$

$$\Rightarrow R^2 = \frac{8r^2}{9}$$

$$\frac{d^2V}{dR^2} = \frac{2\pi r}{3} + \frac{3\sqrt{r^2 - R^2}(2\pi r^2 - 9\pi R^2) - (2\pi R^3 - 3\pi Rr^2)(-6R)}{9(r^2 - R^2)} \frac{1}{2\sqrt{r^2 - R^2}^2}$$

$$= \frac{2\pi r}{3} + \frac{3\sqrt{r^2 - R^2}(2\pi r^2 - 9\pi R^2) - (2\pi R^3 - 3\pi Rr^2)(3R)}{9(r^2 - R^2)} \frac{1}{2\sqrt{r^2 - R^2}^2}$$

$$\text{when } R^2 = \frac{8r^2}{9}, \frac{d^2V}{dR^2} < 0.$$

$$\text{volume is the maximum when } R^2 = \frac{8r^2}{9}. R^2 = \frac{8r^2}{9},$$

$$\text{height of the cone} = r + \sqrt{r^2 - \frac{8R^2}{9}} = r + \sqrt{\frac{r^2}{9}} = r + \frac{r}{3} = \frac{4r}{3}$$

**Question 13**

Let  $f$  be a function defined on  $[a, b]$  such that  $f'(x) > 0$  for all  $x \in (a, b)$ . Then prove that  $f$  is an increasing function on  $(a, b)$ .

**Solution**

Ans: Consider  $I$  be the interval  $(a, b)$

Given that  $f'(x) > 0$  for all  $x$  in an interval  $I$ . Consider  $x_1, x_2 \in I$  with  $x_1 < x_2$ .

By Lagrange's Mean Value Theorem, we have,

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) \text{ where } x_1 < c < x_2$$

$$\Rightarrow f(x_2) - f(x_1) = (x_2 - x_1)f'(c) \text{ where } x_1 < c < x_2$$

Now  $x_1 < x_2$

$$\Rightarrow x_2 - x_1 > 0 \dots \dots (1)$$

Also,  $f'(x) > 0$  for all  $x$  in an interval  $I$

$$\Rightarrow f'(c) > 0$$

From equation (1),  $f(x_2) - f(x_1) > 0$

$$\Rightarrow f(x_1) < f(x_2)$$

Thus, for every pair of points  $x_1, x_2 \in I$  with  $x_1 < x_2$

$$\Rightarrow f(x_1) < f(x_2)$$

Therefore,  $f(x)$  is strictly increasing in  $I$ .

**Question 14**

Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius  $R$  is  $\frac{2R}{\sqrt{3}}$ , also find the maximum volume.

**Solution**

$$\text{Ans: } h = 2\sqrt{R^2 - r^2}$$

$$V = \pi r^2 h = 2\pi r^2 \sqrt{R^2 - r^2}$$

$$\therefore \frac{dV}{dr} = 4\pi r \sqrt{R^2 - r^2} + \frac{2\pi r^2(-2r)}{2\sqrt{R^2 - r^2}}$$

$$= 4\pi r \sqrt{R^2 - r^2} - \frac{2\pi r^3}{\sqrt{R^2 - r^2}}$$

$$= \frac{4\pi r (R^2 - r^2) - 2\pi r^3}{\sqrt{R^2 - r^2}}$$

$$= \frac{4\pi r R^2 - 6\pi r^3}{\sqrt{R^2 - r^2}}$$

$$\text{Now, } \frac{dV}{dr} = 0 \Rightarrow 4\pi r R^2 - 6\pi r^3 = 0$$

$$\Rightarrow r^2 = \frac{2R^2}{3}$$

$$\frac{d^2V}{dr^2} = \frac{\sqrt{R^2 - r^2} (4\pi R^2 - 18\pi r^2) - (4\pi r R^2 - 6\pi r^3) \frac{(-2r)}{2\sqrt{R^2 - r^2}}}{(R^2 - r^2)}$$

$$= \frac{(R^2 - r^2) (4\pi R^2 - 18\pi r^2) + r (4\pi r R^2 - 6\pi r^3)}{\frac{3}{(R^2 - r^2)^{\frac{3}{2}}}}$$

$$= \frac{4\pi R^4 - 22\pi r^2 R^2 + 12\pi r^4 + 4\pi r^2 R^2}{\frac{3}{(R^2 - r^2)^{\frac{3}{2}}}}$$

$$r^2 = \frac{2R^2}{3}, \frac{d^2V}{dr^2} < 0.$$

$$\text{volume is maximum when } r^2 = \frac{2R^2}{3}. \quad r^2 = \frac{2R^2}{3}.$$

$$\text{height of the cylinder is } 2\sqrt{R^2 - \frac{2R^2}{3}} = 2\sqrt{\frac{R^2}{3}} = \frac{2R}{\sqrt{3}}.$$

$$\text{the volume of the cylinder is maximum when the height of the cylinder is } \frac{2R}{\sqrt{3}}$$

### Question 15

Show that the height of the cylinder of greatest volume which can be inscribed in a right circular cone of height  $h$  and semi vertical angle  $a$  is one-third that of the cone and the greatest volume of the cylinder is  $\frac{4}{27}\pi h^2 \tan^2 a$ .

### Solution

Ans:

$$r = h \tan a$$

since  $\triangle AOG$  is similar to  $\triangle CEG$ ,

$$\frac{AO}{OG} = \frac{CE}{EG}$$

$$\Rightarrow \frac{h}{r} = \frac{H}{r - R}$$

$$\Rightarrow H = \frac{h}{r}(r - R) = \frac{h}{h \tan a}(h \tan a - R) = \frac{1}{\tan a}(h \tan a - R)$$

$$\text{volume of the cylinder is } V = \pi R^2 H = \frac{\pi R^2}{\tan a}(h \tan a - R)$$

$$= \pi R^2 h - \frac{\pi R^3}{\tan a}$$

$$\therefore \frac{dV}{dR} = 2\pi R h - \frac{3\pi R^2}{\tan a}$$

$$\frac{dV}{dR} = 0$$

$$\Rightarrow 2\pi R h = \frac{3\pi R^2}{\tan a}$$

$$\Rightarrow 2h \tan a = 3R$$

$$\Rightarrow R = \frac{2h}{3} \tan a$$

$$\frac{d^2V}{dR^2} = 2\pi h - \frac{6\pi R}{\tan a}$$

And, for  $R = \frac{2h}{3} \tan a$ , we have:

$$\frac{d^2V}{dR^2} = 2\pi h - \frac{6\pi}{\tan a} \left( \frac{2h}{3} \tan a \right) = 2\pi h - 4\pi h = -2\pi h < 0$$

volume of the cylinder is greatest when  $R = \frac{2h}{3} \tan a$ .

$$R = \frac{2h}{3} \tan a, H = \frac{1}{\tan a} \left( h \tan a - \frac{2h}{3} \tan a \right) = \frac{1}{\tan a} \left( \frac{h \tan a}{3} \right) = \frac{h}{3}.$$

the maximum volume of the cylinder can be obtained as

$$\pi \left( \frac{2h}{3} \tan a \right)^2 \left( \frac{h}{3} \right) = \pi \left( \frac{4h^2}{9} \tan^2 a \right) \left( \frac{h}{3} \right) = \frac{4}{27} \pi h^3 \tan^2 a$$

**Question 16**

A cylindrical tank of radius 10 m is being filled with wheat at the rate of 314 cubic meters per hour. Then the depth of the wheat is increasing at the rate of

**A.**

1 m / h

**B.**

0.1 m / h

**C.**

1.1 m / h

**D.**

0.5 m / h

**Solution**

Ans.  $V = \pi(\text{radius})^2 \times \text{height}$

$$= \pi(10)^2 h \quad (\text{radius} = 10 \text{ m})$$

$$= 100\pi h$$

$$\frac{dV}{dt} = 100\pi \frac{dh}{dt}$$

Tank is being filled with wheat at rate of 314 cubic meters per hour.

$$\frac{dV}{dt} = 314 \text{ m}^3/\text{h}$$

$$314 = 100\pi \frac{dh}{dt}$$

$$\Rightarrow \frac{dh}{dt} = \frac{314}{100(3.14)} = \frac{314}{314} = 1$$

The depth of wheat is increasing at 1 m/h.

The correct answer is **A**.