

Chapter 2: Inverse Trigonometric Functions

NCERT · Class 12 · Maths

Exercise 2.1

Question 1

Find the principal value of $\sin^{-1}\left(-\frac{1}{2}\right)$

Solution

Ans: Let's assume when $\sin^{-1}\left(-\frac{1}{2}\right) = y$, Then $\sin y = \left(-\frac{1}{2}\right) = -\sin\left(\frac{\pi}{6}\right) = \sin\left(-\frac{\pi}{6}\right)$.

As we know that the range of the principal value branch of $\sin^{-1}$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\sin\left(-\frac{\pi}{6}\right) = \frac{1}{2}$

Hence, the principal value of $\sin^{-1}\left(-\frac{1}{2}\right)$ is $-\frac{\pi}{6}$.

Question 2

Find the principal value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

Solution

Ans: Let's consider, $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = y$. Then $\cos y = \frac{\sqrt{3}}{2} = \cos\left(\frac{\pi}{6}\right)$

As we know that the range of the principal value branch of $\cos^{-1}$ is $[0, \pi]$ and $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

Therefore, the principal value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ is $\frac{\pi}{6}$.

Question 3

Find the principal value of $\operatorname{cosec}^{-1}(2)$

Solution

Ans: Let's consider, $\operatorname{cosec}^{-1}(2) = y$. Then, $\operatorname{cosec} y = 2 = \operatorname{cosec}\left(\frac{\pi}{6}\right)$.

As we know that the range of the principal value branch of $\operatorname{cosec}^{-1}$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$.

Therefore, the principal value of $\operatorname{cosec}^{-1}(2)$ is $\frac{\pi}{6}$.

Question 4

Find the principal value of $\tan^{-1}(-\sqrt{3})$

Solution

Ans: Let's consider $\tan^{-1}(-\sqrt{3}) = y$. Then, $\tan y = -\sqrt{3} = -\tan \frac{\pi}{3} = \tan\left(-\frac{\pi}{3}\right)$.

As we know that the range of the principal value branch of $\tan^{-1}$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\tan\left(-\frac{\pi}{3}\right)$ is $-\sqrt{3}$.

Hence, the principal value of $\tan^{-1}(-\sqrt{3})$ is $-\frac{\pi}{3}$.

Question 5

Find the principal value of $\cos^{-1}\left(-\frac{1}{2}\right)$

Solution

Ans: Let's consider, $\cos^{-1}\left(-\frac{1}{2}\right) = y$.

$$\text{Then, } \cos y = -\frac{1}{2} = -\cos\left(\frac{\pi}{3}\right) = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\left(\frac{2\pi}{3}\right).$$

As we know that the range of the principal value branch of $\cos^{-1}$ is $[0, \pi]$ and $\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$

Therefore, the principal value of $\cos^{-1}\left(-\frac{1}{2}\right)$ is $\left(\frac{2\pi}{3}\right)$.

Question 6

Find the principal value of $\tan^{-1}(-1)$

Solution

Ans: Let's assume that $\tan^{-1}(-1) = y$.

$$\text{Then, } \tan y = -1 = -\tan\left(\frac{\pi}{4}\right) = \tan\left(-\frac{\pi}{4}\right).$$

As we know that the range of the principal value branch of $\tan^{-1}$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\tan\left(-\frac{\pi}{4}\right) = -1$

Therefore, the principal value of $\tan^{-1}(-1)$ is $-\frac{\pi}{4}$.

Question 7

Find the principal value of $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$

Solution

$$\text{Ans: Let's consider } \sec^{-1}\left(\frac{2}{\sqrt{3}}\right) = y.$$

$$\text{Then, } \sec y = \frac{2}{\sqrt{3}} = \sec\left(\frac{\pi}{6}\right).$$

As we know that the range of the principal value branch of $\sec^{-1}$ is $[0, \pi] - \left\{\frac{\pi}{2}\right\}$ and

$$\sec\left(\frac{\pi}{6}\right) = \frac{2}{\sqrt{3}}$$

Question 8

Find the principal value of $\cot^{-1}(\sqrt{3})$

Solution

Ans: Let's consider $\cot^{-1}(\sqrt{3}) = y$. Then $\cot y = \sqrt{3} = \cot\left(\frac{\pi}{6}\right)$.

As we know that the range of the principal value branch of $\cot^{-1}$ is $(0, \pi)$ and $\cot\left(\frac{\pi}{6}\right) = \sqrt{3}$.

Hence, the principal value of $\cot^{-1}(\sqrt{3})$ is $\frac{\pi}{6}$.

Question 9

Find the principal value of $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

Solution

Ans: Let's $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = y$.

Then $\cos y = -\frac{1}{\sqrt{2}} = -\cos\left(\frac{\pi}{4}\right) = \cos\left(\pi - \frac{\pi}{4}\right) = \cos\left(\frac{3\pi}{4}\right)$.

As we know that the range of the principal value branch of $\cos^{-1}$ is $[0, \pi]$ and $\cos\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}$.

Therefore, the principal value of $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ is $\frac{3\pi}{4}$.

Question 10

Find the principal value of $\operatorname{cosec}^{-1}(-\sqrt{2})$

Solution

Ans: Let's consider, $\operatorname{cosec}^{-1}(-\sqrt{2}) = y$. Then, $\operatorname{cosec} y = -\sqrt{2} = -\operatorname{cosec}\left(\frac{\pi}{4}\right) = \operatorname{cosec}\left(-\frac{\pi}{4}\right)$

As we know that the range of the principal value branch of $\operatorname{cosec}^{-1}$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$ and

$$\operatorname{cosec}\left(-\frac{\pi}{4}\right) = -\sqrt{2}$$

Therefore, the principal value of $\operatorname{cosec}^{-1}(-\sqrt{2})$ is $-\frac{\pi}{4}$.

Question 11

Find the value of $\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)$

Solution

Ans: Let's consider $\tan^{-1}(1) = x$. Then, $\tan x = 1 = \tan\left(\frac{\pi}{4}\right)$.

$$\therefore \tan^{-1}(1) = \frac{\pi}{4}$$

Let's assume, $\cos^{-1}\left(-\frac{1}{2}\right) = y$.

$$\text{Then, } \cos y = -\frac{1}{2} = -\cos\left(\frac{\pi}{3}\right) = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\left(\frac{2\pi}{3}\right).$$

$$\therefore \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$$

Let's again assume that $\sin^{-1}\left(-\frac{1}{2}\right) = z$.

$$\text{Then, } \sin z = -\frac{1}{2} = -\sin\left(\frac{\pi}{6}\right) = \sin\left(-\frac{\pi}{6}\right).$$

$$\therefore \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$$

$$\begin{aligned}\therefore \tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right) \\&= \frac{\pi}{4} + \frac{2\pi}{3} - \frac{\pi}{6} \\&= \frac{3\pi + 8\pi - 2\pi}{12} = \frac{9\pi}{12} = \frac{3\pi}{4}\end{aligned}$$

Question 12

Find the value of $\cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right)$

Solution

Ans: Let's consider, $\cos^{-1}\left(\frac{1}{2}\right) = x$.

Then, $\cos x = \frac{1}{2} = \cos\left(\frac{\pi}{3}\right)$.

$$\therefore \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

Let's assume $\sin^{-1}\left(\frac{1}{2}\right) = y$.

Then, $\sin y = \frac{1}{2} = \sin\left(\frac{\pi}{6}\right)$.

$$\therefore \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

$$\therefore \cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3} + 2 \times \frac{\pi}{6} = \frac{\pi}{3} + \frac{\pi}{3} = \frac{2\pi}{3}$$

Question 13

If $\sin^{-1}x = y$, then

A.

$$0 \leq y \leq \pi$$

B.

$$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

C.

$$0 < y < \pi$$

D.

$$-\frac{\pi}{2} < y < \frac{\pi}{2}$$

Solution

Ans: It is given that $\sin^{-1}x = y$. As we know that the range of the principal value branch of $\sin^{-1}$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Therefore, $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.

Hence option (B) is correct.

Question 14

$\tan^{-1}\sqrt{3} - \sec^{-1}(-2)$ is equal to

A.

$$\pi$$

B.

$$-\pi/3$$

C.

$$\pi/3$$

D.

$$2\pi/3$$

Solution

Ans: Let's consider, $\tan^{-1}\sqrt{3} = x$.

$$\text{Then, } \tan x = \sqrt{3} = \tan \frac{\pi}{3}$$

As we know that the range of the principal value branch of $\tan^{-1}$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \therefore \tan^{-1}\sqrt{3} = \frac{\pi}{3}$

Let assume, $\sec^{-1}(-2) = y$.

$$\text{Then, } \sec y = -2 = -\sec\left(\frac{\pi}{3}\right) = \sec\left(\pi - \frac{\pi}{3}\right) = \sec \frac{2\pi}{3}.$$

As we know that the range of the principal value branch of $\sec^{-1}$ is $[0, \pi] - \left\{\frac{\pi}{2}\right\}$.

$$\therefore \sec^{-1}(-2) = \frac{2\pi}{3}$$

$$\text{Thus, } \tan^{-1}(\sqrt{3}) - \sec^{-1}(-2)$$

$$= \frac{\pi}{3} - \frac{2\pi}{3} = -\frac{\pi}{3}$$

Exercise 2.2

Question 1

$$\text{Prove } 3\sin^{-1}x = \sin^{-1}(3x - 4x^3), x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$$

Solution

$$\text{Ans: To Prove } 3\sin^{-1}x = \sin^{-1}(3x - 4x^3), \text{ where } x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$$

$$\text{Let } x = \sin \theta. \text{ Then, } \sin^{-1}x = \theta.$$

We have given that,

$$\text{R.H.S } \sin^{-1}(3x - 4x^3) = \sin^{-1}(3\sin \theta - 4\sin^3 \theta)$$

$$= \sin^{-1}(\sin 3\theta)$$

$$= 3\theta$$

$$= 3\sin^{-1}x = L.H.S$$

Hence proved.

Question 2

Prove $3\cos^{-1}x = \cos^{-1}(4x^3 - 3x), x \in \left[\frac{1}{2}, 1\right]$

Solution

Ans: To Prove $3\cos^{-1}x = \cos^{-1}(4x^3 - 3x), x \in \left[\frac{1}{2}, 1\right]$

Let $x = \cos \theta$. Then, $\cos^{-1}x = \theta$

We have given that,

$$R.H.S = \cos^{-1}(4x^3 - 3x)$$

$$= \cos^{-1}(4\cos^3\theta - 3\cos\theta)$$

$$= \cos^{-1}(\cos 3\theta)$$

$$= 3\theta$$

$$= 3\cos^{-1}x = L.H.S$$

Hence proved.

Question 3

Write the function in the simplest form: $\tan^{-1} \frac{\sqrt{1+x^2} - 1}{x}, x \neq 0$

Solution

Ans: $\tan^{-1} \frac{\sqrt{1+x^2} - 1}{x}$

By putting $x = \tan \theta \Rightarrow \theta = \tan^{-1}x$

$$\begin{aligned}
\therefore \tan^{-1} \frac{\sqrt{1+x^2}-1}{x} &= \tan^{-1} \frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta} \\
&= \tan^{-1} \left(\frac{\sec\theta-1}{\tan\theta} \right) = \tan^{-1} \left(\frac{1-\cos\theta}{\sin\theta} \right) \\
&= \tan^{-1} \left(\frac{2\sin^2\left(\frac{\theta}{2}\right)}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} \right) \\
&= \tan^{-1} \left(\tan\frac{\theta}{2} \right) = \frac{\theta}{2} = \frac{1}{2}\tan^{-1}x
\end{aligned}$$

Question 4

Write the function in the simplest form: $\tan^{-1} \left(\sqrt{\frac{1-\cos x}{1+\cos x}} \right), x < \pi$

Solution

Ans: Given that, $\tan^{-1} \left(\sqrt{\frac{1-\cos x}{1+\cos x}} \right), x < \pi$

$$\begin{aligned}
&\tan^{-1} \left(\sqrt{\frac{1-\cos x}{1+\cos x}} \right) \\
&= \tan^{-1} \left(\sqrt{\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}} \right) \\
&= \tan^{-1} \left(\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} \right) \\
&= \tan^{-1} \left(\tan\frac{x}{2} \right) \\
&= \frac{x}{2}
\end{aligned}$$

Question 5

Write the function in the simplest form: $\tan^{-1} \left(\frac{\cos x - \sin x}{\cos x + \sin x} \right), 0 < x < \pi$

Solution

$$\begin{aligned}
 \text{Ans: We have given that, } & \tan^{-1} \left(\frac{\cos x - \sin x}{\cos x + \sin x} \right) \\
 = & \tan^{-1} \left(\frac{1 - \left(\frac{\sin x}{\cos x} \right)}{1 + \left(\frac{\sin x}{\cos x} \right)} \right) \\
 = & \tan^{-1} \left(\frac{1 - \tan x}{1 + \tan x} \right) \\
 = & \tan^{-1}(1) - \tan^{-1}(\tan x) \quad \left[\because \frac{-y}{x - xy} = \tan^{-1}x - \tan^{-1}y \right] \\
 = & \frac{\pi}{4} - x
 \end{aligned}$$

Question 6

Write the function in the simplest form: $\tan^{-1} \frac{x}{\sqrt{a^2 - x^2}}, |x| < a$

Solution

$$\begin{aligned}
 \text{Ans: We have given that, } & \tan^{-1} \frac{x}{\sqrt{a^2 - x^2}} \\
 \text{Let's consider, } x = a \sin \theta \Rightarrow & \frac{x}{a} = \sin \theta \Rightarrow \sin^{-1} \left(\frac{x}{a} \right) \\
 \therefore \tan^{-1} \frac{x}{\sqrt{a^2 - x^2}} & \\
 = & \tan^{-1} \left(\frac{a \sin \theta}{\sqrt{a^2 - a^2 \sin^2 \theta}} \right) \\
 = & \tan^{-1} \left(\frac{a \sin \theta}{a \sqrt{1 - \sin^2 \theta}} \right)
 \end{aligned}$$

$$= \tan^{-1} \left(\frac{a \sin \theta}{a \cos \theta} \right)$$

$$= \tan^{-1}(\tan \theta) = \theta = \sin^{-1} \frac{x}{a}$$

Question 7

Write the function in the simplest form: $\tan^{-1} \left(\frac{3a^2x - x^3}{a^3 - 3ax^2} \right)$, $a > 0$; $\frac{-a}{\sqrt{3}} \leq x \leq \frac{a}{\sqrt{3}}$

Solution

Ans: Consider, $\tan^{-1} \left(\frac{3a^2x - x^3}{a^3 - 3ax^2} \right)$

Let's consider, $x = a \tan \theta \Rightarrow \frac{x}{a} = \tan \theta \Rightarrow \theta = \tan^{-1} \left(\frac{x}{a} \right)$

$$\tan^{-1} \left(\frac{3a^2x - x^3}{a^3 - 3ax^2} \right)$$

$$= \tan^{-1} \left(\frac{3a^2 \cdot a \tan \theta - a^3 \tan^3 \theta}{a^3 - 3a \cdot a^2 \tan^2 \theta} \right)$$

$$= \tan^{-1} \left(\frac{3a^3 \tan \theta - a^3 \tan^3 \theta}{a^3 - 3a^3 \tan^2 \theta} \right)$$

$$= \tan^{-1}(\tan 3\theta)$$

$$= 3\theta$$

$$= 3 \tan^{-1} \frac{x}{a}$$

Question 8

Find the value of $\tan^{-1} \left[2 \cos \left(2 \sin^{-1} \frac{1}{2} \right) \right]$

Solution

Ans: Let's consider, $\sin^{-1} \frac{1}{2} = x$

$$\text{Then } \sin x = \frac{1}{2} = \sin\left(\frac{\pi}{6}\right)$$

$$\therefore \sin^{-1} \frac{1}{2} = \frac{\pi}{6}$$

$$\therefore \tan^{-1} \left[2 \cos \left(2 \sin^{-1} \frac{1}{2} \right) \right]$$

$$= \tan^{-1} \left[2 \cos \left(2 \times \frac{\pi}{6} \right) \right]$$

$$= \tan^{-1} \left[2 \cos \frac{\pi}{3} \right]$$

$$= \tan^{-1} \left[2 \times \frac{1}{2} \right]$$

$$= \tan^{-1} 1 = \frac{\pi}{4}$$

Question 9

Find the value of $\tan \frac{1}{2} \left[\sin^{-1} \frac{2x}{1+x^2} + \cos^{-1} \frac{1-y^2}{1+y^2} \right]$, $|x| < 1$, $y > 0$ and $xy < 1$

Solution

Ans: Let consider, $x = \tan \theta$.

Then, $\theta = \tan^{-1} x$.

$$\therefore \sin^{-1} \frac{2x}{1+x^2} = \sin^{-1} \left(\frac{2 \tan \theta}{1 + \tan^2 \theta} \right)$$

$$= \sin^{-1}(\sin 2\theta)$$

$$= 2\theta$$

$$= 2 \tan^{-1} x$$

Let's assume, $y = \tan \theta$. Then, $\theta = \tan^{-1} y$.

$$\therefore \cos^{-1} \left(\frac{1-y^2}{1+y^2} \right)$$

$$\begin{aligned} &= \cos^{-1} \left(\frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} \right) \\ &= \cos^{-1}(\cos 2\theta) \\ &= 2\theta = 2\tan^{-1}y \\ \therefore \tan \frac{1}{2} \left[\sin^{-1} \left(\frac{2x}{1+x^2} \right) + \cos^{-1} \left(\frac{1-y^2}{1+y^2} \right) \right] \\ &= \tan \frac{1}{2} [2\tan^{-1}x + 2\tan^{-1}y] \\ &\left[\text{As, } \tan^{-1}x + \tan^{-1}y = \tan^{-1} \frac{x+y}{1-xy} \right] \\ &= \tan \left[\tan^{-1} \left(\frac{x+y}{1-xy} \right) \right] \\ &= \frac{x+y}{1-xy} \end{aligned}$$

Question 10

Find the values of $\sin^{-1} \left(\sin \frac{2\pi}{3} \right)$

Solution

Ans: Let's consider, $\sin^{-1} \left(\sin \frac{2\pi}{3} \right)$

As we know that $\sin^{-1}(\sin x) = x$

If $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$, which is the principal value branch of $\sin^{-1}x$.

Here, $\frac{2\pi}{3} \notin \left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$

Now, $\sin^{-1} \left(\sin \frac{2\pi}{3} \right)$ can be written as:

$$\sin^{-1} \left(\sin \frac{2\pi}{3} \right)$$

$$\begin{aligned} &= \sin^{-1} \left[\sin \left(\pi - \frac{2\pi}{3} \right) \right] \\ &= \sin^{-1} \left(\sin \frac{\pi}{3} \right), \text{ where } \frac{\pi}{3} \in \left[\frac{-\pi}{2}, \frac{\pi}{2} \right] \\ \therefore \sin^{-1} \left(\sin \frac{2\pi}{3} \right) &= \sin^{-1} \left[\sin \frac{\pi}{3} \right] = \frac{\pi}{3} \end{aligned}$$

Question 11

Find the values of $\tan^{-1} \left(\tan \frac{3\pi}{4} \right)$

Solution

Ans: Let's Consider, $\tan^{-1} \left(\tan \frac{3\pi}{4} \right)$

As we know that $\tan^{-1}(\tan x) = x$

If $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right)$, which is the principal value branch of $\tan^{-1}x$. Here, $\frac{3\pi}{4} \notin \left(-\frac{\pi}{2}, \frac{\pi}{2} \right)$.

Now, $\tan^{-1} \left(\tan \frac{3\pi}{4} \right)$ can be written as:

$$\tan^{-1} \left(\tan \frac{3\pi}{4} \right) = \tan^{-1} \left[-\tan \left(\frac{-3\pi}{4} \right) \right] = \tan^{-1} \left[-\tan \left(\pi - \frac{\pi}{4} \right) \right]$$

$$\tan^{-1} \left(-\tan \frac{\pi}{4} \right) = \tan^{-1} \left[\tan \left(\frac{-\pi}{4} \right) \right] \text{ where } -\frac{\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right)$$

$$\therefore \tan^{-1} \left(\tan \frac{3\pi}{4} \right) = \tan^{-1} \left[\tan \left(\frac{-\pi}{4} \right) \right] = \frac{-\pi}{4}$$

Question 12

Find the values of $\tan \left(\sin^{-1} \frac{3}{5} + \cot^{-1} \frac{3}{2} \right)$

Solution

Ans: Let's consider, $\sin^{-1} \frac{3}{5} = x$.

$$\text{Then, } \sin x = \frac{3}{5}$$

$$\Rightarrow \cos x = \sqrt{1 - \sin^2 x} = \frac{4}{5}$$

$$\Rightarrow \sec x = \frac{5}{4}$$

$$\therefore \tan x = \sqrt{\sec^2 x - 1} = \sqrt{\frac{25}{16} - 1} = \frac{3}{4}$$

$$\therefore x = \tan^{-1} \frac{3}{4}$$

$$\therefore \sin^{-1} \frac{3}{5} = \tan^{-1} \frac{3}{4} \quad \dots (i)$$

$$\begin{aligned} \text{Therefore, } \tan \left(\sin^{-1} \frac{3}{5} + \cot^{-1} \frac{3}{2} \right) \\ = \tan \left(\tan^{-1} \frac{3}{4} + \tan^{-1} \frac{2}{3} \right) \quad [\text{Using (i) and (ii)}] \end{aligned}$$

$$= \tan \left[\tan^{-1} \left(\frac{\frac{3}{4} + \frac{2}{3}}{1 - \frac{3}{4} \cdot \frac{2}{3}} \right) \right]$$

$$\left[\text{As, } \tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x + y}{1 - xy} \right]$$

$$= \tan \left(\tan^{-1} \frac{9 + 8}{12 - 6} \right)$$

$$= \tan \left(\tan^{-1} \frac{17}{6} \right) = \frac{17}{6}$$

Question 13

Find the values of $\cos^{-1} \left(\cos \frac{7\pi}{6} \right)$ is equal to

A.

$$\frac{7\pi}{6}$$

B.

$$\frac{5\pi}{6}$$

C.

$$\frac{\pi}{3}$$

D.

$$\frac{\pi}{6}$$

Solution

Ans: We know that $\cos^{-1}(\cos x) = x$ if $x \in [0, \pi]$, which is the principal value branch of $\cos^{-1}x$.

Here, $\frac{7\pi}{6} \notin [0, \pi]$.

Now, $\cos^{-1}\left(\cos \frac{7\pi}{6}\right)$ can be written as:

$$\cos^{-1}\left(\cos \frac{7\pi}{6}\right) = \cos^{-1}\left(\cos \frac{-7\pi}{6}\right) = \cos^{-1}\left[\cos\left(2\pi - \frac{7\pi}{6}\right)\right] \quad [\because$$

$$\cos(2\pi - x) = \cos x]$$

$$\therefore \cos^{-1}\left(\cos \frac{7\pi}{6}\right) = \cos^{-1}\left(\cos \frac{5\pi}{6}\right) = \frac{5\pi}{6}$$

The correct answer is B.

Question 14

Find the values of $\sin\left(\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right)$ is equal to

A.

$$\frac{1}{2}$$

B.

$$\frac{1}{3}$$

C.

$$\frac{1}{4}$$

D.

$$1$$

Solution

Ans: Let's consider, $\sin^{-1}\left(\frac{-1}{2}\right) = x$.

$$\text{Then, } \sin x = \frac{-1}{2} = -\sin \frac{\pi}{6} = \sin\left(\frac{-\pi}{6}\right).$$

As we know that the range of the principal value branch of $\sin^{-1}$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. $\sin^{-1}\left(\frac{-1}{2}\right) = \frac{\pi}{6}$

$$\therefore \sin\left(\frac{\pi}{3} - \sin^{-1}\left(\frac{-1}{2}\right)\right) = \sin\left(\frac{\pi}{3} + \frac{\pi}{6}\right) = \sin\left(\frac{3\pi}{6}\right) = \sin\left(\frac{\pi}{2}\right) = 1$$

The correct answer is D.

Question 15

$\tan^{-1}(\sqrt{3}) - \cot^{-1}(-\sqrt{3})$ is equal to

A.

$$\pi$$

B.

$$-\frac{\pi}{2}$$

C.

0

D.

$2\sqrt{3}$

Solution

Ans:

Suppose that,

$$A = \tan^{-1}(\sqrt{3}) - \cot^{-1}(-\sqrt{3})$$

As we know that,

$$\cot^{-1}(-x) = \pi - \cot^{-1}(x)$$

So,

$$\cot^{-1}(-\sqrt{3}) = \pi - \cot^{-1}(\sqrt{3})$$

Put the value in the equation (i),

$$A = \tan^{-1}(\sqrt{3}) - \pi + \cot^{-1}(\sqrt{3})$$

As we know that,

$$\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$$

So,

$$\cot^{-1}(\sqrt{3}) = \frac{\pi}{2} - \tan^{-1}(\sqrt{3})$$

Put the value in the equation (ii),

$$A = \tan^{-1}(\sqrt{3}) - \pi + \frac{\pi}{2} - \tan^{-1}(\sqrt{3})$$

$$A = -\frac{\pi}{2}$$

Hence, the correct option is B.

Miscellaneous Exercise

Question 1

Evaluate

$$\cos^{-1} \left(\cos \frac{13\pi}{6} \right)$$

Solution

Ans: Consider $\cos \frac{13\pi}{6}$

$$\cos \frac{13\pi}{6}$$

$$= \cos \left(2\pi + \frac{\pi}{6} \right)$$

$$= \cos \left(\frac{\pi}{6} \right)$$

Further solving,

$$\cos^{-1} \left(\cos \frac{13\pi}{6} \right)$$

$$= \cos^{-1} \left(\cos \left(\frac{\pi}{6} \right) \right)$$

$$= \frac{\pi}{6}$$

$$\text{Therefore, } \cos^{-1} \left(\cos \frac{13\pi}{6} \right) = \frac{\pi}{6}$$

Question 2

Evaluate

$$\tan^{-1} \left(\tan \frac{7\pi}{6} \right)$$

Solution

Ans: Consider $\tan \frac{7\pi}{6}$

$$\tan \frac{7\pi}{6}$$

$$= \tan \left(\pi + \frac{\pi}{6} \right)$$

$$= \tan \left(\frac{\pi}{6} \right)$$

Further solving,

$$\tan^{-1} \left(\tan \frac{7\pi}{6} \right)$$

$$= \tan^{-1} \left(\tan \left(\frac{\pi}{6} \right) \right)$$

$$= \frac{\pi}{6}$$

$$\text{Therefore, } \tan^{-1} \left(\tan \frac{7\pi}{6} \right) = \frac{\pi}{6}$$

Question 3

Using inverse trigonometric identities, prove that $2\sin^{-1} \frac{3}{5} = \tan^{-1} \frac{24}{7}$

Solution

Ans: Assuming

$$2\sin^{-1} \frac{3}{5} = \alpha$$

----(1)

$$\sin \frac{\alpha}{2} = \frac{3}{5}$$

$$\cos \frac{\alpha}{2} = \sqrt{1 - \left(\frac{3}{5} \right)^2}$$

$$\cos \frac{\alpha}{2} = \frac{4}{5}$$

Hence,

$$\sin \alpha = 2 \left(\frac{3}{5} \right) \left(\frac{4}{5} \right)$$

$$\sin \alpha = \frac{24}{25}$$

$$\cos \alpha = \sqrt{1 - \left(\frac{24}{25} \right)^2}$$

$$\cos \alpha = \frac{7}{25}$$

Therefore,

$$\tan \alpha = \frac{24}{7}$$

$$\alpha = \tan^{-1} \left(\frac{24}{7} \right)$$

From Equation (1), it is proved that

$$2\sin^{-1} \frac{3}{5} = \tan^{-1} \left(\frac{24}{7} \right)$$

Question 4

Using inverse trigonometric identities, prove that $\sin^{-1} \frac{8}{17} + \sin^{-1} \frac{3}{5} = \tan^{-1} \frac{77}{36}$

Solution

Ans: Assuming

$$\sin^{-1} \frac{8}{17} = \alpha$$

$$\sin \alpha = \frac{8}{17}$$

$$\cos\alpha = \sqrt{1 - \left(\frac{8}{17}\right)^2}$$

$$\cos\alpha = \frac{15}{17}$$

Hence,

Therefore,

$$\tan\alpha = \frac{8}{15}$$

$$\alpha = \tan^{-1}\left(\frac{8}{15}\right)$$

$$\sin^{-1}\left(\frac{8}{17}\right) = \tan^{-1}\left(\frac{8}{15}\right)$$

----(1)

Assuming that

$$\sin^{-1}\frac{3}{5} = \beta$$

$$\sin\beta = \frac{3}{5}$$

$$\cos\beta = \sqrt{1 - \left(\frac{3}{5}\right)^2}$$

$$\cos\beta = \frac{4}{5}$$

Therefore,

$$\tan\beta = \frac{3}{4}$$

$$\beta = \tan^{-1}\left(\frac{3}{4}\right)$$

$$\sin^{-1}\left(\frac{3}{5}\right) = \tan^{-1}\left(\frac{3}{4}\right)$$

----(2)

Consider

$$\sin^{-1}\left(\frac{8}{17}\right) + \sin^{-1}\left(\frac{3}{5}\right)$$

From Equations (1) and (2),

$$\begin{aligned} & \sin^{-1}\left(\frac{8}{17}\right) + \sin^{-1}\left(\frac{3}{5}\right) \\ &= \tan^{-1}\left(\frac{8}{15}\right) + \tan^{-1}\left(\frac{3}{4}\right) \\ &= \tan^{-1}\left(\frac{\frac{8}{15} + \frac{3}{4}}{1 - \left(\frac{8}{15} \times \frac{3}{4}\right)}\right) \\ &= \tan^{-1}\left(\frac{77}{36}\right) \end{aligned}$$

Hence, it is proved that

$$\sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5} = \tan^{-1}\frac{77}{36}$$

Question 5

Using inverse trigonometric identities, prove that $\cos^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13} = \cos^{-1}\frac{33}{65}$

Solution

Ans: Assuming

$$\cos^{-1}\frac{4}{5} = \alpha$$

$$\cos\alpha = \frac{4}{5}$$

$$\sin\alpha = \sqrt{1 - \left(\frac{4}{5}\right)^2}$$

$$\sin\alpha = \frac{3}{5}$$

Therefore,

$$\tan\alpha = \frac{3}{4}$$

$$\alpha = \tan^{-1}\left(\frac{3}{4}\right)$$

$$\cos^{-1}\left(\frac{4}{5}\right) = \tan^{-1}\left(\frac{3}{4}\right)$$

----(1)

Assuming that

$$\cos^{-1}\frac{12}{13} = \beta$$

$$\cos\beta = \frac{12}{13}$$

$$\sin\beta = \sqrt{1 - \left(\frac{12}{13}\right)^2}$$

$$\sin\beta = \frac{5}{13}$$

Therefore,

$$\tan\beta = \frac{5}{12}$$

$$\beta = \tan^{-1}\left(\frac{5}{12}\right)$$

$$\cos^{-1}\left(\frac{12}{13}\right) = \tan^{-1}\left(\frac{5}{12}\right)$$

----(2)

Consider

$$\cos^{-1}\left(\frac{4}{5}\right) + \cos^{-1}\left(\frac{12}{13}\right)$$

From Equations (1) and (2),

$$\begin{aligned} & \cos^{-1}\left(\frac{4}{5}\right) + \cos^{-1}\left(\frac{12}{13}\right) \\ &= \tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{5}{12}\right) \\ &= \tan^{-1}\left(\frac{\frac{3}{4} + \frac{5}{12}}{1 - \left(\frac{3}{4} \times \frac{5}{12}\right)}\right) \\ &= \tan^{-1}\left(\frac{56}{33}\right) \\ &= \cos^{-1}\left(\frac{33}{65}\right) \end{aligned}$$

Hence, it is proved that

$$\cos^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13} = \cos^{-1}\frac{33}{65}$$

Question 6

Using inverse trigonometric identities, prove that $\cos^{-1}\frac{12}{13} + \sin^{-1}\frac{3}{5} = \sin^{-1}\frac{56}{65}$

Solution

Ans: Assuming

$$\cos^{-1}\frac{12}{13} = \alpha$$

$$\cos\alpha = \frac{12}{13}$$

$$\sin\alpha = \sqrt{1 - \left(\frac{12}{13}\right)^2}$$

$$\sin\alpha = \frac{5}{13}$$

Therefore,

$$\tan\alpha = \frac{5}{12}$$

$$\alpha = \tan^{-1}\left(\frac{5}{12}\right)$$

$$\cos^{-1}\left(\frac{12}{13}\right) = \tan^{-1}\left(\frac{5}{12}\right)$$

----(1)

Assuming that

$$\sin^{-1}\frac{3}{5} = \beta$$

$$\sin\beta = \frac{3}{5}$$

$$\cos\beta = \sqrt{1 - \left(\frac{3}{5}\right)^2}$$

$$\cos\beta = \frac{4}{5}$$

Therefore,

$$\tan\beta = \frac{3}{4}$$

$$\beta = \tan^{-1}\left(\frac{3}{4}\right)$$

$$\sin^{-1}\left(\frac{3}{5}\right) = \tan^{-1}\left(\frac{3}{4}\right)$$

----(2)

Consider

$$\cos^{-1}\left(\frac{12}{13}\right) + \sin^{-1}\left(\frac{3}{5}\right)$$

From Equations (1) and (2),

$$\begin{aligned}
& \cos^{-1} \left(\frac{12}{13} \right) + \sin^{-1} \left(\frac{3}{5} \right) \\
&= \tan^{-1} \left(\frac{5}{12} \right) + \tan^{-1} \left(\frac{3}{4} \right) \\
&= \tan^{-1} \left(\frac{\frac{3}{4} + \frac{5}{12}}{1 - \left(\frac{3}{4} \times \frac{5}{12} \right)} \right) \\
&= \tan^{-1} \left(\frac{56}{33} \right) \\
&= \sin^{-1} \left(\frac{56}{65} \right)
\end{aligned}$$

Hence, it is proved that

$$\cos^{-1} \frac{12}{13} + \sin^{-1} \frac{3}{5} = \sin^{-1} \frac{56}{65}$$

Question 7

Using inverse trigonometric identities, prove that $\tan^{-1} \frac{63}{16} = \sin^{-1} \frac{5}{13} + \cos^{-1} \frac{3}{5}$

Solution

Ans: Assuming

$$\sin^{-1} \frac{5}{13} = \alpha$$

$$\sin \alpha = \frac{5}{13}$$

$$\cos \alpha = \sqrt{1 - \left(\frac{5}{13} \right)^2}$$

$$\cos \alpha = \frac{12}{13}$$

Therefore,

$$\tan \alpha = \frac{5}{12}$$

$$\alpha = \tan^{-1} \left(\frac{5}{12} \right)$$

$$\sin^{-1}\left(\frac{5}{13}\right) = \tan^{-1}\left(\frac{5}{12}\right)$$

----(1)

Assuming that

$$\cos^{-1}\frac{3}{5} = \beta$$

$$\cos\beta = \frac{3}{5}$$

$$\sin\beta = \sqrt{1 - \left(\frac{3}{5}\right)^2}$$

$$\sin\beta = \frac{4}{5}$$

Therefore,

$$\tan\beta = \frac{4}{3}$$

$$\beta = \tan^{-1}\left(\frac{4}{3}\right)$$

$$\cos^{-1}\left(\frac{3}{5}\right) = \tan^{-1}\left(\frac{4}{3}\right)$$

----(2)

Consider

$$\sin^{-1}\left(\frac{5}{13}\right) + \cos^{-1}\left(\frac{3}{5}\right)$$

From Equations (1) and (2),

$$\sin^{-1}\left(\frac{5}{13}\right) + \cos^{-1}\left(\frac{3}{5}\right)$$

$$= \tan^{-1}\left(\frac{5}{12}\right) + \tan^{-1}\left(\frac{4}{3}\right)$$

$$= \tan^{-1}\left(\frac{\frac{4}{3} + \frac{5}{12}}{1 - \left(\frac{4}{3} \times \frac{5}{12}\right)}\right)$$

$$= \tan^{-1}\left(\frac{63}{16}\right)$$

Hence, it is proved that

$$\tan^{-1} \frac{63}{16} = \sin^{-1} \frac{5}{13} + \cos^{-1} \frac{3}{5}$$

Question 8

Using inverse trigonometric identities, prove that $\tan^{-1} \sqrt{x} = \frac{1}{2} \cos^{-1} \left(\frac{1-x}{1+x} \right)$, $x \in [0, 1]$

Solution

Ans: Assuming

$$x = \tan^2 \alpha$$

Consider

$$\begin{aligned} \tan^{-1} \sqrt{x} &= \tan^{-1} \sqrt{\tan^2 \alpha} \\ &= \tan^{-1} \tan \alpha \\ &= \alpha \end{aligned}$$

$$\tan^{-1} \sqrt{x} = \alpha$$

----(1)

Consider

$$\begin{aligned} \frac{1}{2} \cos^{-1} \left(\frac{1-x}{1+x} \right) &= \frac{1}{2} \cos^{-1} \left(\frac{1-\tan^2 \alpha}{1+\tan^2 \alpha} \right) \\ &= \frac{1}{2} \cos^{-1} \cos 2\alpha \\ &= \alpha \end{aligned}$$

$$\frac{1}{2} \cos^{-1} \left(\frac{1-x}{1+x} \right) = \alpha$$

----- (2)

From Equations (1) and (2),

It is proved that

$$\tan^{-1} \sqrt{x} = \frac{1}{2} \cos^{-1} \left(\frac{1-x}{1+x} \right)$$

Question 9

Using inverse trigonometric identities, prove that $\cot^{-1} \left(\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}} \right) = \frac{x}{2}, x \in \left(0, \frac{\pi}{4} \right)$

Solution

Ans: We know that

$$\sqrt{1+\sin x} = \cos \frac{x}{2} + \sin \frac{x}{2}$$

$$\sqrt{1-\sin x} = \cos \frac{x}{2} - \sin \frac{x}{2}$$

Consider

$$\begin{aligned} & \cot^{-1} \left(\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}} \right) \\ &= \cot^{-1} \left(\frac{\left(\cos \frac{x}{2} + \sin \frac{x}{2} \right) + \left(\cos \frac{x}{2} - \sin \frac{x}{2} \right)}{\left(\cos \frac{x}{2} + \sin \frac{x}{2} \right) - \left(\cos \frac{x}{2} - \sin \frac{x}{2} \right)} \right) \end{aligned}$$

$$\cot^{-1} \left(\cot \frac{x}{2} \right)$$

$$= \frac{x}{2}$$

It is proved that

$$\cot^{-1} \left(\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}} \right) = \frac{x}{2}$$

Question 10

Using inverse trigonometric identities, prove that

$$\tan^{-1} \left(\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}} \right) = \frac{\pi}{4} - \frac{1}{2} \cos^{-1} x, -\frac{1}{\sqrt{2}} \leq x \leq 1$$

Solution

Ans: Assuming

$$x = \cos 2\beta$$

$$\sqrt{1+x}$$

$$= \sqrt{1+\cos 2\beta}$$

$$= \sqrt{2} \cos \beta$$

$$\sqrt{1-x}$$

$$= \sqrt{1-\cos 2\beta}$$

$$= \sqrt{2} \sin \beta$$

Consider

$$\tan^{-1} \left(\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}} \right)$$

$$= \tan^{-1} \left(\frac{\sqrt{2} \cos \beta - \sqrt{2} \sin \beta}{\sqrt{2} \cos \beta + \sqrt{2} \sin \beta} \right)$$

$$= \tan^{-1} \left(\frac{1 - \tan \beta}{1 + \tan \beta} \right)$$

$$= \tan^{-1} \left(\tan \left(\frac{\pi}{4} - \beta \right) \right)$$

$$= \frac{\pi}{4} - \beta$$

$$= \frac{\pi}{4} - \frac{1}{2} \cos^{-1} x$$

It is proved that

$$\tan^{-1} \left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}} \right) = \frac{\pi}{4} - \frac{1}{2} \cos^{-1} x$$

Question 11

For what value of x does the equation $2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$ satisfy?

Solution

Ans: Consider

$$2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$$

$$\tan^{-1} \left(\frac{2\cos x}{1-\cos^2 x} \right) = \tan^{-1}(2\operatorname{cosec} x)$$

$$\frac{2\cos x}{1-\cos^2 x} = 2\operatorname{cosec} x$$

$$\sin x \cos x = \sin^2 x$$

$$\sin x (\cos x - \sin x) = 0$$

$$\sin x = 0, \cos x - \sin x = 0$$

$$x = 0, \frac{\pi}{4}$$

Therefore,

$$2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$$

is satisfied for

$$x = 0, \frac{\pi}{4}$$

Question 12

For what value of x does the equation $\tan^{-1} \frac{1-x}{1+x} = \frac{1}{2} \tan^{-1} x, (x > 0)$ satisfy?

Solution

Ans: Consider

$$\tan^{-1} \frac{1-x}{1+x} = \frac{1}{2} \tan^{-1} x$$

$$\tan^{-1} \left(\tan \left(\frac{\pi}{4} - \tan^{-1} x \right) \right) = \frac{1}{2} \tan^{-1} x$$

$$\frac{\pi}{4} - \tan^{-1} x = \frac{1}{2} \tan^{-1} x$$

$$\frac{\pi}{4} = \frac{3}{2} \tan^{-1} x$$

$$\tan^{-1} x = \frac{\pi}{6}$$

$$x = \frac{1}{\sqrt{3}}$$

Therefore,

$$\tan^{-1} \frac{1-x}{1+x} = \frac{1}{2} \tan^{-1} x$$

is satisfied for

$$x = \frac{1}{\sqrt{3}}$$

Question 13

The expression $\sin (\tan^{-1} x), |x| < 1$ is equal to

A.

$$\frac{x}{\sqrt{1-x^2}}$$

B.

$$\frac{1}{\sqrt{1-x^2}}$$

C.

$$\frac{1}{\sqrt{1+x^2}}$$

D.

$$\frac{x}{\sqrt{1+x^2}}$$

Solution

Ans: Assuming

$$x = \tan \beta$$

$$\beta = \tan^{-1} x$$

$$\sin (\tan^{-1} x)$$

$$= \sin \beta$$

$$= \frac{x}{\sqrt{1+x^2}}$$

Hence,

$$\sin (\tan^{-1} x) = \frac{x}{\sqrt{1+x^2}}$$

Therefore, the correct option is option D

Question 14

For what value of x does the equation $\sin^{-1} (1-x) - 2\sin^{-1} x = \frac{\pi}{2}$ satisfy?

A.

$$0, \frac{1}{2}$$

B.

$$1, \frac{1}{2}$$

C.

$$0$$

D.

$$\frac{1}{2}$$

Solution

Ans: Consider

$$\sin^{-1}(1-x) - 2\sin^{-1}x = \frac{\pi}{2}$$

$$-2\sin^{-1}x = \frac{\pi}{2} - \sin^{-1}(1-x)$$

$$-2\sin^{-1}x = \cos^{-1}(1-x)$$

-----(1)

Assuming

$$\cos^{-1}(1-x) = \beta$$

$$\cos\beta = 1-x$$

$$\sin\beta = \sqrt{1-(1-x)^2}$$

$$\sin\beta = \sqrt{2x-x^2}$$

$$\beta = \sin^{-1}\sqrt{2x-x^2}$$

$$\cos^{-1}1-x = \sin^{-1}\sqrt{2x-x^2}$$

-----(2)

Substituting Equation (2) in (1)

$$-2\sin^{-1}x = \sin^{-1}\sqrt{2x-x^2}$$

$$\sin^{-1}(-2x\sqrt{1-x^2}) = \sin^{-1}\sqrt{2x-x^2}$$

$$-2x\sqrt{1-x^2} = \sqrt{2x-x^2}$$

$$4x^2 - 4x^4 = 2x - x^2$$

$$4x^4 - 5x^2 + 2x = 0$$

$$x = 0, \frac{1}{2}, \frac{-1 \pm \sqrt{17}}{4}$$

But considering the given options and also when

$$x = \frac{1}{2}$$

, the equation

$$\sin^{-1}(1-x) - 2\sin^{-1}x = \frac{\pi}{2}$$

doesn't satisfy

Thus, $x=0$ is the only solution.

Hence the correct option is C