

Chapter 11: Three dimensional Geometry

NCERT · Class 12 · Maths

Exercise 11.1

Question 1

If a line makes angles

$$90^\circ, 135^\circ, 45^\circ$$

with the

$$x, y$$

and

$$z$$

axis respectively, find its direction cosines.

Solution

Ans: Let the direction of cosines of the given line be

$$l, m$$

and

$$n$$

.

Therefore,

$$l = \cos 90^\circ$$

$$l = 0$$

$$m = \cos 135^\circ$$

$$m = -\frac{1}{\sqrt{2}}$$

$$n = \cos 45^\circ$$

$$n = \frac{1}{\sqrt{2}}$$

Therefore, the direction of cosines are

$$0, -\frac{1}{\sqrt{2}}$$

and

$$\frac{1}{\sqrt{2}}$$

.

Question 2

Find the direction cosines of a line which makes equal angles with the coordinate axes.

Solution

Ans: Let the direction of the line that makes an angle

$$\alpha$$

with each of the coordinate axes.

Therefore,

$$\begin{aligned}l &= \cos \alpha \\m &= \cos \alpha \\n &= \cos \alpha\end{aligned}$$

As, we know that,

$$l^2 + m^2 + n^2 = 1$$

So,

$$\begin{aligned}\cos^2 \alpha + \cos^2 \alpha + \cos^2 \alpha &= 1 \\3\cos^2 \alpha &= 1\end{aligned}$$

$$\cos^2 \alpha = \frac{1}{3}$$

$$\cos \alpha = \pm \frac{1}{\sqrt{3}}$$

Therefore, the direction of cosines of the line, which is equally inclined to the coordinate axes are

$$\pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}}$$

and

$$\pm \frac{1}{\sqrt{3}}$$

.

Question 3

If a line has the direction ratios $-18, 12, -4$, then what are its direction cosines ?

Solution

Ans: In this question it is given the direction ratio a, b and c which is

$$-18, 12$$

and

$$-4$$

respectively.

So,

$$a = -18$$

$$b = 12$$

$$c = -4$$

The direction cosines is given as,

$$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}}$$

$$l = \frac{-18}{\sqrt{(-18)^2 + (12)^2 + (-4)^2}}$$

$$l = -\frac{18}{22}$$

$$m = \frac{b}{\sqrt{a^2 + b^2 + c^2}}$$

$$m = \frac{12}{\sqrt{(-18)^2 + (12)^2 + (-4)^2}}$$

$$m = \frac{12}{22}$$

$$n = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

$$n = \frac{-4}{\sqrt{(-18)^2 + (12)^2 + (-4)^2}}$$

$$n = -\frac{2}{22}$$

Therefore, the direction cosines are

$$-\frac{18}{22}, \frac{12}{22}$$

and

$$-\frac{2}{22}$$

.

Question 4**Show that the points**

$$(2, 3, 4), (-1, -2, 1), (5, 8, 7)$$

are collinear.**Solution**

Ans: The given points are

$$A(2, 3, 4), B(-1, -2, 1)$$

and

$$C(5, 8, 7)$$

.

The direction ratio of line joining the two coordinates

$$(x_1, y_1, z_1)$$

and

$$(x_2, y_2, z_2)$$

is given as

$$(x_2 - x_1), (y_2 - y_1)$$

and

$$(z_1 - z_2)$$

.

The direction ratio of line

$$AB$$

is given as

$$(-1 - 2), (-2 - 3)$$

and

$$(1 - 4)$$

.

So, the direction ratio of line

$$AB$$

is

$$-3, -5$$

and

$$-3$$

.

The direction ratio of line

$$BC$$

is given as

$$(5 - (-1)), (8 - (-2))$$

and

$$(7 - 1)$$

.

So, the direction ratio of line

$$BC$$

is 6, 10 and 6 .

On comparing the direction ratio of

$$AB$$

and

$$BC$$

, it can be seen that the direction ratio of

$$BC$$

is

$$-2$$

times of

$$AB$$

i.e. they are proportional.

$$AB = \lambda (BC)$$

Therefore,

$$AB \parallel BC$$

. As point B is common to both

$$AB$$

and

$$BC$$

.

Therefore, given points

$$A(2, 3, 4), B(-1, -2, 1)$$

and

$$C(5, 8, 7)$$

are collinear.

Question 5

Find the direction cosines of the sides of the triangle whose vertices are

$(3, 5, -4)$, $(-1, 1, 2)$ and $(-5, -5, -2)$.

Solution

Ans: The given vertices of

$$\triangle ABC$$

are

$$A(3, 5, -4), B(-1, 1, 2)$$

and

$$C(-5, -5, -2)$$

.

Calculating direction cosines of side

$$AB$$

.

The direction ratio of line joining the two coordinates

$$(x_1, y_1, z_1)$$

and

$$(x_2, y_2, z_2)$$

is given as

$$(x_2 - x_1), (y_2 - y_1)$$

and

$$(z_1 - z_2)$$

.

The direction ratio of

$$AB$$

is given as

$$(-1 - 3), (1 - 5)$$

and

$$(2 - (-4))$$

.

The direction ratio of

$$AB$$

is

$$-4, -4$$

and

$$6$$

.

The direction cosines of side

$$AB$$

using direction ratio is given as,

$$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}}$$

$$l = \frac{-4}{\sqrt{(-4)^2 + (-4)^2 + (6)^2}}$$

$$l = -\frac{4}{2\sqrt{17}}$$

$$l = -\frac{2}{\sqrt{17}}$$

$$m = \frac{a}{\sqrt{a^2 + b^2 + c^2}}$$

$$m = \frac{-4}{\sqrt{(-4)^2 + (-4)^2 + (6)^2}}$$

$$m = -\frac{4}{2\sqrt{17}}$$

$$m = -\frac{2}{\sqrt{17}}$$

$$n = \frac{a}{\sqrt{a^2 + b^2 + c^2}}$$

$$n = \frac{6}{\sqrt{(-4)^2 + (-4)^2 + (6)^2}}$$

$$n = \frac{6}{2\sqrt{17}}$$

$$n = \frac{3}{\sqrt{17}}$$

So, the direction cosines of

$$AB$$

is

$$-\frac{2}{\sqrt{17}}, -\frac{2}{\sqrt{17}}$$

and

$$\frac{3}{\sqrt{17}}$$

.

Calculating the direction cosines of side

$$BC$$

The direction ratio of line joining the two coordinates

$$(x_1, y_1, z_1)$$

and

$$(x_2, y_2, z_2)$$

is given as

$$(x_2 - x_1), (y_2 - y_1)$$

and

$$(z_1 - z_2)$$

.

The direction ratio of

$$BC$$

is given as

$$(-1 - 3), (1 - 5)$$

and

$$(2 - (-4))$$

.

The direction ratio of

$$BC$$

is

$$-4, -4$$

and

$$6$$

.

The direction cosines of side

$$BC$$

using direction ratio is given as,

$$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}}$$

$$l = \frac{-4}{\sqrt{(-4)^2 + (-6)^2 + (-4)^2}}$$

$$l = -\frac{4}{2\sqrt{17}}$$

$$l = -\frac{2}{\sqrt{17}}$$

$$m = \frac{a}{\sqrt{a^2 + b^2 + c^2}}$$

$$m = \frac{-6}{\sqrt{(-4)^2 + (-6)^2 + (-4)^2}}$$

$$m = -\frac{6}{2\sqrt{17}}$$

$$m = -\frac{3}{\sqrt{17}}$$

$$n = \frac{a}{\sqrt{a^2 + b^2 + c^2}}$$

$$n = \frac{-4}{\sqrt{(-4)^2 + (-6)^2 + (-4)^2}}$$

$$n = -\frac{4}{2\sqrt{17}}$$

$$n = -\frac{2}{\sqrt{17}}$$

So, the direction cosines of

BC

is

$$-\frac{2}{\sqrt{17}}, -\frac{3}{\sqrt{17}}$$

and

$$-\frac{2}{\sqrt{17}}$$

.

Calculating the direction cosines of side

$$AC$$

The direction ratio of line joining the two coordinates

$$(x_1, y_1, z_1)$$

and

$$(x_2, y_2, z_2)$$

is given as

$$(x_2 - x_1), (y_2 - y_1)$$

and

$$(z_1 - z_2)$$

.

The direction ratio of

$$AC$$

is given as

$$(-5 - 3), (-5 - 5)$$

and

$$(-2 - (-4))$$

.

The direction ratio of

$$AC$$

is

$$-8, -10$$

and 2 .

The direction cosines of side

$$AC$$

using direction ratio is given as,

$$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}}$$

$$l = \frac{-8}{\sqrt{(-8)^2 + (-10)^2 + (2)^2}}$$

$$l = -\frac{8}{2\sqrt{42}}$$

$$l = -\frac{4}{\sqrt{42}}$$

$$m = \frac{a}{\sqrt{a^2 + b^2 + c^2}}$$

$$m = \frac{-10}{\sqrt{(-8)^2 + (-10)^2 + (2)^2}}$$

$$m = -\frac{10}{2\sqrt{42}}$$

$$m = -\frac{5}{\sqrt{42}}$$

$$n = \frac{a}{\sqrt{a^2 + b^2 + c^2}}$$

$$n = \frac{2}{\sqrt{(-8)^2 + (-10)^2 + (2)^2}}$$

$$n = \frac{2}{2\sqrt{42}}$$

$$n = \frac{1}{\sqrt{42}}$$

So, the direction cosines of

BC

is

$$-\frac{4}{\sqrt{42}}, -\frac{5}{\sqrt{42}}$$

and

$$\frac{1}{\sqrt{42}}$$

.

Therefore,

Direction cosines of

$$AB$$

is

$$-\frac{2}{\sqrt{17}}, -\frac{2}{\sqrt{17}}$$

and

$$\frac{3}{\sqrt{17}}$$

.

Direction cosines of

$$BC$$

is

$$-\frac{2}{\sqrt{17}}, -\frac{3}{\sqrt{17}}$$

and

$$-\frac{2}{\sqrt{17}}$$

.

Direction cosines of

BC

is

$$-\frac{4}{\sqrt{42}}, -\frac{5}{\sqrt{42}}$$

and

$$\frac{1}{\sqrt{42}}$$

.

Exercise 11.2

Question 1

Show that the three lines with direction cosines $\frac{12}{13}, \frac{-3}{13}, \frac{-4}{13}; \frac{4}{13}, \frac{12}{13}, \frac{3}{13}; \frac{3}{13}, \frac{-4}{13}, \frac{12}{13}$ are mutually perpendicular.

Solution

Ans: Two lines with direction cosines (l_1, m_1, n_1) and (l_2, m_2, n_2) are perpendicular to each other if

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$$

(i) For the lines with direction cosines

$\frac{12}{13}, \frac{-3}{13}, \frac{-4}{13}$ and $\frac{4}{13}, \frac{12}{13}, \frac{3}{13}$, we obtain

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = \frac{12}{13} \times \frac{4}{13} + \frac{-3}{13} \times \frac{12}{13} + \frac{-4}{13} \times \frac{3}{13}$$

$$= \frac{48}{169} - \frac{36}{169} - \frac{12}{169}$$

$$= 0$$

Therefore, the lines are perpendicular.

(ii) For the lines with direction cosines

$\frac{4}{13}, \frac{12}{13}, \frac{3}{13}$ and $\frac{3}{13}, \frac{-4}{13}, \frac{12}{13}$, we obtain

$$\begin{aligned}l_1l_2 + m_1m_2 + n_1n_2 &= \frac{4}{13} \times \frac{-3}{13} + \frac{12}{13} \times \frac{-4}{13} + \frac{3}{13} \times \frac{12}{13} \\&= \frac{12}{169} - \frac{48}{169} + \frac{36}{169} \\&= 0\end{aligned}$$

Therefore, the lines are perpendicular.

(iii) For the lines with direction cosines

$\frac{3}{13}, \frac{-4}{13}, \frac{12}{13}$ and $\frac{12}{13}, \frac{-3}{13}, \frac{-4}{13}$, we obtain

$$\begin{aligned}l_1l_2 + m_1m_2 + n_1n_2 &= \frac{3}{13} \times \frac{12}{13} + \frac{-4}{13} \times \frac{-3}{13} + \frac{12}{13} \times \frac{-4}{13} \\&= \frac{36}{169} + \frac{12}{169} - \frac{48}{169} \\&= 0\end{aligned}$$

Therefore, the lines are perpendicular.

Question 2

Show that the line through the points

$$(1, -1, 2)$$

and

$$(3, 4, -2)$$

is perpendicular to the line through the points

$$(0, 3, 2)$$

and

$$(3, 5, 6)$$

.

Solution

Ans: Let AB be the line joining the points,

$$(1, -1, 2)$$

and

$$(3, 4, -2)$$

,

and CD be the line through the points

$$(0, 3, 2)$$

and

$$(3, 5, 6)$$

.

The direction ratios a_1, b_1, c_1 of AB are

$$(3 - 1), (4 - (-1)), (-2 - 2)$$

i.e.,

$$2, 5, -4$$

The direction ratios a_2, b_2, c_2 of CD are

$$(3 - 0), (5 - 3), (6 - 2)$$

i.e.,

$$3, 2, 4$$

AB and CD will be perpendicular to each other if

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

$$(2 \times 3) + (5 \times 2) + (-4 \times 4)$$

$$= 6 + 10 - 16$$

$$= 0$$

Therefore, AB and CD are perpendicular to each other.

Question 3

Show that the line through the points

$$(4, 7, 8)$$

,

$$(2, 3, 4)$$

is parallel to the line through the points

$$(-1, -2, 1)$$

,

$$(1, 2, 5)$$

.

Solution

Ans: Let AB be the line through the points

$$(4, 7, 8)$$

and

$$(2, 3, 4)$$

,

CD be the line through the points,

$$(-1, -2, 1)$$

and

$$(1, 2, 5)$$

.

The directions ratios a_1, b_1, c_1 , of AB are

$$(2 - 4), (3 - 7), (4 - 8) \text{ i.e.,}$$

$$-2, -4, -4$$

The direction ratios a_2, b_2, c_2 of CD are

$$(1 - (-1)), (2 - (-2)), (5 - 1) \text{ i.e.,}$$

$$2, 4, 4$$

AB will be parallel to CD, if

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

$$\frac{a_1}{a_2} = \frac{-2}{2} = -1 \dots (1)$$

$$\frac{b_1}{b_2} = \frac{-4}{4} = -1 \dots (2)$$

$$\frac{c_1}{c_2} = \frac{-4}{4} = -1 \dots (3)$$

From Equations (1), (2) and (3), we get

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = -1$$

Thus, AB is parallel to CD.

Question 4

Find the equation of the line which passes through point

$$(1, 2, 3)$$

and is parallel to the vector $3\hat{i} + 2\hat{j} - 2\hat{k}$.

Solution

Ans: It is given that the line passes through the point A

$$(1, 2, 3)$$

. Therefore, the position vector through A is

$$\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$$

Also,

$$\vec{b} = 3\hat{i} + 2\hat{j} - 2\hat{k}$$

It is known that the line which passes through point A and parallel to $\vec{b}$ is given by $\vec{r} = \vec{a} + \lambda\vec{b}$ where λ is a constant.

$$\vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + \lambda(3\hat{i} + 2\hat{j} - 2\hat{k})$$

This is the required equation of the line.

Question 5

Find the equation of the line in vector and in Cartesian form that passes through the point with position vector $2\hat{i} - \hat{j} + 4\hat{k}$ and is in the direction $\hat{i} + 2\hat{j} - \hat{k}$.

Solution

Ans: It is given that the line passes through the point with position vector

$$\vec{a} = 2\hat{i} - \hat{j} + 4\hat{k}$$

$$\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$$

It is known that the line which passes through point A and parallel to $\vec{b}$ is given by $\vec{r} = \vec{a} + \lambda\vec{b}$

$$\vec{r} = 2\hat{i} - \hat{j} + 4\hat{k} + \lambda(\hat{i} + 2\hat{j} - \hat{k})$$

This is the required equation in the vector form

$$\vec{r} = (2 + \lambda)\hat{i} + (2\lambda - 1)\hat{j} + (4 - \lambda)\hat{k}$$

Eliminating λ , we obtain the Cartesian form of equation as

$$\frac{x - 2}{1} = \frac{y + 1}{2} = \frac{z - 4}{-1}$$

This is the required equation of line in the cartesian form.

Question 6

Find the Cartesian equation of the line which passes through the point

$$(-2, 4, -5)$$

and parallel to the line given by $\frac{x + 3}{3} = \frac{y - 4}{5} = \frac{z + 8}{6}$.

Solution

Ans: It is given that the line passes through the point

$$(-2, 4, -5)$$

and is parallel to $\frac{x + 3}{3} = \frac{y - 4}{5} = \frac{z + 8}{6}$

The direction ratios of the line, $\frac{x + 3}{3} = \frac{y - 4}{5} = \frac{z + 8}{6}$ are 3, 5, 6.

The required line is parallel to $\frac{x + 3}{3} = \frac{y - 4}{5} = \frac{z + 8}{6}$

Therefore, its direction ratios are

$$3k, 5k, 6k$$

, when

$$k \neq 0$$

It is known that the equation of the line through the point (x_1, y_1, z_1) and with direction ratios (a, b, c) , is given by

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

Therefore, the equation of the required line is

$$\frac{x + 2}{3k} = \frac{y - 4}{5k} = \frac{z + 5}{6k}$$

$$\Rightarrow \frac{x + 2}{3} = \frac{y - 4}{5} = \frac{z + 5}{6} = k$$

Question 7

The Cartesian equation of a line is

$$\frac{x - 5}{3} = \frac{y + 4}{7} = \frac{z - 6}{2}$$

. Write its vector form.

Solution

Ans: The Cartesian equation of the line is

$$\frac{x - 5}{3} = \frac{y + 4}{7} = \frac{z - 6}{2}$$

.

The given line passes through the point

$$(5, -4, 6)$$

. The position vector of this point is $\vec{a} = 5\hat{i} - 4\hat{j} + 6\hat{k}$

Also, the direction ratios of the given line are

$$3, 7, 2$$

.

This means that the line is in the direction of the vector $\vec{b} = 3\hat{i} + 7\hat{j} + 2\hat{k}$

It is known that the line which passes through point A and parallel to $\vec{b}$ is given by $\vec{r} = \vec{a} + \lambda\vec{b}$

$$\vec{r} = (5\hat{i} - 4\hat{j} + 6\hat{k}) + \lambda(3\hat{i} + 7\hat{j} + 2\hat{k})$$

This is the required equation of the given line in vector form.

Question 8

Find the angle between the following pairs of lines:

(i)

$$\vec{r} = (2\hat{i} - 5\hat{j} + 1\hat{k}) + \lambda(3\hat{i} - 2\hat{j} + 6\hat{k})$$

and

$$\vec{r} = (7\hat{i} - 6\hat{k}) + \mu(\hat{i} + 2\hat{j} + 2\hat{k})$$

(ii)

$$\vec{r} = (3\hat{i} + \hat{j} - 2\hat{k}) + \lambda(\hat{i} - \hat{j} - 2\hat{k})$$

and

$$\vec{r} = (2\hat{i} - \hat{j} - 56\hat{k}) + \mu(3\hat{i} - 5\hat{j} - 4\hat{k})$$

Solution

Ans: (i) Let Q be the angle between the given lines.

The angle between the given pairs of lines is given by, $\cos Q = \left| \frac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1| \cdot |\vec{b}_2|} \right|$

The given lines are parallel to the vectors, $\vec{b}_1 = 3\hat{i} - 2\hat{j} + 6\hat{k}$ and $\vec{b}_2 = \hat{i} + 2\hat{j} + 2\hat{k}$, respectively.

$$|b_1| = \sqrt{3^2 + 2^2 + 6^2} = 7$$

$$|b_2| = \sqrt{1^2 + 2^2 + 2^2} = 3$$

$$b_1 \cdot b_2 = (3\hat{i} - 2\hat{j} + 6\hat{k}) \cdot (\hat{i} + 2\hat{j} + 2\hat{k})$$

$$= 3 \times 1 + 2 \times 2 + 6 \times 2$$

$$= 3 + 4 + 12$$

$$= 19$$

$$\cos Q = \frac{19}{7 \times 3}$$

$$\Rightarrow Q = \cos^{-1} \left(\frac{19}{21} \right)$$

(ii) Let Q be the angle between the given lines.

$$\text{The angle between the given pairs of lines is given by, } \cos Q = \left| \frac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1| \cdot |\vec{b}_2|} \right|$$

The given lines are parallel to the vectors, $\vec{b}_1 = \hat{i} - \hat{j} - 2\hat{k}$ and $\vec{b}_2 = 3\hat{i} - 5\hat{j} - 4\hat{k}$, respectively.

$$|b_1| = \sqrt{1^2 + (-1)^2 + (-2)^2} = \sqrt{6}$$

$$|b_2| = \sqrt{3^2 + (-5)^2 + (-4)^2} = \sqrt{50} = 5\sqrt{2}$$

$$b_1 \cdot b_2 = (\hat{i} - \hat{j} - 2\hat{k}) \cdot (3\hat{i} - 5\hat{j} - 4\hat{k})$$

$$= 1 \times 3 + (-1) \times (-5) + (-2) \times (-4)$$

$$= 3 + 5 + 8$$

$$= 16$$

$$\cos Q = \frac{16}{\sqrt{6} \times 5\sqrt{2}}$$

$$\Rightarrow Q = \cos^{-1} \left(\frac{8}{5\sqrt{3}} \right)$$

Question 9

Find the angle between following pairs of lines.

$$(i) \frac{x-2}{2} = \frac{y-1}{5} = \frac{z+3}{-3} \text{ and } \frac{x+2}{-1} = \frac{y-4}{8} = \frac{z-5}{4}$$

$$(ii) \frac{x}{2} = \frac{y}{2} = \frac{z}{1} \text{ and } \frac{x-5}{-4} = \frac{y-2}{1} = \frac{z-3}{8}$$

Solution

Ans: (i) Let Q be the angle between the given lines.

$$\text{The angle between the given pairs of lines is given by, } \cos Q = \left| \frac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1| \cdot |\vec{b}_2|} \right|$$

The given lines are parallel to the vectors, $\vec{b}_1 = 2\hat{i} + 5\hat{j} - 3\hat{k}$ and $\vec{b}_2 = -\hat{i} + 8\hat{j} + 4\hat{k}$, respectively.

$$|b_1| = \sqrt{2^2 + 5^2 + (-3)^2} = \sqrt{38}$$

$$|b_2| = \sqrt{(-1)^2 + 8^2 + 4^2} = 9$$

$$b_1 \cdot b_2 = (2\hat{i} + 5\hat{j} - 3\hat{k}) \cdot (-\hat{i} + 8\hat{j} + 4\hat{k})$$

$$= 2 \times (-1) + 5 \times 8 + (-3) \times 4$$

$$= -2 + 40 - 12$$

$$= 26$$

$$\cos Q = \frac{26}{9\sqrt{38}}$$

$$\Rightarrow Q = \cos^{-1} \left(\frac{26}{9\sqrt{38}} \right)$$

(i) Let Q be the angle between the given lines.

$$\text{The angle between the given pairs of lines is given by, } \cos Q = \left| \frac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1| \cdot |\vec{b}_2|} \right|$$

The given lines are parallel to the vectors, $\vec{b}_1 = 2\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{b}_2 = 8\hat{i} + \hat{j} + 8\hat{k}$, respectively.

$$|b_1| = \sqrt{2^2 + 2^2 + 1^2} = 3$$

$$|b_2| = \sqrt{4^2 + 1^2 + 8^2} = 9$$

$$b_1 \cdot b_2 = (2\hat{i} + 2\hat{j} + \hat{k}) \cdot (4\hat{i} + \hat{j} + 8\hat{k})$$

$$= 2 \times 4 + 2 \times 1 + 1 \times 8$$

$$= 8 + 2 + 8$$

$$= 18$$

$$\cos Q = \frac{18}{3 \times 9}$$

$$\Rightarrow Q = \cos^{-1} \left(\frac{2}{3} \right)$$

Question 10

Find the values of p so that the lines

$$\frac{1-x}{3} = \frac{7y-14}{2p} = \frac{z-3}{2}$$

and

$$\frac{7-7x}{3p} = \frac{y-5}{1} = \frac{6-z}{5}$$

are at right angles.

Solution

Ans: The given equations can be written in the standard form as

$$\frac{x-1}{-3} = \frac{y-2}{\frac{2p}{7}} = \frac{z-3}{2}$$

and

$$\frac{x-1}{-\frac{3p}{7}} = \frac{y-5}{1} = \frac{z-6}{-5}$$

The direction ratios of the lines are

$$-3, \frac{2p}{7}, 2 \text{ and}$$

$$-\frac{3p}{7}, 1, -5$$

Two lines will be perpendicular to each other if

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

$$\left(-3 \times -\frac{3p}{7}\right) + \left(\frac{2p}{7} \times 1\right) + (2 \times (-5)) = 0$$

$$\Rightarrow \frac{9p}{7} + \frac{2p}{7} - 10 = 0$$

$$\Rightarrow 9p + 2p - 70 = 0$$

$$\Rightarrow 11p - 70 = 0$$

$$\Rightarrow 11p = 70$$

$$\Rightarrow p = \frac{70}{11}$$

Thus, the value of $p = \frac{70}{11}$.

Question 11

Show that the lines $\frac{x-5}{7} = \frac{y+2}{-5} = \frac{z}{1}$ and $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ are perpendicular to each other.

Solution

Ans: We are given the equations of lines as

$$\frac{x-5}{7} = \frac{y+2}{-5} = \frac{z}{1} \text{ and } \frac{x}{1} = \frac{y}{2} = \frac{z}{3}$$

The direction ratios of the given lines are

$$7, -5, 1 \text{ and } 1, 2, 3$$

Two lines will be perpendicular to each other if

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

$$7 \times 1 + (-5) \times 2 + 1 \times 3$$

$$= 7 - 10 + 3$$

$$= 0$$

Therefore, the given lines are perpendicular to each other.

Question 12

Find the shortest distance between the lines $\vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda (\hat{i} - \hat{j} + \hat{k})$ and $\vec{r} = (2\hat{i} - \hat{j} - \hat{k}) + \mu (2\hat{i} + \hat{j} + 2\hat{k})$.

Solution

Ans: The equations of given lines are

$$\vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda (\hat{i} - \hat{j} + \hat{k}) \text{ and } \vec{r} = (2\hat{i} - \hat{j} - \hat{k}) + \mu (2\hat{i} + \hat{j} + 2\hat{k}).$$

It is known that the shortest distance between the lines $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu \vec{b}_2$ is given by

$$d = \left| \frac{(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|} \right|$$

...(1)

On comparing the equations, we obtain,

$$\vec{a}_1 = (\hat{i} + 2\hat{j} + \hat{k})$$

$$\vec{b}_1 = (\hat{i} - \hat{j} + \hat{k})$$

$$\vec{a}_2 = (2\hat{i} - \hat{j} - \hat{k})$$

$$\vec{b}_2 = (2\hat{i} + \hat{j} + 2\hat{k})$$

$$\vec{a}_2 - \vec{a}_1 = (2\hat{i} - \hat{j} - \hat{k}) - (\hat{i} + 2\hat{j} + \hat{k})$$

$$(\hat{i} - 3\hat{j} - 2\hat{k})$$

$$\begin{aligned} \vec{b}_1 \times \vec{b}_2 &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 2 & 1 & 2 \end{vmatrix} \\ &= (-2 - 1)\hat{i} - (2 - 2)\hat{j} + (1 + 2)\hat{k} \\ &= -3\hat{i} + 3\hat{k} \end{aligned}$$

$$\Rightarrow |\vec{b}_1 \times \vec{b}_2| = \sqrt{(-3)^2 + 3^2}$$

$$= \sqrt{9 + 9} = \sqrt{18}$$

$$= 3\sqrt{2}$$

Substituting all values obtained in equation (1),

$$d = \left| \frac{(-3\hat{i} + 3\hat{k}) \cdot (\hat{i} - 3\hat{j} - 2\hat{k})}{3\sqrt{2}} \right|$$

$$= \left| \frac{(-3)1 + 3(-2)}{3\sqrt{2}} \right|$$

$$= \left| \frac{-9}{3\sqrt{2}} \right|$$

$$= \frac{3}{\sqrt{2}}$$

Therefore, the shortest distance between the lines given is $\frac{3}{\sqrt{2}}$ units.

Question 13

Find the shortest distance between the lines $\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$ and $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$.

Solution

Ans: The given lines are $\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$ and $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$.

It is known that the shortest distance between the lines

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1} \text{ and } \frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2} \text{ is given by}$$

$$d = \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1 c_2 - b_2 c_1)^2 + (c_1 a_2 - c_2 a_1)^2 + (a_1 b_2 - a_2 b_1)^2}}$$

....(1)

Comparing the equations, we obtain,

$$x_1 = -1, y_1 = -1, z_1 = -1$$

$$a_1 = 7, b_1 = -6, c_1 = 1$$

$$x_2 = 3, y_2 = 5, z_2 = 7$$

$$a_2 = 1, b_2 = -2, c_2 = 1$$

Then,

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} 4 & 6 & 8 \\ 7 & -6 & 1 \\ 1 & -2 & 1 \end{vmatrix}$$

$$= 4(-16 + 2) - 6(7 - 1) + 8(-14 + 6)$$

$$= -16 - 36 - 64$$

$$= -116$$

Also,

$$\sqrt{(b_1 c_2 - b_2 c_1)^2 + (c_1 a_2 - c_2 a_1)^2 + (a_1 b_2 - a_2 b_1)^2}$$

$$= \sqrt{(-6+2)^2 + (1+7)^2 + (-14+6)^2}$$

$$= \sqrt{116}$$

$$= 2\sqrt{29}$$

Substituting all the values in equation (1),

$$d = \left| \frac{-116}{2\sqrt{29}} \right|$$

$$= \left| \frac{-58}{\sqrt{29}} \right|$$

$$= |-2\sqrt{29}|$$

$$= 2\sqrt{29}$$

Since the distance is always non-negative, the distance between the given lines is $-2\sqrt{29}$ units.

Question 14

Find the shortest distance between the lines whose vector equations are

$$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda (\hat{i} - 3\hat{j} + 2\hat{k}) \text{ and } \vec{r} = (4\hat{i} + 5\hat{j} + 6\hat{k}) + \mu (2\hat{i} + 3\hat{j} + \hat{k}).$$

Solution

Ans: The equations of given lines are

$$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda (\hat{i} - 3\hat{j} + 2\hat{k}) \text{ and } \vec{r} = (4\hat{i} + 5\hat{j} + 6\hat{k}) + \mu (2\hat{i} + 3\hat{j} + \hat{k}).$$

It is known that the shortest distance between the lines $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu \vec{b}_2$ is given by

$$d = \left| \frac{(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|} \right|$$

...(1)

On comparing the equations, we obtain,

$$\vec{a}_1 = (\hat{i} + 2\hat{j} + 3\hat{k})$$

$$\vec{b}_1 = (\hat{i} - 3\hat{j} + 2\hat{k})$$

$$\vec{a}_2 = (4\hat{i} + 5\hat{j} + 6\hat{k})$$

$$\vec{b}_2 = (2\hat{i} + 3\hat{j} + \hat{k})$$

$$\vec{a}_2 - \vec{a}_1 = (4\hat{i} + 5\hat{j} + 6\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k})$$

$$= (3\hat{i} + 3\hat{j} + 3\hat{k})$$

$$\begin{aligned} \vec{b}_1 \times \vec{b}_2 &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 2 \\ 2 & 3 & 1 \end{vmatrix} \\ &= (-3 - 6)\hat{i} - (1 - 4)\hat{j} + (3 + 6)\hat{k} \\ &= -9\hat{i} + 3\hat{j} + 9\hat{k} \end{aligned}$$

$$\Rightarrow |\vec{b}_1 \times \vec{b}_2| = \sqrt{(-9)^2 + 3^2 + 9^2}$$

$$= \sqrt{81 + 9 + 81} = \sqrt{171}$$

$$= 3\sqrt{19}$$

Substituting all values obtained in equation (1),

$$\begin{aligned}
 d &= \left| \frac{(-9\hat{i} + 3\hat{j} + 9\hat{k}) \cdot (3\hat{i} + 3\hat{j} + 3\hat{k})}{3\sqrt{19}} \right| \\
 &= \left| \frac{(-9)3 + 3 \times 3 + 9 \times 3}{3\sqrt{19}} \right| \\
 &= \left| \frac{9}{3\sqrt{19}} \right| \\
 &= \frac{3}{\sqrt{19}}
 \end{aligned}$$

Therefore, the shortest distance between the lines given is

$$\frac{3}{\sqrt{19}}$$

units.

Question 15

Find the shortest distance between the lines whose vector equations are

$$\vec{r} = (1 - t)\hat{i} + (t - 2)\hat{j} + (3 - 2t)\hat{k}$$

$$\vec{r} = (s + 1)\hat{i} + (2s - 1)\hat{j} - (2s + 1)\hat{k}$$

Solution

Ans: Firstly, let's consider the given equations

$$\Rightarrow \vec{r} = (1 - t)\hat{i} + (t - 2)\hat{j} + (3 - 2t)\hat{k}$$

$$\vec{r} = \hat{i} - t\hat{i} + t\hat{j} - 2\hat{j} + 3\hat{k} - 2t\hat{k}$$

$$\vec{r} = \hat{i} - 2\hat{j} + 3\hat{k} + t(-\hat{i} + \hat{j} - 2\hat{k})$$

$$\Rightarrow \vec{r} = (s + 1)\hat{i} + (2s - 1)\hat{j} - (2s + 1)\hat{k}$$

$$\vec{r} = (s + 1)\hat{i} + (2s - 1)\hat{j} - (2s + 1)\hat{k}$$

$$\vec{r} = s\hat{i} + \hat{i} + 2s\hat{j} - \hat{j} - 2s\hat{k} - \hat{i}$$

$$\vec{r} = \hat{i} - \hat{j} - \hat{k} + s(\hat{i} + 2\hat{j} - 2\hat{k})$$

So, now we need to find the shortest distance between

$$\vec{r} = \hat{i} - 2\hat{j} + 3\hat{k} + t(-\hat{i} + \hat{j} - 2\hat{k}) \text{ and } \vec{r} = \hat{i} - \hat{j} - \hat{k} + s(\hat{i} + 2\hat{j} - 2\hat{k})$$

We know that shortest distance between two lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$

$$d = \left| \frac{(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 \times \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|} \right| \dots\dots (1)$$

By comparing the equations we get,

$$\vec{a}_1 = \hat{i} - 2\hat{j} + 3\hat{k}, \vec{b}_1 = -\hat{i} + \hat{j} - 2\hat{k}$$

$$\vec{a}_2 = \hat{i} - \hat{j} - \hat{k}, \vec{b}_2 = \hat{i} + 2\hat{j} - 2\hat{k}$$

Since,

$$(X_1\hat{i} + Y_1\hat{j} + Z_1\hat{k}) - (X_2\hat{i} + Y_2\hat{j} + Z_2\hat{k})$$

$$\Rightarrow (X_1 - X_2)\hat{i} + (Y_1 - Y_2)\hat{j} + (Z_1 - Z_2)\hat{k}$$

$$\text{So, } \vec{a}_2 - \vec{a}_1 = (\hat{i} - \hat{j} - \hat{k}) - (\hat{i} - 2\hat{j} + 3\hat{k}) = \hat{j} - 4\hat{k} \dots\dots(2)$$

And,

$$\vec{b}_1 \times \vec{b}_2 = (-\hat{i} + \hat{j} - 2\hat{k}) \times (\hat{i} + 2\hat{j} - 2\hat{k})$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & -2 \\ 1 & 2 & -2 \end{vmatrix}$$

$$\vec{b}_1 \times \vec{b}_2 = 2\hat{i} - 4\hat{j} - 3\hat{k} \dots\dots(3)$$

$$\Rightarrow |\vec{b}_1 \times \vec{b}_2| = \sqrt{2^2 + (-4)^2 + (-3)^2} = \sqrt{4 + 16 + 9} = 29$$

Now multiplying (2) and (3) we get,

$$(a_1\hat{i} + b_1\hat{j} + c_1\hat{k}) \cdot (a_2\hat{i} + b_2\hat{j} + c_2\hat{k}) = a_1a_2 + b_1b_2 + c_1c_2$$

$$(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1) = (2\hat{i} - 4\hat{j} - 3\hat{k}) \cdot (\hat{j} - 4\hat{k})$$

$$= -4 + 18 = 8 \dots\dots(5)$$

By substituting all the values in (1), we obtain

The shortest distance between the two lines,

$$d = \left| \frac{8}{\sqrt{29}} \right| = \frac{8}{\sqrt{29}}$$

∴ The shortest distance is $\frac{8}{\sqrt{29}}$

Miscellaneous Exercise

Question 1

find the angle between the lines whose direction ratios are a, b, c and $b-c, c-a, a-b$.

Solution

Ans: As we know that, for any angle θ , with direction cosines, a, b, c and $b-c, c-a, a-b$ can be found by,

$$\cos\theta = \left| \frac{a(b-c) + b(b-c) + c(c-a)}{\sqrt{a^2 + b^2 + c^2} \sqrt{(b-c)^2 + (c-a)^2 + (a-b)^2}} \right|$$

Solving this we get, $\cos\theta = 0$

$$\theta = \cos^{-1} 0$$

$$\Rightarrow \theta = 90^\circ$$

Therefore, the angle between the two lines will be 90° .

Question 2

Find the equation of a line parallel to x-axis line passing through the origin.

Solution

Ans: As it is given that the line is passing through the origin and is also parallel to x-axis is x-axis,

Now,

Let us consider a point on x-axis be A

So, the coordinates of A will be (a,0,0)

Now, the direction ratios of OA will be,

$$\Rightarrow (a-0) = a, 0, 0$$

$$\text{The equation of OA} \Rightarrow \frac{x-0}{a} = \frac{y-0}{0} = \frac{z-0}{0} \Rightarrow \frac{x}{1} = \frac{y}{0} = \frac{z}{0} = a$$

Therefore, the equation of the line passing through origin and parallel to x-axis is $\frac{x}{1} = \frac{y}{0} = \frac{z}{0}$.

Question 3

if the lines $\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}$ and $\frac{x-1}{3k} = \frac{y-1}{1} = \frac{z-6}{-5}$ are perpendicular Find the value of k

Solution

Ans: From the given equation we can say that $a_1 = -3, b_1 = 2k, c_1 = 2$ and $a_2 = 3k, b_2 = 1, c_2 = -5$.

We know that the two lines are perpendicular, if $a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$

$$-3(3k) + 2k \times 1 + 2(-5) = 0$$

$$\Rightarrow -9k + 2k - 10 = 0$$

$$\Rightarrow 7k = -10$$

$$\Rightarrow k = -\frac{10}{7}$$

Therefore, the value of k is $-\frac{10}{7}$

Question 4

Find the shortest distance between these two lines $\vec{r} = 6\hat{i} + 2\hat{j} + 2\hat{k} + \lambda(\hat{i} - 2\hat{j} + 2\hat{k})$

$$\vec{r} = -4\hat{i} - \hat{k} + \mu(3\hat{i} - 2\hat{j} - 2\hat{k})$$

Solution

Ans: According to the question, we need to find the distance between the lines,

$$\vec{r} = 6\hat{i} + 2\hat{j} + 2\hat{k} + \lambda (\hat{i} - 2\hat{j} + 2\hat{k})$$

$$\vec{r} = -4\hat{i} - \hat{k} + \mu (3\hat{i} - 2\hat{j} - 2\hat{k})$$

As we know we can find the shortest distance by,

$$d = \left| \frac{(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_1 - \vec{a}_2)}{|\vec{b}_1 \times \vec{b}_2|} \right|$$

Now, from the equation of lines we get

$$\vec{a}_1 = 6\hat{i} + 2\hat{j} + 2\hat{k}$$

$$\vec{b}_1 = \hat{i} - 2\hat{j} + 2\hat{k}$$

$$\vec{a}_2 = -4\hat{i} - \hat{k}$$

$$\vec{b}_2 = 3\hat{i} - 2\hat{j} - 2\hat{k}$$

$$\Rightarrow \vec{a}_2 - \vec{a}_1 = (-4\hat{i} - \hat{k}) - (6\hat{i} + 2\hat{j} + 2\hat{k}) = -10\hat{i} - 2\hat{j} - 3\hat{k}$$

$$\Rightarrow \vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 2 \\ 3 & -2 & -2 \end{vmatrix} = (4+4)\hat{i} - (-2-6)\hat{j} + (-2+6)\hat{k}$$

$$(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1) = (8\hat{i} + 8\hat{j} + 4\hat{k}) \cdot (-10\hat{i} - 2\hat{j} - 3\hat{k})$$

$$= -80 - 16 - 12$$

$$= -108$$

Now, putting these values in $d = \left| \frac{(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_1 - \vec{a}_2)}{|\vec{b}_1 \times \vec{b}_2|} \right|$, we get

$$d = \left| \frac{-108}{12} \right| = 9$$

Therefore, the shortest distance between the above two lines is of 9 units.

Question 5

Find the vector equation of the line passing through the points $(1,2,-4)$ and perpendicular to the two lines $\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7}$ and $\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$

Solution

Ans: According to the question, we get that $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{a} = \hat{i} + 2\hat{j} - 4\hat{k}$

We know that the equation of the line passing through point and also parallel to vector, we get

$$\vec{r} = \hat{i} + 2\hat{j} - 4\hat{k} + \lambda (b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) \dots (1)$$

Now, the equation of the two lines will be

$$\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7} \dots (2)$$

$$\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5} \dots (3)$$

As we know that line (1) and (2) are perpendicular to each other, we get

$$3b_1 - 16b_2 + 7b_3 = 0 \dots (4)$$

Also, we know that the line (1) and (3) are perpendicular to each other, we get

$$3b_1 + 8b_2 - 5b_3 = 0 \dots (5)$$

Now, from equation (4) and (5) we get that

$$\frac{b_1}{(-16)(-5) - 8(7)} = \frac{b_2}{7(3) - 3(-5)} = \frac{b_3}{3(8) - 3(-16)}$$

$$\Rightarrow \frac{b_1}{24} = \frac{b_2}{36} = \frac{b_3}{72} \Rightarrow \frac{b_1}{2} = \frac{b_2}{3} = \frac{b_3}{6}$$

Therefore, direction ratios of $\vec{b}$ are 2,3,6

$$\text{Which means } \vec{b} = 2\hat{i} + 3\hat{j} + 6\hat{k}$$

Putting $\vec{b} = 2\hat{i} + 3\hat{j} + 6\hat{k}$ in equation (1), we get

$$\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda (2\hat{i} + 3\hat{j} + 6\hat{k})$$