

Chapter 10: Vector Algebra

NCERT · Class 12 · Maths

Exercise 10.1

Question 1

Represent graphically a displacement of 40km, 30° east of north.

Solution

Ans:

(Image will be uploaded soon)

Here, vector OP represents the displacement of 40 km, 30° East of North.

Question 2

Classify the following measures as scalars and vectors.

(i) 10 kg

(ii) 2 meters north-west

(iii) 40°

(iv) 40 watt

(v) 10^{-19} coulomb

(vi) 20 m/s^2

Solution

Ans: 10 kg is a scalar quantity because it involves only magnitude.

Ans: 2 meters north-west is a vector quantity as it involves both magnitude and direction.

Ans: 40° is a scalar quantity as it involves only magnitude.

Ans: 40 watts is a scalar quantity as it involves only magnitude.

Ans: 10^{-19} Coulomb is a scalar quantity as it involves only magnitude.

Ans: 20 m/s^2 is a vector quantity as it involves magnitude as well as direction.

Question 3

Classify the following as scalar and vector quantities.

(i) Time Period

(ii) Distance

(iii) Force

(iv) Velocity

(v) Work Done

Solution

Ans: Time period is a scalar quantity as it involves only magnitude.

Ans: Distance is a scalar quantity as it involves only magnitude.

Ans: Force is a vector quantity as it involves both magnitude and direction.

Ans: Velocity is a vector quantity as it involves both magnitude as well as direction.

Ans: Work done is a scalar quantity as it involves only magnitude.

Question 4

In Figure, identify the following vectors.

(Image will be uploaded soon)

(i) Co-initial

(ii) Equal

(iii) Collinear but not equal

Solution

Ans: Vectors $\vec{a}$ and $\vec{d}$ are co-initial because they have the same initial point.

Ans: Vectors $\vec{b}$ and $\vec{d}$ are equal because they have the same magnitude and direction.

Ans: Vectors $\vec{a}$ and $\vec{c}$ are collinear but not equal. This is because although they are parallel, their directions are not the same

Question 5

Answer the following as true or false:

(i) $\vec{a}$ and $-\vec{a}$ are collinear.

(ii) Two collinear vectors are always equal in magnitude.

(iii) Two vectors having same magnitude are collinear.

(iv) Two collinear vectors having the same magnitude are equal.

Solution

Ans: True.

Vectors $\mathbf{a}$ and $-\mathbf{a}$ can be parallel or coinciding vectors. Either way the vectors will have same magnitude but opposite in direction and will be parallel to the same line.

Ans: False.

Collinear vectors are those vectors that are parallel to the same line.

Ans: False.

It is not necessary for two vectors having the same magnitude to be parallel to the same line.

Ans: False.

Two vectors are said to be equal if they have the same magnitude and direction, regardless of the positions of their initial points.

Exercise 10.2

Question 1

Compute the magnitude of the following vectors:

$$\vec{a} = \hat{i} + \hat{j} + \hat{k}; \vec{b} = 2\hat{i} - 7\hat{j} - 3\hat{k}; \vec{c} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} - \frac{1}{\sqrt{3}}\hat{k}$$

Solution

Ans: The given vectors are,

$$\vec{a} = \hat{i} + \hat{j} + \hat{k}; \vec{b} = 2\hat{i} - 7\hat{j} - 3\hat{k}; \vec{c} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} - \frac{1}{\sqrt{3}}\hat{k}$$

Magnitude of a vector say,

$$\vec{a}_1 = \hat{i}_1 + \hat{j}_1 + \hat{k}_1$$

is,

$$|\vec{a}_1| = \sqrt{(i_1)^2 + (j_1)^2 + (k_1)^2}$$

Therefore, the magnitude of the following vectors are,

$$|\vec{a}| = \sqrt{(1)^2 + (1)^2 + (1)^2} = \sqrt{3}$$

Now, for $\vec{b}$,

$$|\vec{b}| = \sqrt{(2)^2 + (-7)^2 + (-3)^2} = \sqrt{4 + 49 + 9} = \sqrt{62}$$

Again, for $\vec{c}$,

$$|\vec{c}| = \sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{-1}{\sqrt{3}}\right)^2} = 1$$

Question 2

Write two different vectors having the same magnitude.

Solution

Ans: Let us assume the two vectors to be,

and

$$\vec{b} = 2\hat{i} - \hat{j} - 3\hat{k}$$

Magnitude of a vector say,

$$\vec{a}_1 = \hat{i}_1 + \hat{j}_1 + \hat{k}_1$$

is,

$$|\vec{a}_1| = \sqrt{(i_1)^2 + (j_1)^2 + (k_1)^2} \vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$$

Therefore, the magnitude of the following vectors are,

$$|\vec{a}| = \sqrt{(1)^2 + (2)^2 + (3)^2} = \sqrt{1 + 4 + 9} = \sqrt{14}$$

Now, for $\vec{b}$,

$$|\vec{b}| = \sqrt{(2)^2 + (-1)^2 + (-3)^2} = \sqrt{4 + 1 + 9} = \sqrt{14}$$

Both, vectors have got an equal magnitude that is $\sqrt{14}$ units but are completely different and not equal vectors since they are going in different directions.

Question 3

Write two different vectors having the same direction.

Solution

Ans: Let us assume the two vectors as,

$$\vec{p} = \hat{i} + \hat{j} + \hat{k}$$

and

$$\vec{q} = 2\hat{i} + 2\hat{j} + 2\hat{k}$$

The direction cosines of the two vectors can be calculated as,

In case of $\vec{p}$,

$$l = \frac{1}{\sqrt{(1)^2 + (1)^2 + (1)^2}} = \frac{1}{\sqrt{3}}$$

$$m = \frac{1}{\sqrt{(1)^2 + (1)^2 + (1)^2}} = \frac{1}{\sqrt{3}}$$

$$n = \frac{1}{\sqrt{(1)^2 + (1)^2 + (1)^2}} = \frac{1}{\sqrt{3}}$$

Now, for

$$\vec{q}$$

,

$$l = \frac{2}{\sqrt{(2)^2 + (2)^2 + (2)^2}} = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$m = \frac{2}{\sqrt{(2)^2 + (2)^2 + (2)^2}} = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$n = \frac{2}{\sqrt{(2)^2 + (2)^2 + (2)^2}} = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}}$$

It can be clearly observed that the direction cosines obtained in case of both the vectors that is, $\vec{p}$ and $\vec{q}$ are same and therefore they have got the same direction but different vectors.

Question 4

Find the values of x and y so that the vectors

$$2\hat{i} + 3\hat{j}$$

and

$$x\hat{i} + y\hat{j}$$

are equal.

Solution

Ans: Observe that it is required for the corresponding coefficients of $\hat{i}$ and $\hat{j}$ to be equal in order to have two equal vectors.

Therefore,

$$x = 2, y = 3$$

Question 5

Find the scalar and vector components of the vector with initial point (2, 1) and terminal point

$$(-5, 7)$$

.

Solution

Ans: The given initial and terminal point are respectively, (2, 1) and

$$(-5, 7)$$

.

Therefore, the vector is,

$$\overrightarrow{PQ} = (-5 - 2)\hat{i} + (7 - 1)\hat{j} = -7\hat{i} + 6\hat{j}$$

Question 6

Find the sum of the vectors with initial point $\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$, $\vec{b} = -2\hat{i} + 4\hat{j} + 5\hat{k}$ and $\vec{c} = \hat{i} - 6\hat{j} - 7\hat{k}$

.

Solution

Ans: The given vectors are respectively, $\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$, $\vec{b} = -2\hat{i} + 4\hat{j} + 5\hat{k}$ and $\vec{c} = \hat{i} - 6\hat{j} - 7\hat{k}$.

Therefore, the sum of the following vectors can be calculated as,

$$\begin{aligned}\vec{a} + \vec{b} + \vec{c} &= \hat{i} - 2\hat{j} + \hat{k} + (-2\hat{i}) + 4\hat{j} + 5\hat{k} + \hat{i} - 6\hat{j} - 7\hat{k} \\ \Rightarrow \vec{a} + \vec{b} + \vec{c} &= (1 - 2 + 1)\hat{i} + (-2 + 4 - 6)\hat{j} + (1 + 5 - 7)\hat{k} \\ \Rightarrow \vec{a} + \vec{b} + \vec{c} &= -4\hat{j} - \hat{k}\end{aligned}$$

Question 7

Find the unit vector in the direction of the vector $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$.

Solution

Ans: The given vectors is, $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$.

Therefore, the unit vector in the direction of the vector, that is, $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$ is represented as $\hat{a}$ and can be calculated as,

$$|\vec{a}| = \sqrt{(1)^2 + (1)^2 + (2)^2} = \sqrt{6}$$

$$\therefore \hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{6}} = \frac{1}{\sqrt{6}}\hat{i} + \frac{1}{\sqrt{6}}\hat{j} + \frac{2}{\sqrt{6}}\hat{k}$$

Question 8

Find the unit vector in the direction of the vector, $\vec{PQ}$, where P and Q are the points (1, 2, 3) and (4, 5, 6) respectively.

Solution

Ans: The given points are P and Q which are (1, 2, 3) and (4, 5, 6) respectively.

Now, the vector $\vec{PQ}$ can be calculated as,

$$\vec{PQ} = (4 - 1)\hat{i} + (5 - 2)\hat{j} + (6 - 3)\hat{k} = 3\hat{i} + 3\hat{j} + 3\hat{k}$$

Therefore, the unit vector in the direction of the vector, that is, $\vec{PQ} = 3\hat{i} + 3\hat{j} + 3\hat{k}$ is represented as $\widehat{PQ}$ and can be calculated as,

$$|\vec{PQ}| = \sqrt{(3)^2 + (3)^2 + (3)^2} = \sqrt{27} = 3\sqrt{3}$$

$$\therefore \widehat{PQ} = \frac{\vec{PQ}}{|\vec{PQ}|} = \frac{3\hat{i} + 3\hat{j} + 3\hat{k}}{3\sqrt{3}} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}$$

Question 9

For the given vectors $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = -\hat{i} + \hat{j} + \hat{k}$, find the unit vector in the direction of the vector, $\vec{a} + \vec{b}$.

Solution

Ans: It is given that the vectors are $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = -\hat{i} + \hat{j} + \hat{k}$ respectively.

Now, the vector $\vec{a + b}$ can be calculated as,

$$\therefore \vec{a + b} = (2 - 1)\hat{i} + (-1 + 1)\hat{j} + (2 - 1)\hat{k} = \hat{i} + \hat{k}$$

Therefore, the unit vector in the direction of the vector, that is, $\vec{a + b} = \hat{i} + \hat{k}$ is represented as $\widehat{a + b}$ and can be calculated as,

$$|\vec{a + b}| = \sqrt{(1)^2 + (0)^2 + (1)^2} = \sqrt{2} = \sqrt{2}$$

$$\therefore \widehat{a + b} = \frac{\vec{a + b}}{|\vec{a + b}|} = \frac{\hat{i} + \hat{k}}{\sqrt{2}} = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{k}$$

Question 10

Find a vector in the direction of the vector $5\hat{i} - \hat{j} + 2\hat{k}$ which has magnitude 8 units.

Solution

Ans: Assume the given vector as, $\vec{a} = 5\hat{i} - \hat{j} + 2\hat{k}$.

Therefore, the unit vector in the direction of the vector, that is, $\vec{a} = 5\hat{i} - \hat{j} + 2\hat{k}$ is represented as $\hat{a}$ and can be calculated as,

$$|\vec{a}| = \sqrt{(5)^2 + (-1)^2 + (2)^2} = \sqrt{30}$$

$$\therefore \hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{5\hat{i} - \hat{j} + 2\hat{k}}{\sqrt{10}} = \frac{5}{\sqrt{10}}\hat{i} - \frac{1}{\sqrt{10}}\hat{j} + \frac{2}{\sqrt{10}}\hat{k}$$

Now, a vector in the direction of the vector $5\hat{i} - \hat{j} + 2\hat{k}$ which has magnitude 8 units can be calculated as,

$$\therefore 8\hat{a} = 8 \left(\frac{5}{\sqrt{10}}\hat{i} - \frac{1}{\sqrt{10}}\hat{j} + \frac{2}{\sqrt{10}}\hat{k} \right) = \frac{40}{\sqrt{10}}\hat{i} - \frac{8}{\sqrt{10}}\hat{j} + \frac{16}{\sqrt{10}}\hat{k}$$

Question 11

Show that the vectors

$$2\hat{i} - 3\hat{j} + 4\hat{k}$$

and

$$-4\hat{i} + 6\hat{j} - 8\hat{k}$$

are collinear.

Solution

Ans: Assume the given vector as, $\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}$ and

$$\vec{b} = -4\hat{i} + 6\hat{j} - 8\hat{k}$$

respectively.

Therefore, it can be observed that $\vec{b}$ can be deduced in terms of $\vec{a}$ as,

$$\vec{b} = -4\hat{i} + 6\hat{j} - 8\hat{k} = -2(2\hat{i} - 3\hat{j} + 4\hat{k}) = -2\vec{a}$$

$$\therefore \vec{b} = \lambda \vec{a}$$

The value of $\lambda = -2$ in this case and therefore the given two vectors collinear.

Question 12

Find the direction cosines of the vector

$$\hat{i} + 2\hat{j} + 3\hat{k}$$

.

Solution

Ans: Assume the given vector as, $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$.

Therefore, the magnitude of the vector, that is, $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$ can be calculated as,

$$|\vec{a}| = \sqrt{(1)^2 + (2)^2 + (3)^2} = \sqrt{14}$$

Therefore, the direction cosines of the vector, that is, $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$ can be calculated as,

$$\left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right).$$

Question 13

Find the direction cosines of the vector joining the points A (1, 2, -3) and B (-1, -2, 1) directed from A to B.

Solution

Ans: Observe that the given points are, A (1, 2, -3) and B (-1, -2, 1).

Hence, the vector $\vec{AB}$ can be calculated as,

$$\therefore \vec{AB} = (-1 - 1)\hat{i} + (-2 - 2)\hat{j} + (1 - (-3))\hat{k} = -2\hat{i} - 4\hat{j} + 4\hat{k}$$

Therefore, the magnitude of the vector, that is, $\vec{AB} = -2\hat{i} - 4\hat{j} + 4\hat{k}$ can be calculated as,

$$|\vec{AB}| = \sqrt{(-2)^2 + (-4)^2 + (4)^2} = 6$$

Therefore, the direction cosines of the vector, that is, $\vec{AB} = -2\hat{i} - 4\hat{j} + 4\hat{k}$ can be calculated as, $\left(\frac{-1}{3}, \frac{-2}{3}, \frac{2}{3} \right).$

Question 14

Show that the vector $\hat{i} + \hat{j} + \hat{k}$ is equally inclined to the axes OX, OY and OZ.

Solution

Ans: Consider the given vector as,

$$\vec{a} = \hat{i} + \hat{j} + \hat{k}$$

.

Therefore, the magnitude and the direction cosines of the vector, that is,

$$\vec{a} = \hat{i} + \hat{j} + \hat{k}$$

can be calculated as,

$$|\vec{a}| = \sqrt{(1)^2 + (1)^2 + (1)^2} = \sqrt{3}$$

$$\therefore \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right).$$

Assume, α , β and γ to be the angles formed by $\vec{a}$ with the axes OX , OY and OZ .

$$\therefore \cos \alpha = \frac{1}{\sqrt{3}}, \cos \beta = \frac{1}{\sqrt{3}}, \cos \gamma = \frac{1}{\sqrt{3}}$$

This shows that the given vector is equally inclined to the the positive direction of all the three axes.

Question 15

Find the position vector of a point R which divides the line segment joining two points P and Q whose position vectors are $\hat{i} + 2\hat{j} - \hat{k}$ and $-\hat{i} + \hat{j} + \hat{k}$ respectively in the ratio 2:1

(i) internally

(ii) externally

Solution

Ans: Observe that the given position vectors of the two points, P and Q are,

$$\vec{OP} = \hat{i} + 2\hat{j} - \hat{k}$$

and

$$\vec{OQ} = -\hat{i} + \hat{j} + \hat{k}$$

.

Therefore, the position vector of R dividing the line joining the two given points through an internal division in the ratio of 2 : 1 can be calculated using the formula, $\frac{m\vec{b} + n\vec{a}}{m+n}$ (where the ratio is represented by $m : n$), as shown below,

$$\overrightarrow{OR} = \frac{2(-\hat{i} + \hat{j} + \hat{k}) + 1(\hat{i} + 2\hat{j} - \hat{k})}{2 + 1} = \frac{-\hat{i} + 4\hat{j} + \hat{k}}{3} = \frac{-1}{3}\hat{i} + \frac{4}{3}\hat{j} + \frac{1}{3}\hat{k}$$

Ans: Observe that the given position vectors of the two points, P and Q are,

$$\overrightarrow{OP} = \hat{i} + 2\hat{j} - \hat{k}$$

and

$$\overrightarrow{OQ} = -\hat{i} + \hat{j} + \hat{k}$$

.

Therefore, the position vector of R dividing the line joining the two given points through an external division in the ratio of $2 : 1$ can be calculated using the formula, $\frac{m\vec{b} - n\vec{a}}{m - n}$ (where the ratio is represented by $m : n$), as shown below,

$$\overrightarrow{OR} = \frac{2(-\hat{i} + \hat{j} + \hat{k}) - 1(\hat{i} + 2\hat{j} - \hat{k})}{2 - 1} = \frac{-3\hat{i} - \hat{k}}{1} = -3\hat{i} + 3\hat{k}$$

.

Question 16

Find the position vector of the midpoint of the vector joining the points $P(2, 3, 4)$ and $Q(4, 1, -2)$.

Solution

Ans: Consider the midpoint of the vector joining the position vectors of the two points,

$$\overrightarrow{OP} = 2\hat{i} + 3\hat{j} + 4\hat{k}$$

and

$$\overrightarrow{OQ} = 4\hat{i} + \hat{j} - 2\hat{k}$$

to be R .

Therefore, the position vector of R dividing the line joining the two given points through an internal division in the ratio of $1 : 1$ can be calculated as,

$$\overrightarrow{OR} = \frac{(2\hat{i} + 3\hat{j} + 4\hat{k}) + (4\hat{i} + \hat{j} - 2\hat{k})}{1 + 1} = \frac{6\hat{i} + 4\hat{j} + 2\hat{k}}{2} = 3\hat{i} + 2\hat{j} + \hat{k}$$

Question 17

Show that the positions

A, B

and

C

with position vectors $\vec{a} = 3\hat{i} - 4\hat{j} - 4\hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{c} = \hat{i} - 3\hat{j} - 5\hat{k}$ respectively form the vertices of a right-angled triangle.

Solution

Ans: Observe that the position vectors of the points

A, B

and

C

are, $\vec{a} = 3\hat{i} - 4\hat{j} - 4\hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{c} = \hat{i} - 3\hat{j} - 5\hat{k}$.

Now,

$\overrightarrow{AB}, \overrightarrow{BC}$

and

$\overrightarrow{CA}$

can be calculated as,

$$\therefore \overrightarrow{AB} = \vec{b} - \vec{a} = (2 - 3)\hat{i} + (-1 + 4)\hat{j} + (1 + 4)\hat{k} = -\hat{i} + 3\hat{j} + 5\hat{k}$$

$$\overrightarrow{BC} = \vec{c} - \vec{b} = (1 - 2)\hat{i} + (-3 + 1)\hat{j} + (-5 - 1)\hat{k} = -\hat{i} - 2\hat{j} - 6\hat{k}$$

$$\overrightarrow{CA} = \vec{a} - \vec{c} = (3 - 1)\hat{i} + (-4 + 3)\hat{j} + (-4 + 5)\hat{k} = 2\hat{i} - \hat{j} + \hat{k}$$

Again, calculate the squares of magnitudes as shown below,

$$\therefore |AB|^2 = (-1)^2 + (3)^2 + (5)^2 = 35$$

$$\therefore |BC|^2 = (-1)^2 + (-2)^2 + (-6)^2 = 41$$

$$\therefore |CA|^2 = (2)^2 + (-1)^2 + (1)^2 = 6$$

Now, use Pythagoras theorem it can be shown that this triangle is right-angled triangle as shown below,

$$|AB|^2 + |CA|^2 = 35 + 6 = 41 = |BC|^2$$

Question 18

In triangle ABC which of the following is not true:

A.

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{0}$$

B.

$$\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{AC} = \vec{0}$$

C.

$$\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{CA} = \vec{0}$$

D.

$$\overrightarrow{AB} - \overrightarrow{CB} + \overrightarrow{CA} = \vec{0}$$

Solution

Ans:

On considering the triangle law of addition for this triangle it can be observed that,

$$\begin{aligned}\therefore \vec{AB} + \vec{BC} &= \vec{AC} \\ \Rightarrow \vec{AB} + \vec{BC} &= -\vec{CA} \\ \Rightarrow \vec{AB} + \vec{BC} + \vec{CA} &= \vec{0}\end{aligned}$$

Therefore, the equation in (A) is true.

Again,

$$\begin{aligned}\therefore \vec{AB} + \vec{BC} &= \vec{AC} \\ \Rightarrow \vec{AB} + \vec{BC} - \vec{AC} &= \vec{0}\end{aligned}$$

Therefore, the equation in (B) is true.

Again,

$$\begin{aligned}\therefore \vec{AB} + \vec{BC} &= \vec{AC} \\ \Rightarrow \vec{AB} - \vec{CB} + \vec{CA} &= \vec{0}\end{aligned}$$

Therefore, the equation in (D) is true.

Now, for the equation in (C) also,

$$\begin{aligned}\therefore \vec{AB} + \vec{BC} &= \vec{AC} \\ \Rightarrow \vec{AB} + \vec{BC} &= -\vec{CA} \\ \Rightarrow \vec{AB} + \vec{BC} + \vec{CA} &= \vec{0} \\ \Rightarrow \vec{AB} + \vec{BC} - \vec{CA} &\neq \vec{0}\end{aligned}$$

Therefore, the equation in (C) is incorrect and henceforth, it is the correct answer.

Question 19

If $\vec{a}$ and $\vec{b}$ are two collinear vectors, then which of the following are incorrect. A.A.

A.

$$\vec{b} = \lambda \vec{a}$$

, for some scalar

$$\lambda$$

.

B.

$$\vec{a} = \pm \vec{b}$$

C.

the respective components of

$$\vec{a}$$

and

$$\vec{b}$$

are not proportional.

D.

both the vectors

$$\vec{a}$$

and

$$\vec{b}$$

have same direction, but different magnitudes.

Solution

Ans: It can be observed that if

$$\vec{a}$$

and

$$\vec{b}$$

are collinear vectors then, they are parallel.

$$\vec{b} = \lambda \vec{a}$$

, for some scalar

$$\lambda$$

.

Again, if

$$\lambda = \pm 1$$

, then $\vec{a} = \pm \vec{b}$.

Again if is considered that, $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then

$$\vec{b} = \lambda \vec{a}$$

.

$$\begin{aligned} b_1\hat{i} + b_2\hat{j} + b_3\hat{k} &= \lambda (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \\ \Rightarrow b_1\hat{i} + b_2\hat{j} + b_3\hat{k} &= \lambda a_1\hat{i} + \lambda a_2\hat{j} + \lambda a_3\hat{k} \\ \Rightarrow b_1 &= \lambda a_1, b_2 = \lambda a_2, b_3 = \lambda a_3 \\ \Rightarrow \lambda &= \frac{b_1}{a_1} = \frac{b_2}{a_2} = \frac{b_3}{a_3} \end{aligned}$$

This shows that the respective components of $\vec{a}$ and $\vec{b}$, are proportional but the vectors can have different directions.

Therefore, the statement in (D) is incorrect and henceforth, it is the correct answer.

Exercise 10.3

Question 1

Find the angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitudes $\sqrt{3}$ and 2, respectively having $\vec{a} \cdot \vec{b} = \sqrt{6}$

Solution

Ans: It is given in the question that

$$|\vec{a}| = \sqrt{3}$$

$$|\vec{b}| = 2$$

$$\vec{a} \cdot \vec{b} = \sqrt{6}$$

Also we know that

$$\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta \text{ where } \theta \text{ is the angle between the vectors}$$

$$\Rightarrow \sqrt{6} = 2\sqrt{3} \cos \theta$$

$$\Rightarrow \cos \theta = \frac{1}{\sqrt{2}}$$

Henc the angle between the vectors is $\frac{\pi}{4}$

Question 2

Find the angle between the vectors $\hat{i} - 2\hat{j} + 3\hat{k}$ and $3\hat{i} - 2\hat{j} + \hat{k}$

Solution

Ans: Let the given vectors be

$$\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$$

$$\vec{b} = 3\hat{i} - 2\hat{j} + \hat{k}$$

We know that

$$\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta \text{ where } \theta \text{ is the angle between the vectors}$$

$$\Rightarrow (\hat{i} - 2\hat{j} + 3\hat{k}) \cdot (3\hat{i} - 2\hat{j} + \hat{k}) = \sqrt{1^2 + 2^2 + 3^2} \sqrt{3^2 + 2^2 + 1^2}$$

$$\Rightarrow 10 = 14 \cos \theta$$

$$\text{Hence, we found } \theta = \cos^{-1} \left(\frac{5}{7} \right)$$

Question 3

Find the projection of the vector $\hat{i} - \hat{j}$ on the vector $\hat{i} + \hat{j}$

Solution

Ans: Given we have vectors

$$\vec{a} = \hat{i} - \hat{j} \text{ and}$$

$$\vec{b} = \hat{i} + \hat{j}$$

Now the projection of $\vec{a}$ on $\vec{b}$ is given by

$$\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = \frac{(\hat{i} - \hat{j}) \cdot (\hat{i} + \hat{j})}{\sqrt{2}}$$

Hence, the projection of $\hat{i} - \hat{j}$ on $\hat{i} + \hat{j}$ is 0

Question 4

Find the projection of the vector $\hat{i} + 3\hat{j} + 7\hat{k}$ on the vector $7\hat{i} - \hat{j} + 8\hat{k}$

Solution

Ans: : Given we have vectors

$$\vec{a} = \hat{i} + 3\hat{j} + 7\hat{k} \text{ and}$$

$$\vec{b} = 7\hat{i} - \hat{j} + 8\hat{k}$$

Now the projection of $\vec{a}$ on $\vec{b}$ is given by

$$\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = \frac{(\hat{i} + 3\hat{j} + 7\hat{k}) \cdot (\hat{i} + 3\hat{j} + 7\hat{k})}{\sqrt{114}}$$

Hence, the projection of $\hat{i} - \hat{j}$ on $\hat{i} + \hat{j}$ is $\frac{60}{\sqrt{114}}$

Question 5

Show that each of the given three vectors is a unit vector:

$$\frac{1}{7}(2\hat{i} + 3\hat{j} + 6\hat{k}), \frac{1}{7}(3\hat{i} - 6\hat{j} + 2\hat{k}), \frac{1}{7}(6\hat{i} + 2\hat{j} - 3\hat{k})$$

Also, show that they are mutually perpendicular to each other.

Solution

Ans: let us have following notations

$$7\vec{a} = 2\hat{i} + 3\hat{j} + 6\hat{k}$$

$$7\vec{b} = 3\hat{i} - 6\hat{j} + 2\hat{k}$$

$$7\vec{c} = 6\hat{i} + 2\hat{j} - 3\hat{k}$$

Its magnitude is given by

$$7a = \sqrt{2^2 + 3^2 + 6^2}$$

$$\Rightarrow a = \frac{7}{7} = 1$$

Hence, the magnitude is 1, it is a unit vector

Similarly, magnitude of $7\vec{b}$ is given by

$$7b = \sqrt{3^2 + 6^2 + 2^2}$$

$$\Rightarrow b = \frac{7}{7} = 1$$

Hence, the magnitude is 1, it is a unit vector

Similarly, magnitude of $7\vec{c}$ is given by

$$7c = \sqrt{6^2 + 2^2 + 3^2}$$

$$\Rightarrow c = \frac{7}{7} = 1$$

Hence, the magnitude is 1, it is a unit vector

Now we will calculate the followings

$$\vec{a} \cdot \vec{b} = \frac{1}{7} (2\hat{i} + 3\hat{j} + 6\hat{k}) \cdot (3\hat{i} - 6\hat{j} + 2\hat{k}) = 0$$

$$\vec{c} \cdot \vec{b} = \frac{1}{7} (6\hat{i} + 2\hat{j} - 3\hat{k}) \cdot (3\hat{i} - 6\hat{j} + 2\hat{k}) = 0$$

$$\vec{a} \cdot \vec{c} = \frac{1}{7} (2\hat{i} + 3\hat{j} + 6\hat{k}) \cdot (6\hat{i} + 2\hat{j} - 3\hat{k}) = 0$$

Hence, all these vectors are mutually perpendicular

Question 6

Find $|\vec{a}|$ and $|\vec{b}|$, if $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 8$ and $|\vec{a}| = 8|\vec{b}|$

Solution

Ans: Given in the question that

$$(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 8 \dots (1)$$

$$|\vec{a}| = 8|\vec{b}| \dots (2)$$

Now from (1)

$$|\vec{a}|^2 - |\vec{b}|^2 = 8$$

Also from (2)

$$|\vec{b}|^2 (64 - 1) = 8$$

$$\Rightarrow \left| \vec{b} \right| = \sqrt{\frac{8}{63}}$$

$$\text{Hence, } \left| \vec{a} \right| = 8\sqrt{\frac{8}{63}}$$

Question 7

Evaluate the product $\left(3\vec{a} - 5\vec{b} \right) \cdot \left(2\vec{a} + 7\vec{b} \right)$

Solution

Ans: We are given with two vectors $3\vec{a} - 5\vec{b}$ and $2\vec{a} + 7\vec{b}$

$$\text{Now } \left(3\vec{a} - 5\vec{b} \right) \cdot \left(2\vec{a} + 7\vec{b} \right) = 6a^2 + 11\vec{a} \cdot \vec{b} - 35b^2$$

Question 8

Find the magnitude of two vectors $\vec{a}$ and $\vec{b}$, having the same magnitude and such that the angle between them is 60° and their scalar product is $\frac{1}{2}$

Solution

Ans: It is given in the question

$$\theta = 60^\circ$$

$$\vec{a} \cdot \vec{b} = \frac{1}{2}$$

According to the question

$$\frac{1}{2} = \left| \vec{a} \right| \left| \vec{b} \right| \frac{1}{2}$$

$$\Rightarrow \left| \vec{a} \right| = \left| \vec{b} \right| = 1$$

Question 9

Find $|\vec{x}|$, if for a unit vector $\vec{a}$, $(\vec{x} - \vec{a}) \cdot (\vec{x} + \vec{a}) = 12$

Solution

Ans: It is given that $|\vec{a}| = 1$

$$(\vec{x} - \vec{a}) \cdot (\vec{x} + \vec{a}) = 12$$

$$\Rightarrow |\vec{x}|^2 - 1 = 12 \quad (\text{since } |\vec{a}| = 1)$$

Hence, we found that $|\vec{x}| = \sqrt{13}$

Question 10

If $\vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$, $\vec{c} = 3\hat{i} + \hat{j}$ are such that $\vec{a} + \lambda \vec{b}$ perpendicular to $\vec{c}$, then find the value of λ

Solution

Ans: The given vectors are

$$\vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k}$$

$$\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$$

$$\therefore \vec{a} + \lambda \vec{b} = (2 - \lambda)\hat{i} + (2 + 2\lambda)\hat{j} + (3 - \lambda)\hat{k}$$

$$\vec{c} = 3\hat{i} + \hat{j}$$

According to the question

$$(\vec{a} + \lambda \vec{b}) \cdot (3\hat{i} + \hat{j}) = 0$$

$$\Rightarrow ((2 - \lambda)\hat{i} + (2 + 2\lambda)\hat{j} + (3 - \lambda)\hat{k}) \cdot (3\hat{i} + \hat{j}) = 0$$

Hence, we found that $\lambda = 8$

Question 11**Show that:**

$\frac{\vec{a}}{|\vec{a}|} \frac{\vec{b}}{|\vec{b}|} + \frac{\vec{b}}{|\vec{b}|} \frac{\vec{a}}{|\vec{a}|}$ is perpendicular to $\frac{\vec{a}}{|\vec{a}|} \frac{\vec{b}}{|\vec{b}|} - \frac{\vec{b}}{|\vec{b}|} \frac{\vec{a}}{|\vec{a}|}$ For any two nonzero vectors $\vec{a}$ and $\vec{b}$

Solution

Ans: Let us suppose the two vectors as shown

$$\vec{\eta} = \frac{\vec{a}}{|\vec{a}|} \frac{\vec{b}}{|\vec{b}|} + \frac{\vec{b}}{|\vec{b}|} \frac{\vec{a}}{|\vec{a}|}$$

$$\vec{\kappa} = \frac{\vec{a}}{|\vec{a}|} \frac{\vec{b}}{|\vec{b}|} - \frac{\vec{b}}{|\vec{b}|} \frac{\vec{a}}{|\vec{a}|}$$

$$\text{Now } \vec{\eta} \cdot \vec{\kappa} = \left(\frac{\vec{a}}{|\vec{a}|} \frac{\vec{b}}{|\vec{b}|} + \frac{\vec{b}}{|\vec{b}|} \frac{\vec{a}}{|\vec{a}|} \right) \cdot \left(\frac{\vec{a}}{|\vec{a}|} \frac{\vec{b}}{|\vec{b}|} - \frac{\vec{b}}{|\vec{b}|} \frac{\vec{a}}{|\vec{a}|} \right)$$

$$\Rightarrow \vec{\eta} \cdot \vec{\kappa} = \frac{|\vec{a}|^2 |\vec{b}|^2}{|\vec{a}|^2 |\vec{b}|^2} - \frac{|\vec{a}|^2 |\vec{b}|^2}{|\vec{a}|^2 |\vec{b}|^2} = 0$$

Hence, proved

Question 12

If $\vec{a} \cdot \vec{a} = 0$ and $\vec{a} \cdot \vec{b} = 0$, then what can be concluded about the vector $\vec{b}$?

Solution

Ans: Given we have two equations

$$\vec{a} \cdot \vec{a} = 0 \dots \dots (1)$$

$$\vec{a} \cdot \vec{b} = 0$$

It is clear from the equation (1) that $|\vec{a}|^2 = 0$

Hence, $\vec{b}$ can be any vector

Question 13

If $\vec{a}, \vec{b}, \vec{c}$ are unit vectors such that $\vec{a} + \vec{b} + \vec{c} = 0$, find the value of $\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}$

Solution

Ans: Given we have three unit vectors $\vec{a}, \vec{b}, \vec{c}$ such that

$$\begin{aligned}\vec{a} + \vec{b} + \vec{c} &= 0 \\ \Rightarrow (\vec{a} + \vec{b} + \vec{c}) \cdot (\vec{a} + \vec{b} + \vec{c}) &= 0 \\ \Rightarrow a^2 + b^2 + c^2 + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) &= 0 \\ \Rightarrow (\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) &= \frac{-3}{2}\end{aligned}$$

Question 14

If either $\vec{a} = 0$ or $\vec{b} = 0$ vector, then But the converse need not be true. Justify your answer with an example.

Solution

Ans: Let us suppose two vectors as shown

$$\vec{a} = 2\hat{i} + 4\hat{j} + 3\hat{k}$$

$$\vec{b} = 3\hat{i} + 3\hat{j} - 6\hat{k}$$

$$\text{Now } \vec{a} \cdot \vec{b} = (2\hat{i} + 4\hat{j} + 3\hat{k}) \cdot (3\hat{i} + 3\hat{j} - 6\hat{k})$$

$$\Rightarrow \vec{a} \cdot \vec{b} = 6 + 12 - 18 = 0$$

But clearly neither $\vec{a} = 0$ nor $\vec{b} = 0$

Question 15

If the vertices A,B,C of a triangle ABC are $(1, 2, 3)$, $(-1, 0, 0)$ and $(0, 1, 2)$ respectively then find $\angle ABC$

Solution

Ans: It is given that the vertices of ΔABC are

$$A(1, 2, 3)$$

$$B(-1, 0, 0)$$

$$C(0, 1, 2)$$

Now $\angle ABC$ is the angle between $\overrightarrow{BA}$ and $\overrightarrow{BC}$

$$\therefore \overrightarrow{BA} \cdot \overrightarrow{BC} = (2\hat{i} + 2\hat{j} + 3\hat{k}) \cdot (\hat{i} + \hat{j} + 2\hat{k})$$

$$\Rightarrow 10 = \sqrt{17}\sqrt{6} \cos \angle ABC$$

$$\text{Hence, we found that } \angle ABC = \cos^{-1} \left(\frac{10}{\sqrt{17}\sqrt{6}} \right)$$

Question 16

Show that the points $A(1, 2, 7)$, $B(2, 6, 3)$ and $C(3, 10, -1)$ are collinear

Solution

Ans: Given we have points as shown

$$A(1, 2, 7)$$

$$B(2, 6, 3)$$

$$C(3, 10, -1)$$

$$\therefore \overrightarrow{AB} = \hat{i} + 4\hat{j} - 4\hat{k}$$

$$\therefore \overrightarrow{BC} = (\hat{i} + 4\hat{j} - 4\hat{k})$$

$$\therefore \overrightarrow{AC} = 2(\hat{i} + 4\hat{j} - 4\hat{k})$$

Clearly,

$$|\overrightarrow{AC}| = |\overrightarrow{BC}| + |\overrightarrow{AB}|$$

Hence, vectors are collinear

Question 17

Show that the vectors $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} - 3\hat{j} - 5\hat{k}$, $3\hat{i} - 4\hat{j} - 4\hat{k}$ forms the vertices of the right angled triangle

Solution

Ans: It is given in the question that

$$\overrightarrow{OA} = 2\hat{i} - \hat{j} + \hat{k}$$

$$\overrightarrow{OB} = \hat{i} - 3\hat{j} - 5\hat{k}$$

$$\overrightarrow{OC} = 3\hat{i} - 4\hat{j} - 4\hat{k}$$

Where $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ are position vectors of A,B,C

Now

$$\overrightarrow{AB} = -\hat{i} - 2\hat{j} - 6\hat{k}$$

$$|\overrightarrow{AB}| = \sqrt{(-1)^2 + (-2)^2 + (-6)^2} = \sqrt{41}$$

$$\overrightarrow{BC} = -2\hat{i} - \hat{j} + \hat{k}$$

$$|\overrightarrow{BC}| = \sqrt{(-2)^2 + (-1)^2 + 1^2} = \sqrt{6}$$

$$|\overrightarrow{AC}| = \sqrt{35}$$

$$\text{Clearly } |\overrightarrow{AC}| + |\overrightarrow{BC}| = \sqrt{41} = |\overrightarrow{AB}|$$

Hence, proved

Question 18

If $\vec{a}$ is a non-zero vector of magnitude 'a' and λ a non-zero scalar, then $\lambda \vec{a}$ is unit vector for what value of λ

Solution

Ans: Given we have a vector $\vec{a}$ and a scalar λ

For $\lambda \vec{a}$ to be a unit vector

$$|\lambda \vec{a}| = 1$$

Hence, the value is $|\lambda| = \frac{1}{a}$

Exercise 10.4**Question 1**

Find $|\vec{a} \times \vec{b}|$, if $\vec{a} = \hat{i} - 7\hat{j} + 7\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + 2\hat{k}$

Solution

Ans: The given vectors are

$$\vec{a} = \hat{i} - 7\hat{j} + 7\hat{k} \text{ and } \vec{b} = 3\hat{i} - 2\hat{j} + 2\hat{k}.$$

Then the cross-product between the vectors is given by

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -7 & 7 \\ 3 & -2 & 2 \end{vmatrix}$$

$$= \hat{i}(-14 - 14) - \hat{j}(2 - 21) + \hat{k}(-2 - 21)$$

$$= -28\hat{i} + 19\hat{j} - 23\hat{k}$$

$$|\vec{a} \times \vec{b}| = \sqrt{(-28)^2 + (19)^2 + (-23)^2}$$

$$\end{vmatrix}\end{vmatrix}$$

$$= \hat{i}(-14 + 14) + -\hat{j}(2 - 21) + \hat{k}(-2 + 21)$$

$$= 19\hat{j} + 19\hat{k}.$$

Therefore,

$$\left| \vec{a} \times \vec{b} \right| = \sqrt{19^2 + 19^2}$$

$$= \sqrt{2 \times 19^2}$$

$$= 19\sqrt{2}.$$

Question 2

Find a unit vector perpendicular to each of the vector $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$, where $\vec{a} = 3\hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$.

Solution

Ans: The given vectors are

$$\vec{a} = 3\hat{i} + 2\hat{j} + 2\hat{k} \text{ and } \vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}.$$

Then, adding and subtracting the vectors successively, we have

$$\vec{a} + \vec{b} = 4\hat{i} + 4\hat{j} \text{ and } \vec{a} - \vec{b} = 2\hat{i} + 4\hat{k}.$$

Therefore, their cross-product,

$$\left(\vec{a} + \vec{b} \right) \times \left(\vec{a} - \vec{b} \right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 4 & 0 \\ 2 & 0 & 4 \end{vmatrix}$$

$$\hat{i}(4 \cdot 4 - 0 \cdot 0) - \hat{j}(4 \cdot 4 - 0 \cdot 2) + \hat{k}(4 \cdot 0 - 4 \cdot 2)$$

$$\hat{i}(16) - \hat{j}(16) + \hat{k}(-8)$$

$$4\hat{i} + 4\hat{j} + 0\hat{k}$$

$$2\hat{i} + 0\hat{j} + 4\hat{k}$$

$$\left. \begin{matrix} \end{matrix} \right\} \text{right} \$$$

$$= 16\hat{i} - 16\hat{j} - 8\hat{k}.$$

Then, its magnitude,

$$\left| \left(\vec{a} + \vec{b} \right) \times \left(\vec{a} - \vec{b} \right) \right| = \sqrt{16^2 + (-16)^2 + (-8)^2}$$

$$= \sqrt{2^2 \times 8^2 + 2^2 \times 8^2 + 8^2}$$

$$= 8\sqrt{2^2 + 2^2 + 1}$$

$$= 8\sqrt{9}$$

$$= 8 \times 3$$

$$= 24.$$

Thus, the unit vector perpendicular to each of the vectors $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$

is provided as,

$$= \pm \frac{\left(\vec{a} + \vec{b} \right) \left(\vec{a} - \vec{b} \right)}{\left| \left(\vec{a} + \vec{b} \right) \left(\vec{a} - \vec{b} \right) \right|}$$

$$= \pm \frac{16\hat{i} - 16\hat{j} - 8\hat{k}}{24}$$

$$= \pm \frac{2\hat{i} - 2\hat{j} - \hat{k}}{3}$$

$$= \pm \frac{2}{3}\hat{i} \mp \frac{2}{3}\hat{j} \mp \frac{1}{3}\hat{k}$$

Question 3

If a unit vector $\vec{a}$ makes angles $\frac{\pi}{3}$ with $\vec{i}$, $\frac{\pi}{4}$ with $\vec{j}$ and an acute angle θ with $\hat{k}$, then find θ and hence, the components of $\vec{a}$.

Solution

Ans: Suppose the components of the given unit vector $\vec{a}$ is (a_1, a_2, a_3) .

Then, $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$.

Also, $|\vec{a}| = 1$ as $\vec{a}$ is a unit vector.

Again, we are provided that, the vector $\vec{a}$ makes angles $\frac{\pi}{3}$ with $\vec{i}$, $\frac{\pi}{4}$ with $\vec{j}$ and an acute angle θ with $\hat{k}$. Therefore, it gives

$$\begin{aligned}\cos \frac{\pi}{3} &= \frac{a_1}{|\vec{a}|} \\ \Rightarrow \frac{1}{2} &= a_1\end{aligned}$$

, since $\vec{a}$ is a unit vector.

Also,

$$\cos \frac{\pi}{4} = \frac{a_2}{|\vec{a}|}$$

$\Rightarrow \frac{1}{\sqrt{2}} = a_2$, since $\vec{a}$ is a unit vector.

Now, let

$$\cos \theta = \frac{a_3}{|\vec{a}|}$$

$$\Rightarrow a_3 = \cos \theta$$

Since, $|\vec{a}| = 1$, so

$$\Rightarrow \sqrt{a_1^2 + a_2^2 + a_3^2} = 1$$

$$\Rightarrow \left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \cos^2 \theta = 1$$

$$\Rightarrow \frac{1}{4} + \frac{1}{2} + \cos^2 \theta = 1$$

$$\Rightarrow \frac{3}{4} + \cos^2 \theta = 1$$

$$\Rightarrow \cos^2 \theta = 1 - \frac{3}{4} = \frac{1}{4}$$

$$\Rightarrow \cos \theta = \frac{1}{2}$$

$$\Rightarrow \theta = \frac{\pi}{3}$$

$$\text{Thus, } a_3 = \cos \frac{\pi}{3} = \frac{1}{2}.$$

Therefore, $\theta = \frac{\pi}{3}$ and the components of the vector $\vec{a}$ are $\left(\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}\right)$.

Question 4

Show that

$$\left(\vec{a} - \vec{b}\right) \times \left(\vec{a} + \vec{b}\right) = 2\left(\vec{a} \times \vec{b}\right).$$

Solution

Ans: The given cross-product can be written as

$$\begin{aligned} & \left(\vec{a} - \vec{b}\right) \times \left(\vec{a} + \vec{b}\right) \\ &= \left(\vec{a} - \vec{b}\right) \times \vec{a} + \left(\vec{a} - \vec{b}\right) \times \vec{b}, \text{ (using the distributive property of vector product over addition)} \\ &= \vec{a} \times \vec{a} - \vec{b} \times \vec{a} + \vec{a} \times \vec{b} - \vec{b} \times \vec{b}, \text{ (Distributive property of vector product over addition)} \\ &= \vec{0} + \vec{a} \times \vec{b} + \vec{a} \times \vec{b} - \vec{0} \\ &= 2\vec{a} \times \vec{b}. \end{aligned}$$

Question 5

Find λ and μ if $(2\hat{i} + 6\hat{j} + 27\hat{k}) \times (\hat{i} + \lambda\hat{j} + \mu\hat{k}) = \vec{0}$.

Solution

Ans: The given vector equation can be written as,

$$(2\hat{i} + 6\hat{j} + 27\hat{k}) \times (\hat{i} + \lambda\hat{j} + \mu\hat{k}) = \vec{0}$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 6 & 27 \\ 1 & \lambda & \mu \end{vmatrix} = 0$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 6 & 27 \\ 1 & \lambda & \mu \end{vmatrix} = 0$$

$$2(\mu - 27\lambda) - 6(2\mu - 27) + 27(2\lambda - 6) = 0$$

$$2\mu - 27\lambda - 12\mu + 162 + 54\lambda - 162 = 0$$

$$-10\mu + 27\lambda = 0$$

$$\Rightarrow \hat{i}(6\mu - 27\lambda) - \hat{j}(2\mu - 27) + \hat{k}(2\lambda - 6) = 0\hat{i} + 0\hat{j} + 0\hat{k}$$

Now, compare the scalar components both sides of the equation.

Then, it yields

$$6\mu - 27\lambda = 0 \quad \dots\dots (i)$$

$$2\mu - 27 = 0 \quad \dots\dots (ii)$$

$$2\lambda - 6 = 0 \quad \dots\dots (iii)$$

Hence, on solving the equations (i), (ii), and (iii), we obtain

$$\lambda = 3 \text{ and } \mu = \frac{27}{2}.$$

Question 6

Given that $\vec{a} \cdot \vec{b} = 0$ and $\vec{a} \times \vec{b} = \vec{0}$.

What can you conclude about the vector $\vec{a}$ and $\vec{b}$?

Solution

Ans: Since, $\vec{a} \cdot \vec{b} = 0$, then we can say that

either $|\vec{a}| = 0$, or $|\vec{b}| = 0$, or the vectors $\vec{a}$ and $\vec{b}$ are perpendicular to each other.

Again, since $\vec{a} \times \vec{b} = 0$, so it can be said that

either $|\vec{a}| = 0$, or $|\vec{b}| = 0$, or the vectors $\vec{a}$ and $\vec{b}$ are parallel to each other.

But, since the vectors cannot be perpendicular and parallel at simultaneous, so the only possibility is

either $|\vec{a}| = 0$ or $|\vec{b}| = 0$.

Question 7

Let the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ given as $a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, $c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

Then show that $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$.

Solution

Ans: The given vectors are

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k},$$

$$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k},$$

$$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}.$$

$$\text{So, } (\vec{b} + \vec{c}) = (b_1 + c_1)\hat{i} + (b_2 + c_2)\hat{j} + (b_3 + c_3)\hat{k}$$

Then,

$$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

$$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \text{ and } \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \text{ and } \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \text{ are}$$

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \text{ and } \begin{pmatrix} b_2 \\ b_3 \\ b_1 \end{pmatrix} + \begin{pmatrix} c_2 \\ c_3 \\ c_1 \end{pmatrix} \text{ and } \begin{pmatrix} b_3 \\ b_1 \\ b_2 \end{pmatrix} + \begin{pmatrix} c_3 \\ c_1 \\ c_2 \end{pmatrix}$$

end{matrix} \right] \\$

$$\begin{aligned} &= \hat{i} [a_2 (b_3 + c_3) - a_3 (b_2 + c_2)] - \hat{j} [a_1 (b_3 + c_3) - a_3 (b_1 + c_1)] + \hat{k} [a_1 (b_2 + c_2) - a_2 (b_1 + c_1)] \\ &= \hat{i} (a_2 b_3 + a_2 c_3 - a_3 b_2 - a_3 c_2) + \hat{j} (-a_1 b_3 - a_1 c_3 + a_3 b_1 + a_3 c_1) \\ &\quad + \hat{k} (a_1 b_2 + a_1 c_2 - a_2 b_1 - a_2 c_1) \end{aligned} \quad \dots\dots (i)$$

Therefore,

$$\overrightarrow{a} \times \overrightarrow{b} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$

$$\hat{i} \text{ and } \hat{j} \text{ and } \hat{k} \text{ are}$$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \text{ and } \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \text{ and } \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \text{ are}$$

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \text{ and } \begin{pmatrix} b_2 \\ b_3 \\ b_1 \end{pmatrix} \text{ and } \begin{pmatrix} b_3 \\ b_1 \\ b_2 \end{pmatrix}$$

end{matrix} \right]

$$= \hat{i} (a_2 b_3 - a_3 b_2) + \hat{j} (a_3 b_1 - a_1 b_3) + \hat{k} (a_1 b_2 - a_2 b_1)$$

..... (ii)

Also,

$$\overrightarrow{a} \times \overrightarrow{c} = \begin{pmatrix} a_2 c_3 - a_3 c_2 \\ a_3 c_1 - a_1 c_3 \\ a_1 c_2 - a_2 c_1 \end{pmatrix}$$

$$\hat{i} \text{ and } \hat{j} \text{ and } \hat{k} \text{ are}$$

$$\hat{a}_1 \hat{a}_2 \hat{a}_3 \parallel$$

$$\hat{c}_1 \hat{c}_2 \hat{c}_3$$

$\end{matrix} \right|$

$$= \hat{i} (a_2 c_3 - a_3 c_2) + \hat{j} (a_3 c_1 - a_1 c_3) + \hat{k} (a_1 c_2 - a_2 c_1) \quad \dots\dots (iii)$$

Now, add the equations (ii) and (iii). Then, it yields

$$\begin{aligned} \left(\vec{a} \times \vec{b} \right) + \left(\vec{a} \times \vec{c} \right) &= \hat{i} (a_2 b_3 + a_2 c_3 - a_3 b_2 - a_3 c_2) + \hat{j} (b_1 a_3 + a_3 c_1 - a_1 b_3 - a_1 c_3) \\ &\quad + \hat{k} (a_1 b_2 + a_1 c_2 - a_2 b_1 - a_2 c_1) \quad \dots\dots (iv) \end{aligned}$$

Then, the equations (i) and (iv) together implies that

$$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}.$$

Thus, the required result has been proved.

Question 8

If either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, then $\vec{a} \times \vec{b} = \vec{0}$.

Is the converse true? Justify your answer with an example.

Solution

Ans: Consider any parallel non-zero vectors so that $\vec{a} \times \vec{b} = \vec{0}$.

So, take the nonzero vectors

$$\vec{a} = 2\hat{i} + 3\hat{j} + 4\hat{k}$$

$$\text{and } \vec{b} = 4\hat{i} + 6\hat{j} + 8\hat{k}.$$

Therefore, the cross-product between the vectors,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 4 & 6 & 8 \end{vmatrix}$$

$$\hat{i} \begin{vmatrix} 3 & 4 \\ 6 & 8 \end{vmatrix} - \hat{j} \begin{vmatrix} 2 & 4 \\ 4 & 8 \end{vmatrix} + \hat{k} \begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix}$$

$$2 \hat{i} - 4 \hat{j} + 0 \hat{k}$$

$$4 \hat{i} - 4 \hat{j}$$

$$\end{vmatrix} \right]$$

$$= \hat{i} (24 - 24) - \hat{j} (16 - 16) + \hat{k} (12 - 12)$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k}.$$

Then, the magnitudes of the vectors are given by

$$|\vec{a}| = \sqrt{2^2 + 3^2 + 4^2} = \sqrt{29} \text{ and}$$

$$|\vec{b}| = \sqrt{4^2 + 6^2 + 8^2} = \sqrt{116}.$$

Thus, observing the above results, it is found that $\vec{a} \times \vec{b} = \vec{0}$, while $\vec{a} \neq \vec{0}$ and $\vec{b} \neq \vec{0}$.

Therefore, it is justified that the converse of the given statement need not be true.

Question 9

Find the area of the triangle with vertices A (1, 1, 2), B (2, 3, 5) and C (1, 5, 5).

Solution

Ans: The triangle ABC has the vertices A (1,1,2), B (2,3,5) and C (1,5,5).

The adjacent sides of the triangle ΔABC are $\vec{AB}$ and $\vec{BC}$ such that

$$\overrightarrow{AB} = (2 - 1)\hat{i} + (3 - 1)\hat{j} + (5 - 2)\hat{k} = \hat{i} + 2\hat{j} + 3\hat{k}$$

and

$$\overrightarrow{BC} = (1 - 2)\hat{i} + (5 - 3)\hat{j} + (5 - 5)\hat{k} = -\hat{i} + 2\hat{j}.$$

Now, the cross-product between the vectors is given by

$$\overrightarrow{AB} \times \overrightarrow{BC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ -1 & 2 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ -1 & 2 & 0 \end{vmatrix}$$

$$= \hat{i}(2 \cdot 0 - 3 \cdot 2) - \hat{j}(1 \cdot 0 - 3 \cdot (-1)) + \hat{k}(1 \cdot 2 - (-1) \cdot 2)$$

$$= \hat{i}(-6) - \hat{j}(3) + \hat{k}(2 + 2)$$

$$= -6\hat{i} - 3\hat{j} + 4\hat{k}.$$

$$= -6\hat{i} - 3\hat{j} + 4\hat{k}.$$

$$= -6\hat{i} - 3\hat{j} + 4\hat{k}.$$

Therefore,

$$|\overrightarrow{AB} \times \overrightarrow{BC}| = \sqrt{(-6)^2 + (-3)^2 + 4^2}$$

$$= \sqrt{36 + 9 + 16}$$

$$= \sqrt{61}.$$

Thus, the area of the triangle ΔABC

$$= \frac{1}{2} \times |\overrightarrow{AB} \times \overrightarrow{BC}|$$

$$= \frac{1}{2} \times \sqrt{61}$$

$$= \frac{\sqrt{61}}{2} \text{ square units.}$$

Question 10

Find the area of the parallelogram whose adjacent sides are determined by the vector

$$\vec{a} = \hat{i} - \hat{j} + 3\hat{k}$$

and $\vec{b} = 2\hat{i} - 7\hat{j} + \hat{k}$.

Solution

Ans: The given vectors are

$$\vec{a} = \hat{i} - \hat{j} + 3\hat{k}$$

and $\vec{b} = 2\hat{i} - 7\hat{j} + \hat{k}$ such that they are the adjacent sides of the parallelogram.

Therefore, the cross-product between the vectors,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 3 \\ 2 & -7 & 1 \end{vmatrix}$$

$$= \hat{i} \begin{vmatrix} -1 & 3 \\ -7 & 1 \end{vmatrix} - \hat{j} \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} + \hat{k} \begin{vmatrix} 1 & -1 \\ 2 & -7 \end{vmatrix}$$

$$= \hat{i} (-1 + 21) - \hat{j} (1 - 6) + \hat{k} (-7 + 2)$$

$$= 20\hat{i} + 5\hat{j} - 5\hat{k}$$

$$= \sqrt{20^2 + 5^2 + 5^2}$$

$$= \sqrt{400 + 25 + 25}$$

$$= 15\sqrt{2}$$

Therefore, its magnitude,

$$|\vec{a} \times \vec{b}| = \sqrt{20^2 + 5^2 + 5^2}$$

$$= \sqrt{400 + 25 + 25}$$

$$= 15\sqrt{2}$$

Thus, the area of the parallelogram is $15\sqrt{2}$ square units.

Question 11

Let the vectors $\vec{a}$ and $\vec{b}$ be such that $|\vec{a}| = 3$ and $|\vec{b}| = \frac{\sqrt{2}}{3}$, then $\vec{a} \times \vec{b}$ is a unit vector, if the angle between $\vec{a}$ and $\vec{b}$ is

- (a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

Solution

Ans: We are provided that, $|\vec{a}| = 3$ and $|\vec{b}| = \frac{\sqrt{2}}{3}$.

Now, it is generally known that $\vec{a} \times \vec{b} = |\vec{a}| |\vec{b}| \sin \theta \cdot \hat{n}$, where $\hat{n}$ is a unit vector that is perpendicular to both the vectors $\vec{a}$ and $\vec{b}$; θ is the angle the vectors $\vec{a}$ and $\vec{b}$.

Since, $\vec{a} \times \vec{b}$ is a unit vector, so $|\vec{a} \times \vec{b}| = 1$

$$\Rightarrow \left| |\vec{a}| |\vec{b}| \sin \theta \hat{n} \right| = 1$$

$$\Rightarrow |\vec{a}| |\vec{b}| |\sin \theta| = 1$$

$$\Rightarrow 3 \times \frac{\sqrt{2}}{3} \times \sin \theta = 1$$

$$\Rightarrow \sin \theta = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \theta = \frac{\pi}{4}.$$

Thus, the angle between the vectors $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{4}$.

Hence, the correct answer is option (b).

Question 12

Area of a rectangle having vertices A, B, C, and D with position vectors

$-\hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}$, $\hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}$, $\hat{i} - \frac{1}{2}\hat{j} + 4\hat{k}$ and $-\hat{i} - \frac{1}{2}\hat{j} + 4\hat{k}$ respectively is

- (a) $\frac{1}{2}$ (b) 1 (c) 2 (d) 4

Solution

Ans: The position vectors of the vertices A, B, C, D of the rectangle $ABCD$ are such that

$$\overrightarrow{OA} = -\hat{i} + \frac{1}{2}\hat{j} + 4\hat{k},$$

$$\overrightarrow{OB} = \hat{i} + \frac{1}{2}\hat{j} + 4\hat{k},$$

$$\overrightarrow{OC} = \hat{i} - \frac{1}{2}\hat{j} + 4\hat{k}, \text{ and}$$

$$\overrightarrow{OD} = -\hat{i} - \frac{1}{2}\hat{j} + 4\hat{k}.$$

The adjacent sides of the rectangle $ABCD$ are $\overrightarrow{AB}$ and $\overrightarrow{BC}$ such that

$$\overrightarrow{AB} = (1 + 1)\hat{i} + \left(\frac{1}{2} - \frac{1}{2}\right)\hat{j} + (4 - 4)\hat{k} = 2\hat{j}$$

$$\overrightarrow{BC} = (1 - 1)\hat{i} + \left(-\frac{1}{2} - \frac{1}{2}\right)\hat{j} + (4 - 4)\hat{k} = -\hat{j}.$$

Therefore, the cross-product between these two vectors,

$$\overrightarrow{AB} \times \overrightarrow{BC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 2 & 0 \\ 0 & -1 & 0 \end{vmatrix}$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 2 & 0 \\ 0 & -1 & 0 \end{vmatrix}$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 2 & 0 \\ 0 & -1 & 0 \end{vmatrix}$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 2 & 0 \\ 0 & -1 & 0 \end{vmatrix}$$

$$= \hat{k}(-2)$$

$$= -2\hat{k}.$$

Since, the area of a parallelogram having the adjacent sides $\vec{a}$ and $\vec{b}$ is $|\vec{a} \times \vec{b}|$, so the area of the rectangle $ABCD$ is $|\vec{AB} \times \vec{CD}| = 2$ square units.

Thus, the correct answer is the option (c).

Miscellaneous Exercise

Question 1

Write down a unit vector in the XY-plane, making an angle of 30° with the positive direction of the x-axis.

Solution

Ans: Unit vector is $\vec{r} = \cos \theta \hat{i} + \sin \theta \hat{j}$, where θ is angle with positive X axis.

$$\vec{r} = \cos 30^\circ \hat{i} + \sin 30^\circ \hat{j} = \frac{\sqrt{3}}{2} \hat{i} + \frac{1}{2} \hat{j}$$

Question 2

Find the scalar components and magnitude of the vector joining the points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2)

Solution

Ans: $\vec{PQ} = (x_2 - x_1) \hat{i} + (y_2 - y_1) \hat{j} + (z_2 - z_1) \hat{k}$

$$|\vec{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Question 3

A girl walks 4 km towards the west, then she walks 3 km in a direction 30° east of north and stops. Determine the girl's displacement from her initial point of departure.

Solution

Ans:

$$\overrightarrow{OA} = -4\hat{i}$$

$$\overrightarrow{AB} = \hat{i}|\overrightarrow{AB}|\cos 60^\circ + \hat{j}|\overrightarrow{AB}|\sin 60^\circ$$

$$= \hat{i}3 \times \frac{1}{2} + \hat{j}3 \times \frac{\sqrt{3}}{2}$$

$$= \frac{3}{2}\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}$$

$$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$$

$$= (-4\hat{i}) + \left(\frac{3}{2}\hat{i} + \frac{3\sqrt{3}}{2}\hat{j} \right)$$

$$= \left(-4 + \frac{3}{2} \right) \hat{i} + \frac{3\sqrt{3}}{2}\hat{j}$$

$$= \left(\frac{-8+3}{2} \right) \hat{i} + \frac{3\sqrt{3}}{2}\hat{j}$$

$$= \frac{-5}{2}\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}$$

Question 4

If $\vec{a} = \vec{b} + \vec{c}$, then is it true that $|\vec{a}| = |\vec{b}| + |\vec{c}|$? Justify your answer.

Solution

Ans: In $\triangle ABC$, $\overrightarrow{CB} = \vec{a}$, $\overrightarrow{CA} = \vec{b}$, $\overrightarrow{AB} = \vec{c}$

$\vec{a} = \vec{b} + \vec{c}$, by triangle law of addition for vectors.

$|\vec{a}| < |\vec{b}| + |\vec{c}|$, by triangle inequality law of lengths.

Hence, it's not true that $|\vec{a}| = |\vec{b}| + |\vec{c}|$

Question 5

Find the value of x for which $x(\hat{i} + \hat{j} + \hat{k})$ unit vector.

Solution

Ans: $|x(\hat{i} + \hat{j} + \hat{k})| = 1$

$$\Rightarrow \sqrt{x^2 + x^2 + x^2} = 1$$

$$\Rightarrow \sqrt{3x^2} = 1$$

$$\Rightarrow \sqrt{3}x = 1$$

$$\Rightarrow x = \pm \frac{1}{\sqrt{3}}$$

Question 6

Find a vector of magnitude 5 units, and parallel to the resultant of the vectors $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$

Solution

Ans: $\vec{c} = \vec{a} + \vec{b} = (2 + 1)\hat{i} + (3 - 2)\hat{j} + (-1 + 1)\hat{k} = 3\hat{i} + \hat{j}$

$$|\vec{c}| = \sqrt{3^2 + 1^2} = \sqrt{9 + 1} = \sqrt{10}$$

$$\therefore \hat{c} = \frac{\vec{c}}{|\vec{c}|} = \frac{(3\hat{i} + \hat{j})}{\sqrt{10}}$$

So, a vector of magnitude 5 and parallel to the resultant of $\vec{a}$ and $\vec{b}$ is

$$\pm 5(\hat{c}) = \pm 5 \left(\frac{1}{\sqrt{10}} (3\hat{i} + \hat{j}) \right) = \pm \frac{3\sqrt{10}}{2} \hat{i} \pm \frac{\sqrt{10}}{2} \hat{j}$$

Question 7

If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} + 3\hat{k}$ and $\vec{c} = \hat{i} - 2\hat{j} + \hat{k}$, find a unit vector parallel to the vector $2\vec{a} - \vec{b} + 3\vec{c}$.

Solution

$$\begin{aligned}\text{Ans: } 2\vec{a} - \vec{b} + 3\vec{c} &= 2(\hat{i} + \hat{j} + \hat{k}) - (2\hat{i} - \hat{j} + 3\hat{k}) + 3(\hat{i} - 2\hat{j} + \hat{k}) \\ &= 2\hat{i} + 2\hat{j} + 2\hat{k} - 2\hat{i} + \hat{j} - 3\hat{k} + 3\hat{i} - 6\hat{j} + 3\hat{k} \\ &= 3\hat{i} - 3\hat{j} + 2\hat{k}\end{aligned}$$

$$|2\vec{a} - \vec{b} + 3\vec{c}| = \sqrt{3^2 + (-3)^2 + 2^2} = \sqrt{9 + 9 + 4} = \sqrt{22}$$

$$\text{Thus, required unit vector is } \frac{2\vec{a} - \vec{b} + 3\vec{c}}{|2\vec{a} - \vec{b} + 3\vec{c}|} = \frac{3\hat{i} - 3\hat{j} + 2\hat{k}}{\sqrt{22}} = \frac{3}{\sqrt{22}}\hat{i} - \frac{3}{\sqrt{22}}\hat{j} + \frac{2}{\sqrt{22}}\hat{k}$$

Question 8

Show that the points A(1, -2, -8), B(5, 0, -2) and C(11, 3, 7) are collinear, and find the ratio in which B divides Ac.

Solution

$$\text{Ans: } \overrightarrow{AB} = (5 - 1)\hat{i} + (0 + 2)\hat{j} + (-2 + 8)\hat{k} = 4\hat{i} + 2\hat{j} + 6\hat{k}$$

$$\overrightarrow{BC} = (11 - 5)\hat{i} + (3 - 0)\hat{j} + (7 + 2)\hat{k} = 6\hat{i} + 3\hat{j} + 9\hat{k}$$

$$\overrightarrow{AC} = (11 - 1)\hat{i} + (3 + 2)\hat{j} + (7 + 8)\hat{k} = 10\hat{i} + 5\hat{j} + 15\hat{k}$$

$$|\overrightarrow{AB}| = \sqrt{4^2 + 2^2 + 6^2} = \sqrt{16 + 4 + 36} = \sqrt{56} = 2\sqrt{14}$$

$$|\overrightarrow{BC}| = \sqrt{6^2 + 3^2 + 9^2} = \sqrt{36 + 9 + 81} = \sqrt{126} = 3\sqrt{14}$$

$$|\overrightarrow{AC}| = \sqrt{10^2 + 5^2 + 15^2} = \sqrt{100 + 25 + 225} = \sqrt{350} = 5\sqrt{14}$$

$$\therefore |\overrightarrow{AC}| = |\overrightarrow{AB}| + |\overrightarrow{BC}|$$

So, the points are collinear.

Let B divide AC in ratio $\lambda : 1$. $\overrightarrow{OB} = \frac{\lambda \overrightarrow{OC} + \overrightarrow{OA}}{(\lambda + 1)}$

$$\Rightarrow 5\hat{i} - 2\hat{k} = \frac{\lambda(11\hat{i} + 3\hat{j} + 7\hat{k}) + (\hat{i} - 2\hat{j} - 8\hat{k})}{\lambda + 1}$$

$$\Rightarrow (\lambda + 1)(5\hat{i} - 2\hat{k}) = 11\lambda\hat{i} + 3\lambda\hat{j} + 7\lambda\hat{k} + \hat{i} - 2\hat{j} - 8\hat{k}$$

$$\Rightarrow 5(\lambda + 1)\hat{i} - 2(\lambda + 1)\hat{k} = (11\lambda + 1)\hat{i} + (3\lambda - 2)\hat{j} + (7\lambda - 8)\hat{k}$$

$$\Rightarrow \lambda = \frac{2}{3}$$

So, the required ratio is 2:3

Question 9

Find the position vector of a point R which divides the line joining two points P and Q whose position vectors are $(2\vec{a} + \vec{b})$ and $(\vec{a} - 3\vec{b})$ externally in the ratio 1: 2. Also, show that P is the midpoint of the line segment RQ .

Solution

Ans: $\overrightarrow{OP} = 2\vec{a} + \vec{b}$, $\overrightarrow{OQ} = \vec{a} - 3\vec{b}$

$$\overrightarrow{OR} = \frac{2(2\vec{a} + \vec{b}) - (\vec{a} - 3\vec{b})}{2 - 1} = \frac{4\vec{a} + 2\vec{b} - \vec{a} - 3\vec{b}}{1} = 3\vec{a} + 5\vec{b}$$

So, the position vector of R is $3\vec{a} + 5\vec{b}$

Position vector of midpoint of $RQ = \frac{\overrightarrow{OQ} + \overrightarrow{OR}}{2}$

$$= \frac{(a\sqrt{6}) + (3\vec{a} + 5\vec{b})}{2}$$

$$= 2\vec{a} + \vec{b}$$

$$= \overrightarrow{OP}$$

Thus, P is midpoint of line segment RQ

Question 10

The two adjacent sides of a parallelogram are $2\hat{i} - 4\hat{j} + 5\hat{k}$ and $\hat{i} - 2\hat{j} - 3\hat{k}$. Find the unit vector parallel to its diagonal. Also, find its area.

Solution

Ans: Diagonal of a parallelogram is $\vec{a} + \vec{b}$

$$\vec{a} + \vec{b} = (2 + 1)\hat{i} + (-4 - 2)\hat{j} + (5 - 3)\hat{k} = 3\hat{i} - 6\hat{j} + 2\hat{k}$$

So, the unit vector parallel to diagonal is

$$\frac{\vec{a} + \vec{b}}{|\vec{a} + \vec{b}|} = \frac{3\hat{i} - 6\hat{j} + 2\hat{k}}{\sqrt{3^2 + (-6)^2 + 2^2}} = \frac{3\hat{i} - 6\hat{j} + 2\hat{k}}{\sqrt{9 + 36 + 4}} = \frac{3\hat{i} - 6\hat{j} + 2\hat{k}}{7} = \frac{3}{7}\hat{i} - \frac{6}{7}\hat{j} + \frac{2}{7}\hat{k}$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -4 & 3 \\ 1 & -2 & -3 \end{vmatrix}$$

$$= \hat{i}(12 + 10) - \hat{j}(-6 - 5) + \hat{k}(-4 + 4)$$

$$= 22\hat{i} + 11\hat{j}$$

$$= 11(2\hat{i} + \hat{j})$$

$$\therefore |\vec{a} \times \vec{b}| = 11\sqrt{2^2 + 1^2} = 11\sqrt{5}$$

So, the area of the parallelogram is $11\sqrt{5}$ sq units

Question 11

Show that the direction cosines of a vector equally inclined to the axes OX, OY and OZ are $\frac{1}{\sqrt{3}}$

$$\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$$

Solution

Ans: Let a vector be equally inclined to OX, OY, and OZ at an angle α .

So, the Direction Cosines of the vector are $\cos \alpha$, $\cos \alpha$ and $\cos \alpha$.

$$\cos^2 \alpha + \cos^2 \alpha + \cos^2 \alpha = 1$$

$$\Rightarrow 3\cos^2\alpha = 1$$

$$\Rightarrow \cos\alpha = \frac{1}{\sqrt{3}}$$

So, the DCs of the vector are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.

Question 12

Let $\vec{a} = \hat{i} + 4\hat{j} + 2\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + 7\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$ and $\vec{c} \cdot \vec{d} = 15$

Solution

$$\text{Ans: } \vec{d} = d_1\hat{i} + d_2\hat{j} + d_3\hat{k}$$

$$\vec{d} \cdot \vec{a} = 0 \Rightarrow d_1 + 4d_2 + 2d_3 = 0$$

$$\vec{d} \cdot \vec{b} = 0 \Rightarrow 3d_1 - 2d_2 + 7d_3 = 0$$

$$\vec{c} \cdot \vec{d} = 15 \Rightarrow 2d_1 - d_2 + 4d_3 = 15$$

$$\text{Solving these equations, we get } d_1 = \frac{160}{3}, d_2 = -\frac{5}{3}, d_3 = -\frac{70}{3}$$

$$\therefore \vec{d} = \frac{160}{3}\hat{i} + \frac{5}{3}\hat{j} + \frac{70}{3}\hat{k} = \frac{1}{3}(160\hat{i} + 5\hat{j} + 70\hat{k})$$

Question 13

The scalar product of the vector $\hat{i} + \hat{j} + \hat{k}$ with a unit vector along the sum of vectors $2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\lambda\hat{i} + 2\hat{j} + 3\hat{k}$ is equal to one. Find the value of λ .

Solution

$$\text{Ans: } (2\hat{i} + 4\hat{j} - 5\hat{k}) + (\lambda\hat{i} + 2\hat{j} + 3\hat{k}) = (2 + \lambda)\hat{i} + 6\hat{j} - 2\hat{k}$$

$$\text{So, unit vector along } (2\hat{i} + 4\hat{j} - 5\hat{k}) + (\lambda\hat{i} + 2\hat{j} + 3\hat{k}) \text{ is } \left(\frac{(2 + \lambda)\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{\lambda^2 + 4\lambda + 44}} \right)$$

$$(\hat{i} + \hat{j} + \hat{k}) \cdot \left(\frac{(2 + \lambda)\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{\lambda^2 + 4\lambda + 44}} \right) = 1$$

$$\Rightarrow \frac{(2 + \lambda) + 6 - 2}{\sqrt{\lambda^2 + 4\lambda + 44}} = 1$$

$$\Rightarrow \sqrt{\lambda^2 + 4\lambda + 44} = \lambda + 6$$

$$\Rightarrow \lambda^2 + 4\lambda + 44 = (\lambda + 6)^2$$

$$\Rightarrow \lambda^2 + 4\lambda + 44 = \lambda^2 + 12\lambda + 36$$

$$\Rightarrow 8\lambda = 8$$

$$\Rightarrow \lambda = 1$$

Question 14

If $\vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular vectors of equal magnitudes, show that the vector $\vec{a} + \vec{b} + \vec{c}$ is equally inclined to $\vec{a}$, $\vec{b}$ and $\vec{c}$.

Solution

Ans: $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c} = \vec{c} \cdot \vec{a} = 0$

$$|\vec{a}| = |\vec{b}| = |\vec{c}|$$

Let $\vec{a} + \vec{b} + \vec{c}$ be inclined to $\vec{a}, \vec{b}, \vec{c}$ at angles $\theta_1, \theta_2, \theta_3$ respectively.

$$\cos \theta_1 = \frac{(\vec{a} + \vec{b} + \vec{c}) \cdot \vec{a}}{|\vec{a} + \vec{b} + \vec{c}| |\vec{a}|} = \frac{\vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{a} + \vec{c} \cdot \vec{a}}{|\vec{a} + \vec{b} + \vec{c}| |\vec{a}|} = \frac{|\vec{a}|^2}{|\vec{a} + \vec{b} + \vec{c}| |\vec{a}|} = \frac{|\vec{a}|}{|\vec{a} + \vec{b} + \vec{c}|}$$

$$\cos \theta_2 = \frac{(\vec{a} + \vec{b} + \vec{c}) \cdot \vec{b}}{|\vec{a} + \vec{b} + \vec{c}| |\vec{b}|} = \frac{\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b} + \vec{c} \cdot \vec{b}}{|\vec{a} + \vec{b} + \vec{c}| |\vec{b}|} = \frac{|\vec{b}|^2}{|\vec{a} + \vec{b} + \vec{c}| |\vec{b}|} = \frac{|\vec{b}|}{|\vec{a} + \vec{b} + \vec{c}|}$$

$$\cos \theta_3 = \frac{(\vec{a} + \vec{b} + \vec{c}) \cdot \vec{c}}{|\vec{a} + \vec{b} + \vec{c}| |\vec{c}|} = \frac{\vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{c}}{|\vec{a} + \vec{b} + \vec{c}| |\vec{c}|} = \frac{|\vec{c}|^2}{|\vec{a} + \vec{b} + \vec{c}| |\vec{c}|} = \frac{|\vec{c}|}{|\vec{a} + \vec{b} + \vec{c}|}$$

Since, $|\vec{a}| = |\vec{b}| = |\vec{c}| \Rightarrow \cos \theta_1 = \cos \theta_2 = \cos \theta_3$, So, $\theta_1 = \theta_2 = \theta_3$

Question 15

Prove that, $(\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) = |\vec{a}|^2 + |\vec{b}|^2$ if and only if $\vec{a}, \vec{b}$ are perpendicular, given $\vec{a} \neq \vec{0}, \vec{b} \neq \vec{0}$

Solution

$$\text{Ans: } (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) = |\vec{a}|^2 + |\vec{b}|^2$$

$$\Rightarrow \vec{a} \cdot \vec{a} + \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b} = |\vec{a}|^2 + |\vec{b}|^2$$

$$\Rightarrow |\vec{a}|^2 + 2\vec{a} \cdot \vec{b} + |\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2$$

$$\Rightarrow 2\vec{a} \cdot \vec{b} = 0$$

$$\Rightarrow \vec{a} \cdot \vec{b} = 0$$

So $\vec{a}$ and $\vec{b}$ are perpendicular.

Question 16

If θ is the angle between two vectors $\vec{a}$ and $\vec{b}$, then $\vec{a} \cdot \vec{b} \geq 0$ only when

A.

$$0 < \theta < \frac{\pi}{2}$$

B.

$$0 \leq \theta \leq \frac{\pi}{2}$$

C.

$$0 < \theta < \pi$$

D.

$$0 \leq \theta \leq \pi$$

Solution

$$\text{Ans: } \therefore \vec{a} \cdot \vec{b} \geq 0$$

$$\Rightarrow |\vec{a}| |\vec{b}| \cos \theta \geq 0$$

$$\Rightarrow \cos \theta \geq 0 \quad \because [|\vec{a}| \geq 0 \text{ and } |\vec{b}| \geq 0]$$

$$\Rightarrow 0 \leq \theta \leq \frac{\pi}{2}$$

$$\vec{a} \cdot \vec{b} \geq 0 \text{ if } 0 \leq \theta \leq \frac{\pi}{2}$$

So the right answer is B

Question 17

Let $\vec{a}$ and $\vec{b}$ be two unit vectors and θ is the angle between them. Then $\vec{a} + \vec{b}$ is a unit vector if

A.

$$\theta = \frac{\pi}{4}$$

B.

$$\theta = \frac{\pi}{3}$$

C.

$$\theta = \frac{\pi}{2}$$

D.

$$\theta = \frac{2\pi}{3}$$

Solution

$$\text{Ans: } |\vec{a}| = |\vec{b}| = 1$$

$$|\vec{a} + \vec{b}| = 1$$

$$\Rightarrow (\vec{a} + \vec{b})(\vec{a} + \vec{b}) = 1$$

$$\Rightarrow \vec{a} \cdot \vec{a} + \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b} = 1$$

$$\Rightarrow |\vec{a}|^2 + 2\vec{a} \cdot \vec{b} + |\vec{b}|^2 = 1$$

$$\Rightarrow 1^2 + 2|\vec{a}||\vec{b}|\cos\theta + 1^2 = 1$$

$$\Rightarrow 1 + 2 \cdot 1 \cdot 1 \cos\theta + 1 = 1$$

$$\Rightarrow \cos \theta = -\frac{1}{2}$$

$$\Rightarrow \theta = \frac{2\pi}{3}$$

So, $\vec{a} + \vec{b}$ is unit vector if $\theta = \frac{2\pi}{3}$

The correct answer is D

Question 18

The value of $\hat{i} \cdot (\hat{j} \times \hat{k}) + \hat{j} \cdot (\hat{i} \times \hat{k}) + \hat{k} \cdot (\hat{i} \times \hat{j})$ is

- a. 0 b. -1 c. 1 d. 3

Solution

Ans:

$$\hat{i} \cdot (\hat{j} \times \hat{k}) + \hat{j} \cdot (\hat{i} \times \hat{k}) + \hat{k} \cdot (\hat{i} \times \hat{j})$$

$$= \hat{i} \cdot \hat{i} + \hat{j} \cdot (-\hat{j}) + \hat{k} \cdot \hat{k}$$

$$= 1 - 1 + 1$$

$$= 1$$

The correct answer is C.

Question 19

If θ is the angle between any two vectors $\vec{a}$ and $\vec{b}$, then $|\vec{a}\vec{b}| = |\vec{a} \times \vec{b}|$ when θ is equal to

A.

0

B.

$$\frac{\pi}{4}$$

C.

$$\frac{\pi}{2}$$

D.

n

Solution

$$\text{Ans: } |\vec{a}\vec{b}| = |\vec{a} \times \vec{b}|$$

$$\Rightarrow |\vec{a}||\vec{b}| \cos \theta = |\vec{a}||\vec{b}| \sin \theta$$

$$\Rightarrow \cos \theta = \sin \theta$$

$$\Rightarrow \tan \theta = 1$$

$$\Rightarrow \theta = \frac{\pi}{4}$$

The correct answer is B