

Chapter 8: Sequences and Series

NCERT · Class 11 · Maths

Exercise 8.2

Question 1

Find the 20th and nth terms of the G.P. $\frac{5}{2}, \frac{5}{4}, \frac{5}{8}, \dots$

Solution

Ans:

The G.P. is provided as $\frac{5}{2}, \frac{5}{4}, \frac{5}{8}, \dots$

Clearly, $a = \frac{5}{2}$ is the first term and $r = \frac{\frac{5}{4}}{\frac{5}{2}} = \frac{1}{2}$ is the common ratio of the G.P.

Therefore, the 20th term of the G.P. is

$$\begin{aligned}a_{20} &= ar^{20-1} \\&= \frac{5}{2} \left(\frac{1}{2} \right)^{19} \\&= \frac{5}{2 \times 2^{19}} \\&= \frac{5}{2^{20}}.\end{aligned}$$

Also, the nth term of the G.P. is

$$\begin{aligned}a_n &= ar^{n-1} \\&= \frac{5}{2} \left(\frac{1}{2} \right)^{n-1} \\&= \frac{5}{2 \cdot 2^{n-1}} \\&= \frac{5}{2^n}.\end{aligned}$$

Hence, the 20^{th} and n^{th} terms of the G.P. are respectively $\frac{5}{2^{20}}$ and $\frac{5}{2^n}$.

Question 2

Find the 12^{th} term of the G.P. whose 8^{th} term is 192 and the common ratio is 2.

Solution

Ans:

The G.P. described in the problem has the common ratio $r = 2$.

Suppose that the first term of the G.P. is a .

Then, the 8^{th} term of the G.P. is given by

$$a_8 = ar^{8-1} = ar^7$$

$$\Rightarrow ar^7 = 192$$

$$\Rightarrow a(2)^7 = 192$$

$$\Rightarrow a(2)^7 = (2)^6 \times 3$$

$$\Rightarrow a = \frac{2^6 \times 3}{2^7} = \frac{3}{2}.$$

Thus, the 12^{th} term,

$$a_{12} = ar^{12-1}$$

$$= ar^{11}$$

$$= \frac{3}{2}(2)^{11}$$

$$= 3(2)^{10}$$

$$= 3072.$$

Hence, the 12^{th} term of the G.P. is 3072.

Question 3

The 5th, 8th, and 11th terms of a G.P. are p , q , and s respectively. Show that $q^2 = ps$.

Solution

Ans:

Suppose that a and r are the first term and the common ratio of the given G.P. Then, by the condition provided to us,

the 5th term of G.P.,

$$a_5 = ar^{5-1} = ar^4 = p \quad \dots\dots (i)$$

The 8th term of the G.P.,

$$a_8 = ar^{8-1} = ar^7 = q \quad \dots\dots (ii)$$

Also, the 11th term of the G.P.,

$$a_{11} = ar^{11-1} = ar^{10} = s \quad \dots\dots (iii)$$

Now, divide the equation (ii) by the equation (i). Then, it gives

$$\begin{aligned} \frac{ar^7}{ar^4} &= \frac{q}{p} \\ \Rightarrow r^3 &= \frac{q}{p} \quad \dots\dots (iv) \end{aligned}$$

Again, divide the equation (iii) by the equation (ii). Then, we have

$$\begin{aligned} \frac{ar^{10}}{ar^7} &= \frac{s}{q} \\ \Rightarrow r^3 &= \frac{s}{q} \quad \dots\dots (v) \end{aligned}$$

Therefore, from the equation (iv) and (v) gives

$$\begin{aligned} \frac{q}{p} &= \frac{s}{q} \\ \Rightarrow q^2 &= ps. \end{aligned}$$

Hence, the required result has been shown.

Question 4

The 4th term of a G.P. is a square of its second term, and the first term is -3 . Determine its 7th term.

Solution

Ans:

Suppose that r is the common ratio of the G.P.

It is given that the first term $a = -3$.

Now, by the formula, n^{th} term of a G.P.,

$$a_n = ar^{n-1}.$$

Therefore,

$$a_4 = ar^3 = (-3)r^3 \text{ and } a_2 = ar^1 = (-3)r.$$

So, by the condition provided to us,

$$(-3)r^3 = [(-3)r]^2$$

$$\Rightarrow -3r^3 = 9r^2$$

$$\Rightarrow r = -3.$$

Thus, the 7th term of the G.P. is given by

$$\begin{aligned} a_7 &= (-3)(-3)^{7-1} \\ &= (-3)^7 \\ &= -2187. \end{aligned}$$

Hence, the 7th term of the geometric progression is -2187 .

Question 5

Which term of the following sequences:

(a) $2, 2\sqrt{2}, 4, \dots$ is 128?

(b) $\sqrt{3}, 3, 3\sqrt{3}, \dots$ is 729?

(c) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \dots$ is $\frac{1}{19683}$?

Solution

Ans:

The sequence provided to us is in a G.P. since the first term is $a = 2$ and the common ratio is

$$r = \frac{2\sqrt{2}}{2} = \sqrt{2}.$$

So, suppose that the n^{th} term of the G.P. is 128.

Therefore, we have

$$2(\sqrt{2})^{n-1} = 128$$

$$\Rightarrow 2^{\frac{n-1}{2}+1} = 2^7$$

$$\Rightarrow \frac{n-1}{2} + 1 = 7$$

$$\Rightarrow \frac{n+1}{2} = 6$$

$$\Rightarrow n+1 = 12$$

$$\Rightarrow n = 11.$$

Hence, 128 is the 11^{th} term of the G.P. (sequence) $2, 2\sqrt{2}, 4, \dots$

Ans:

The sequence provided to us is in a G.P. since the first term is $a = \sqrt{3}$, and the common ratio is

$$r = \frac{3}{\sqrt{3}} = \sqrt{3}.$$

Now, suppose that the n^{th} term of the G.P. is 729.

Therefore, we have

$$ar^{n-1} = 729$$

$$\Rightarrow (\sqrt{3})(\sqrt{3})^{n-1} = 729$$

$$\Rightarrow 3^{\frac{1}{2} + \frac{n-1}{2}} = 3^6$$

$$\Rightarrow \frac{1}{2} + \frac{n-1}{2} = 6$$

$$\Rightarrow \frac{1+n-1}{2} = 6$$

$$\Rightarrow n = 12.$$

Hence, 729 is the n^{th} term of the geometric progression (sequence) $\sqrt{3}, 3, 3\sqrt{3}, \dots$

Ans:

The sequence given to us is in a geometric progression (G.P.) since the first term of it is $a = \frac{1}{3}$ and the common ratio is

$$r = \frac{\frac{1}{9}}{\frac{1}{3}} = \frac{1}{3}.$$

Now, suppose that $\frac{1}{19683}$ is the n^{th} term of the G.P.

Therefore, we have

$$ar^{n-1} = \frac{1}{19683}$$

$$\Rightarrow \frac{1}{3} \left(\frac{1}{3} \right)^{n-1} = \frac{1}{19683}$$

$$\Rightarrow \left(\frac{1}{3} \right)^n = \left(\frac{1}{3} \right)^9$$

$$\Rightarrow n = 9.$$

Hence, $\frac{1}{19683}$ is the 9^{th} term of the G.P. (sequence) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \dots$

Question 6

For what values of x , the numbers $-\frac{2}{7}$, x , $-\frac{7}{2}$ are in G.P.?

Solution

Ans:

The numbers provided to us are $-\frac{2}{7}$, x , $-\frac{7}{2}$.

Now, let $a_1 = -\frac{2}{7}$, $a_2 = x$ and $a_3 = -\frac{7}{2}$.

$$\text{Then, } \frac{a_2}{a_1} = \frac{x}{-\frac{2}{7}} = \frac{-7x}{2} \text{ and } \frac{a_3}{a_2} = \frac{-\frac{7}{2}}{x} = -\frac{7}{2x}.$$

Now, if the numbers are in G.P., then

$$\begin{aligned} r &= \frac{a_2}{a_1} = \frac{a_3}{a_2} \\ \Rightarrow \frac{-7x}{2} &= -\frac{7}{2x} \\ \Rightarrow x^2 &= \frac{-2 \times 7}{-2 \times 7} = 1 \\ \Rightarrow x &= \sqrt{1} \\ \Rightarrow x &= \pm 1. \end{aligned}$$

Hence, the numbers $-\frac{2}{7}$, x , $-\frac{7}{2}$ are in G.P. if $x = \pm 1$.

Question 7

Find the sum of the numbers 0.15, 0.015, 0.0015, ... in the G.P. up to 20 terms.

Solution

Ans:

It is provided that the numbers 0.15, 0.015, 0.0015, ... are in geometric progression.

Note that, the first term of the sequence is $a = 0.15$ and the common ratio is $r = \frac{0.015}{0.15} = 0.1$

Now, by the formula, sum of n terms in G.P., we have

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

Therefore,

$$\begin{aligned} S_{20} &= \frac{0.15 [1 - (0.1)^{20}]}{1 - 0.1} \\ &= \frac{0.15}{0.9} [1 - (0.1)^{20}] \\ &= \frac{15}{90} [1 - (0.1)^{20}] \\ &= \frac{1}{6} [1 - (0.1)^{20}]. \end{aligned}$$

Question 8

Find the sum of the numbers $\sqrt{7}, \sqrt{21}, 3\sqrt{7}, \dots$ in the G.P. up to n terms.

Solution

Ans:

It is provided to us that the numbers $\sqrt{7}, \sqrt{21}, 3\sqrt{7}, \dots$ are in geometric progression.

Note that, the first term of the G.P. is $a = \sqrt{7}$ and the common ratio is $r = \frac{\sqrt{21}}{\sqrt{7}} = \sqrt{3}$.

Now, by the formula, sum of the first n terms in G.P.,

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

$$\begin{aligned}
&= \frac{\sqrt{7} \left[1 - (\sqrt{3})^n \right]}{1 - \sqrt{3}} \\
&= \frac{\sqrt{7} \left[1 - (\sqrt{3})^n \right]}{1 - \sqrt{3}} \cdot \frac{1 + \sqrt{3}}{1 + \sqrt{3}} \\
&= \frac{\sqrt{7} \left[1 - (\sqrt{3})^n \right] (1 + \sqrt{3})}{1 - 3} \\
&= \frac{-\sqrt{7} (\sqrt{3} + 1) \left[1 - (\sqrt{3})^n \right]}{2}
\end{aligned}$$

Hence, the sum of the first n terms is given by

$$S_n = \frac{\sqrt{7} (1 + \sqrt{3})}{2} \left[(3)^{\frac{n}{2}} - 1 \right].$$

Question 9

Find the sum of the numbers $1, -a, a^2, -a^3, \dots$ (if $a \neq -1$) in geometric progression up to n terms.

Solution

Ans:

The numbers $1, -a, a^2, -a^3, \dots$ provided to us, is in G.P. with the first term $a = 1$ and the common ratio $r = \frac{-a}{1} = -a$.

Now, by the formula, sum of the first n terms in G.P.,

$$\begin{aligned}
S_n &= \frac{a(1 - r^n)}{1 - r} \\
&= \frac{1[1 - (-a)^n]}{1 - (-a)}
\end{aligned}$$

Hence, the sum of n terms of the sequence is given by

$$S_n = \frac{[1 - (-a)^n]}{1 + a}.$$

Question 10

Find the sum of the numbers $x^3, x^5, x^7, \dots$ (if $x \neq \pm 1$) in the geometric progression up to n terms.

Solution

Ans:

The numbers $x^3, x^5, x^7, \dots$ are in G.P. with its first term $a = x^3$ and the common ratio

$$r = \frac{x^5}{x^3} = x^2.$$

Therefore, by the formula, sum of the first n terms in G.P.,

$$\begin{aligned} S_n &= \frac{a(1 - r^n)}{1 - r} \\ &= \frac{x^3 [1 - (x^2)^n]}{1 - x^2} \\ &= \frac{x^3 (1 - x^{2n})}{1 - x^2}. \end{aligned}$$

Hence, the sum of the first n terms of the sequence $x^3, x^5, x^7, \dots$ is given by $S_n = \frac{x^3 (1 - x^{2n})}{1 - x^2}$.

Question 11

Evaluate $\sum_{k=1}^{11} (2 + 3^k)$.

Solution

Ans:

The given sum can be written as

$$\sum_{k=1}^{11} (2 + 3^k) = \sum_{k=1}^{11} 2 + \sum_{k=1}^{11} 3^k = 22 + \sum_{k=1}^{11} 3^k \quad \dots (i)$$

Again, $\sum_{k=1}^{11} 3^k = 3^1 + 3^2 + 3^3 + \dots + 3^{11}$, which is in a geometric progression with the first term $a = 3$ and the common ratio $r = 3$.

Therefore, we have

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$\Rightarrow S_n = \frac{3[3^{11} - 1]}{3 - 1}$$

$$\Rightarrow S_n = \frac{3}{2}(3^{11} - 1)$$

Thus,

$$\sum_{k=1}^{11} 3^k = \frac{3}{2}(3^{11} - 1). \quad \dots\dots (ii)$$

Hence, from the equation (i) and (ii), it gives

$$\sum_{k=1}^{11} (2 + 3^k) = 22 + \frac{3}{2}(3^{11} - 1).$$

Question 12

The sum of the first three terms of a G.P. is $\frac{39}{10}$ and their product is 1. Find the common ratio and the terms.

Solution

Ans:

Suppose that the first three terms of the geometric progression are $\frac{a}{r}$, a , ar .

Now, by the given conditions,

$$\frac{a}{r} + a + ar = \frac{39}{10} \quad \dots\dots (i)$$

$$\left(\frac{a}{r}\right) \cdot (a) \cdot (ar) = 1 \quad \dots\dots (ii)$$

Simplifying the equation (ii) gives

$$a^3 = 1$$

$\Rightarrow a = 1$, neglecting the imaginary roots.

Thus, replacing a by 1 in the equation (i) gives

$$\frac{1}{r} + 1 + r = \frac{39}{10}$$

$$\Rightarrow 1 + r + r^2 = \frac{39}{10}r$$

$$\Rightarrow 10 + 10r + 10r^2 - 39r = 0$$

$$\Rightarrow 10r^2 - 29r + 10 = 0$$

$$\Rightarrow 10r^2 - 25r - 4r + 10 = 0$$

$$\Rightarrow (5r - 2)(2r - 5) = 0$$

$$\Rightarrow r = \frac{2}{5} \text{ or } \frac{5}{2}.$$

Hence, $\frac{5}{2}$, 1, and $\frac{2}{5}$ are the three terms of the geometric progression.

Question 13

How many terms of G.P. $3, 3^2, 3^3, \dots$ are needed to give the sum 120.

Solution

Ans:

It is provided that the terms $3, 3^2, 3^3, \dots$ are geometric progression with the first term $a = 3$ and the common ratio $r = 3$.

Suppose that to get the sum of the G.P. as 120, n terms are needed to be considered.

So, by the formula, sum of n terms in G.P.,

$$S_n = \frac{a(r^n - 1)}{r - 1}.$$

Therefore,

$$\frac{3(3^n - 1)}{3 - 1} = 120$$

$$\Rightarrow 3^n - 1 = 120 \times \frac{2}{3}$$

$$\Rightarrow 3^n - 1 = 80$$

$$\Rightarrow 3^n = 81$$

$$\Rightarrow 3^n = 3^4$$

$$\Rightarrow n = 4.$$

Hence, to get the sum of $3, 3^2, 3^3, \dots$ as 120, 4 terms are needed to be considered.

Question 14

The sum of the first three terms of a G.P. is 16 and the sum of the next three terms is 128. Determine the first term, the common ratio and the sum of n terms of the G.P.

Solution

Ans:

Suppose that the first six terms of the given geometric progression are

$$a, ar, ar^2, ar^3, ar^4, ar^5.$$

Then, by the condition provided to us,

$$a + ar + ar^2 = 16$$

$$\Rightarrow a(1 + r + r^3) = 16 \quad \dots\dots (i)$$

$$\text{and } ar^3 + ar^4 + ar^5 = 128$$

$$\Rightarrow ar^3(1 + r + r^2) = 128 \quad \dots\dots (ii)$$

Now, divide the equation (ii) by the equation (i). Then, it gives

$$\frac{ar^3(1 + r + r^2)}{a(1 + r + r^2)} = \frac{128}{16}$$

$$\Rightarrow r^3 = 8$$

$$\Rightarrow r^3 = 2^3$$

$$\Rightarrow r = 2 \quad \dots\dots (iii)$$

Therefore, the equation (i) and (iii) together implies

$$a(1 + 2 + 2^2) = 16$$

$$\Rightarrow a \times 7 = 16$$

$$\Rightarrow a = \frac{16}{7}$$

Hence, the sum of n terms of the G.P. is given by

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$= \frac{16}{7} \frac{(2^n - 1)}{2 - 1}$$

$$= \frac{16}{7} (2^n - 1).$$

Question 15

Given a G.P. with $a = 729$ and the 7th term 64, determine S_7 .

Solution

Ans:

The first term of the G.P. is $a = 729$.

Suppose that r is the common ratio of the geometric progression.

Therefore, by the formula, n^{th} term of the sequence in G.P.,

$$a_n = ar^{n-1}$$

$$\text{So, } a_7 = (729) r^{7-1} = 729 \cdot r^6$$

$$\Rightarrow 64 = 729r^6$$

$$\Rightarrow r^6 = \left(\frac{2}{3}\right)^6$$

$$\Rightarrow r = \frac{2}{3}.$$

Thus, by the formula, sum of n terms in G.P.,

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

Therefore, for $n = 7$, we have

$$S_7 = \frac{729 \left[1 - \left(\frac{2}{3} \right)^7 \right]}{1 - \frac{2}{3}}$$

$$= 3^7 \times \frac{3^7 - 2^7}{3^7}$$

$$= 3^7 - 2^7$$

$$= 2187 - 128$$

$$= 2059.$$

Hence, $S_7 = 2059$.

Question 16

Find a G.P. for which the sum of the first two terms is -4 and the fifth term is 4 times the third term.

Solution

Ans:

Suppose that a and r respectively are the first term and the common ratio of the geometric progression.

So, by the condition provided to us,

$$-4 = \frac{a(1 - r^2)}{1 - r} \quad \dots\dots (i)$$

$$a_5 = 4 \times a_3 \quad \dots\dots (ii)$$

Now, we know that, the n^{th} term of a G.P.,

$$a_n = ar^{n-1}.$$

Therefore,

$$a_3 = ar^{3-1} = ar^2 \quad \dots\dots (iii)$$

$$\text{and } a_5 = ar^{5-1} = ar^4. \quad \dots\dots (iv)$$

Thus, from the equation (ii), (iii), and (iv) we have

$$ar^4 = 4 \times ar^2$$

$$\Rightarrow r^2 = 4$$

$$\Rightarrow r = \pm 2.$$

Substituting $r = 2$ into the equation (i), it gives

$$-4 = \frac{a [1 - (2)^2]}{1 - 2}$$

$$\Rightarrow -4 = \frac{a(1 - 4)}{-1}$$

$$\Rightarrow -4 = 3a$$

$$\Rightarrow a = -\frac{4}{3}.$$

Also, substituting $r = -2$ into the equation (i), gives

$$-4 = \frac{a [1 - (-2)^2]}{1 - (-2)}$$

$$\Rightarrow -4 = \frac{a(1 - 4)}{1 + 2}$$

$$\Rightarrow -4 = \frac{a(-3)}{3}$$

$$\Rightarrow a = 4.$$

Hence, the G.P. can be of the form $-\frac{4}{3}, -\frac{8}{3}, -\frac{16}{3}, \dots$ or $4, -8, -16, -32, \dots$

Question 17

If the 4^{th} , 10^{th} , and 16^{th} terms of a G.P. are x, y and z , respectively. Prove that x, y, z are in G.P.

Solution

Ans:

Suppose that a and r respectively are the first term and common ratio of the geometric progression.

Then, by the condition provided to us,

$$a_4 = ar^3 = x \quad \dots\dots (i)$$

$$a_{10} = ar^9 = y \quad \dots\dots (ii)$$

$$a_{16} = ar^{15} = z \quad \dots\dots (iii)$$

Now, divide the equation (ii) by the equation (i). Then, it gives

$$\begin{aligned} \frac{y}{x} &= \frac{ar^9}{ar^3} \\ \Rightarrow \frac{y}{x} &= r^6 \quad \dots\dots(iv) \end{aligned}$$

Again, divide the equation (iii) by the equation (ii). Then, it gives

$$\begin{aligned} \frac{z}{y} &= \frac{ar^{15}}{ar^9} \\ \Rightarrow \frac{z}{y} &= r^6 \quad \dots\dots (v) \end{aligned}$$

Equating the equations (iv) and (v), we have

$$\frac{y}{x} = \frac{z}{y}.$$

Hence, it has been proved that x, y, z are in G.P.

Question 18

Find the sum of n terms of the sequence 8, 88, 888, 8888, ...

Solution

Ans:

The provided sequence 8, 88, 888, 8888, ... is not in a geometric progression (G.P.), but by rearranging and rewriting the terms, it can be converted into a G.P.

The sum of the n terms of the given sequence is given by

$$\begin{aligned}
 S_n &= 8 + 88 + 888 + 8888 + \cdots \text{to } n \text{ terms} \\
 &= \frac{8}{9} [9 + 99 + 999 + 9999 + \cdots \text{to } n \text{ terms}] \\
 &= \frac{8}{9} [(10 - 1) + (10^2 - 1) + (10^3 - 1) + (10^4 - 1) + \cdots \text{to } n \text{ terms}] \\
 &= \frac{8}{9} [(10 + 10^2 + \cdots + n \text{ terms}) - (1 + 1 + 1 + \cdots n \text{ terms})] \\
 &= \frac{8}{9} \left[\frac{10(10^n - 1)}{10 - 1} - n \right] \\
 &= \frac{80}{81}(10^n - 1) - \frac{8}{9}n.
 \end{aligned}$$

Hence, the sum of n terms of the sequence 8, 88, 888, 8888, ... is $\frac{80}{81}(10^n - 1) - \frac{8}{9}n$.

Question 19

Find the sum of the products of the corresponding terms of the sequences 2, 4, 8, 16, 32 and 128, 32, 8, 2, $\frac{1}{2}$.

Solution

Ans:

The sum of the product of the corresponding terms 2, 4, 8, 16, 32 and 128, 32, 8, 2, $\frac{1}{2}$, is given by

$$\begin{aligned}
 &= 2 \times 128 + 4 \times 32 + 8 \times 8 + 16 \times 2 + 32 \times \frac{1}{2} \\
 &= 64 \left[4 + 2 + 1 + \frac{1}{2} + \frac{1}{2^2} \right].
 \end{aligned}$$

Now, note that, the terms 4, 2, 1, $\frac{1}{2}$, $\frac{1}{2^2}$ are in geometric progression, with the first term $a = 4$ and the common ratio $r = \frac{2}{4} = \frac{1}{2}$.

So, by the formula, sum of n terms in G.P.,

$$S_n = \frac{a(1 - r^n)}{1 - r}, r < 1.$$

Therefore,

$$\begin{aligned} S_5 &= \frac{4 \left[1 - \left(\frac{1}{2} \right)^5 \right]}{1 - \frac{1}{2}} \\ &= \frac{4 \left[1 - \frac{1}{32} \right]}{\frac{1}{2}} \\ &= 8 \left(\frac{32 - 1}{32} \right) \\ &= \frac{31}{4}. \end{aligned}$$

Hence, the sum of the products of the corresponding terms of the given sequences is $\frac{31}{4}$.

Question 20

Show that the products of the corresponding terms of the sequences $a, ar, ar^2, \dots, ar^{n-1}$ and $A, AR, AR^2, \dots, AR^{n-1}$ form a G.P. and find the common ratio.

Solution

Ans:

The product of the corresponding terms of the sequences $a, ar, ar^2, \dots, ar^{n-1}$ and $A, AR, AR^2, \dots, AR^{n-1}$ is $aA, arAR, ar^2AR^2, \dots, ar^{n-1}AR^{n-1}$.

Now, we are to be proved that the product is in G.P.

$$\text{So, } \frac{\text{Second term}}{\text{First term}} = \frac{arAR}{aA} = rR \quad \dots (i)$$

Also,

$$\frac{\text{Third term}}{\text{Second term}} = \frac{ar^2 AR^2}{ar AR} = rR \quad \dots\dots (ii)$$

Thus, by equating the equations (i) and (ii), we have

$$\frac{\text{Second term}}{\text{First term}} = \frac{\text{Third term}}{\text{Second term}} = rR, \text{ the common ratio.}$$

Hence, it has been proved that the product of the corresponding terms of the sequences $a, ar, ar^2, \dots, ar^{n-1}$ and $A, AR, AR^2, \dots, AR^{n-1}$ is in G.P. with the common ratio rR .

Question 21

Find four numbers forming a geometric progression in which the third term is greater than the first term by 9, and the second term is greater than the 4th by 18.

Solution

Ans:

Suppose that the first term and the common ratio of the geometric progression respectively are a and r .

Then the first four terms of the sequence are a, ar, ar^2, ar^3 .

Now, according to the condition provided,

$$ar^2 = a + 9 \quad \dots\dots (i)$$

$$ar = ar^3 + 18 \quad \dots\dots (ii)$$

Therefore, rewriting the equations, give

$$a(r^2 - 1) = 9 \quad \dots\dots (iii)$$

$$ar(1 - r^2) = 18 \quad \dots\dots (iv)$$

Now, divide the equation (iv) by the equation (iii). Then, it gives

$$\frac{ar(1 - r^2)}{-a(1 - r^2)} = \frac{18}{9}$$

$$\Rightarrow r = -2 \quad \dots\dots (v)$$

Now, using the equation (v) into the equation (i) gives

$$4a = a + 9$$

$$\Rightarrow 3a = 9$$

$$\Rightarrow a = 3$$

Hence, the first four numbers that forms a geometric progression are 3, $3(-2)$, $3(-2)^2$, and $3(-2)^3$, that is 3, -6, 12, and -24.

Question 22

If the p^{th} , q^{th} and r^{th} terms of a G.P. are a , b , and c respectively. Prove that $a^{q-r} b^{r-p} c^{p-q} = 1$.

Solution

Ans:

First suppose that the first term and the common ratio of the geometric progression are respectively A and R .

Then, by the condition provided to us,

$$AR^{p-1} = a, \quad \dots (i)$$

$$AR^{q-1} = b, \quad \dots (ii)$$

$$\text{and } AR^{r-1} = c. \quad \dots (iii)$$

Therefore, we have

$$a^{q-r} b^{r-p} c^{p-q}$$

$$= A^{q-r} \times R^{(p-1)(q-r)} \times A^{r-p} \times R^{(q-1)(r-p)} \times A^{p-q} \times R^{(r-1)(p-q)}, \text{ using the equations (i), (ii) and (iii).}$$

$$= A^{q-r+r-p+p-q} \times R^{(pr-pr-q+r)(rq-r+p-pq)+(pr-p-qr+q)}$$

$$= A^0 \times R^0$$

$$= 1.$$

Hence, it has been proved that, $a^{q-r} b^{r-p} c^{p-q} = 1$.

Question 23

If the first and the n^{th} term of a G.P. are a and b , respectively, and if P is the product of n terms, prove that $P^2 = (ab)^n$.

Solution

Ans:

First suppose that, r is the common ratio of the geometric progression.

Since, a is the first of the G.P., so it takes the form $a, ar, ar^2, ar^3, \dots, ar^{n-1}$.

Therefore, by the given condition,

$$b = ar^{n-1} \quad \dots\dots (i)$$

Also,

$P =$ Product of n terms

$$= (a)(ar)(ar^2) \dots (ar^{n-1})$$

$$= (a \times a \times a \times \dots \times a)(r \times r^2 \times \dots \times r^{n-1})$$

$$= a^n r^{1+2+\dots+(n-1)} \quad \dots\dots (ii)$$

Now, notice that the numbers $1, 2, \dots, (n-1)$ on the power of r are in A.P.

So, the sum of the $n-1$ terms,

$$1 + 2 + \dots + (n-1) = \frac{n-1}{2} [2 + (n-1-1) \times 1]$$

$$= \frac{n-1}{2} [2 + n - 2]$$

$$= \frac{n(n-1)}{2}.$$

Thus, from the equation (ii) we have

$$P = a^n r^{\frac{n(n-1)}{2}}$$

$$\Rightarrow P^2 = a^{2n} r^{n(n-1)}$$

$$= \left[a^2 r^{(n-1)} \right]^n$$

$$= (a \times ar^{n-1})^n$$

$= (ab)^n$, applying the result of equation (i).

Hence, it has been proved that $P = (ab)^n$.

Question 24

Show that the ratio of the sum of first n terms of a G.P. to the sum of terms from $(n+1)^{\text{th}}$ to $(2n)^{\text{th}}$ term is $\frac{1}{r^n}$.

Solution

Ans:

Suppose that a and r respectively are the first term and the common ratio of the geometric progression.

Therefore, by the formula, sum of the first n terms of a G.P.,

$$S_n = \frac{a(1 - r^n)}{1 - r}.$$

Now, note that from $(n+1)^{\text{th}}$ to $(2n)^{\text{th}}$, there are n terms.

So, the sum of the terms between these two terms,

$$S'_n = \frac{a_{n+1}(1 - r^n)}{1 - r}.$$

Thus, the $(n+1)^{\text{th}}$ term, $a_{n+1} = ar^{n+1-1} = ar^n$.

So, the ratio of the sums,

$$\begin{aligned} \frac{S_n}{S'_n} &= \frac{a(1 - r^n)}{1 - r} \times \frac{(1 - r)}{ar^n(1 - r^n)} \\ &= \frac{1}{r^n}. \end{aligned}$$

Hence, it has been shown that the ratio of the sum of the sum of first n terms of a G.P. to the sum of terms from $(n+1)^{\text{th}}$ to $(2n)^{\text{th}}$ term is $\frac{1}{r^n}$.

Question 25

If a, b, c and d are in G.P., then show that

$$(a^2 + b^2 + c^2)(b^2 + c^2 + d^2) = (ab + bc + cd)^2.$$

Solution

Ans:

It is given that the numbers a, b, c and d are in geometric progression (G.P.). So, we have

$$\frac{a}{b} = \frac{b}{c} = \frac{c}{d}.$$

That is,

$$bc = ad \quad \dots\dots (i)$$

$$b^2 = ac \quad \dots\dots (ii)$$

$$c^2 = bd \quad \dots\dots (iii)$$

Now, Right-Hand-Side of the statement that needs to be proved,

$$= (ab + bc + cd)^2$$

$$= (ab + ad + cd)^2, \text{ applying the result of the equation (i)}$$

$$= [ab + d(a + c)]^2$$

$$= a^2b^2 + 2abd(a + c) + d^2(a + c)^2$$

$$= a^2b^2 + 2a^2bd + 2acbd + d^2(a^2 + 2ac + c^2)$$

$$= a^2b^2 + 2a^2c^2 + 2b^2c^2 + d^2a^2 + 2d^2b^2 + d^2c^2$$

, applying the results of equations (i) and (ii).

$$= a^2b^2 + a^2c^2 + a^2c^2 + b^2c^2 + b^2c^2 + d^2a^2 + d^2b^2 + d^2b^2 + d^2c^2$$

$$= a^2b^2 + a^2c^2 + a^2d^2 + b^2 \cdot b^2 + b^2c^2 + b^2d^2 + c^2b^2 + c^2 \times c^2 + c^2d^2, \text{ applying the results of equations (ii) and (iii) and rewriting the terms.}$$

$$= a^2(b^2 + c^2 + d^2) + b^2(b^2 + c^2 + d^2) + c^2(b^2 + c^2 + d^2)$$

$$= (a^2 + b^2 + c^2)(b^2 + c^2 + d^2)$$

= Left-Hand-Side of the proof statement.

Hence, it has been proved that

$$(a^2+b^2+c^2)(b^2+c^2+d^2)=(ab+bc+cd)^2.$$

Question 26

Insert two numbers between 3 and 81 so that the resulting sequence is G.P.

Solution

Ans:

Suppose the two numbers that needed to be inserted between 3 and 81 are G_1 and G_2 , so that it forms a G.P. sequence 3, G_1 , G_2 , 81.

Also, assume that the first term and the common ratio respectively are a and r .

Then, according to the given condition we have

$$3(r)^{4-1} = 81$$

$$\Rightarrow r^3 = 27$$

$$\Rightarrow r = 3, \text{ neglecting the imaginary roots.}$$

Therefore, the terms

$$G_1 = ar = 3 \times 3 = 9 \text{ and}$$

$$G_2 = ar^2 = 3(3)^2 = 27.$$

Hence, the numbers that need to be inserted between 3 and 81 are 9 and 27.

Question 27

Find the value of n so that $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ may be the geometric mean between a and b .

Solution

Ans:

It is known that the geometric mean between a and b is $\sqrt{ab}$.

Therefore, according to the condition provided,

$$\frac{a^{n+1} + b^{n+1}}{a^n + b^n} = \sqrt{ab}$$

$$\Rightarrow \frac{(a^{n+1} + b^{n+1})^2}{(a^n + b^n)^2} = ab, \text{ squaring both sides of the equation.}$$

$$\Rightarrow a^{2n+2} + 2a^{n+1}b^{n+1} + b^{2n+2} = (ab)(a^{2n} + 2a^n b^n + b^{2n})$$

$$\Rightarrow a^{2n+2} + 2a^{n+1}b^{n+1} + b^{2n+2} = a^{2n+1}b + 2a^{n+1}b^{n+1} + ab^{2n+1}$$

$$\Rightarrow a^{2n+2} + b^{2n+2} = a^{2n+1}b + ab^{2n+1}$$

$$\Rightarrow a^{2n+2} - a^{2n+1}b = ab^{2n+1} - b^{2n+2}.$$

$$\Rightarrow a^{2n+1}(a - b) = b^{2n+1}(a - b)$$

$$\Rightarrow \left(\frac{a}{b}\right)^{2n+1} = 1 = \left(\frac{a}{b}\right)^0$$

$$\Rightarrow 2n + 1 = 0$$

$$\Rightarrow n = -\frac{1}{2}.$$

Hence, for $n = -\frac{1}{2}$, $\frac{a^{n+1} + b^{n+1}}{a^n + b^n}$ may be the geometric mean between a and b .

Question 28

The sum of two numbers is 6 times their geometric mean, show that numbers are in the ratio $(3 + 2\sqrt{2}) : (3 - 2\sqrt{2})$.

Solution

Ans:

Suppose that a and b are the two numbers.

Then, their geometric mean $= \sqrt{ab}$.

So, by the condition provided to us,

$$a + b = 6\sqrt{ab} \quad \dots\dots (i)$$

Squaring both sides of the equation (i), we have

$$(a + b)^2 = 36ab \quad \dots\dots (ii)$$

Now,

$$\begin{aligned} (a - b)^2 &= (a + b)^2 - 4ab \\ &= 36ab - 4ab, \text{ using the equation (ii).} \end{aligned}$$

$$= 32ab.$$

$$\Rightarrow a - b = \sqrt{32}\sqrt{ab} = 4\sqrt{2}\sqrt{ab} \quad \dots\dots (iii)$$

Therefore, adding the equation (i) and (iii), gives

$$\begin{aligned} 2a &= (6 + 4\sqrt{2})\sqrt{ab} \\ \Rightarrow a &= (3 + 2\sqrt{2})\sqrt{ab} \quad \dots\dots (iv) \end{aligned}$$

Now, using the equation (iv) into the equation (i), gives

$$\begin{aligned} b &= 6\sqrt{ab} - (3 + 2\sqrt{2})\sqrt{ab} \\ \Rightarrow b &= (3 - 2\sqrt{2})\sqrt{ab} \quad \dots\dots (v) \end{aligned}$$

Thus, dividing the equation (iv) by (v) we have,

$$\frac{a}{b} = \frac{(3 + 2\sqrt{2})\sqrt{ab}}{(3 - 2\sqrt{2})\sqrt{ab}} = \frac{3 + 2\sqrt{2}}{3 - 2\sqrt{2}}$$

Hence, the ratio of the numbers is $(3 + 2\sqrt{2}) : (3 - 2\sqrt{2})$.

Question 29

If A and G be A.M. and G.M., respectively between two positive numbers, prove that the numbers are $A \pm \sqrt{(A + G)(A - G)}$.

Solution

Ans:

Suppose that a and b are the two positive numbers.

Then, according to the definition, A.M and G.M of the numbers are

$$A = \frac{a + b}{2} \quad \dots\dots (i)$$

$$G = \sqrt{ab} \quad \dots\dots (ii)$$

Rewriting the equations (i) and (ii), gives

$$a + b = 2A \quad \dots\dots (iii)$$

$$ab = G^2 \quad \dots\dots (iv)$$

Now, it is known that,

$$\begin{aligned} (a - b)^2 &= (a + b)^2 - 4ab \\ &= (2A)^2 - 4G^2, \text{ using the equations (iii) and (iv).} \\ &= 4A^2 - 4G^2 \\ &= 4(A^2 - G^2) \end{aligned}$$

Therefore,

$$\begin{aligned} (a - b)^2 &= 4(A + G)(A - G) \\ \Rightarrow a - b &= 2\sqrt{(A + G)(A - G)} \quad \dots\dots (v) \end{aligned}$$

Adding the equations (iii) and (v) we have

$$\begin{aligned} 2a &= 2A + 2\sqrt{(A + G)(A - G)} \\ \Rightarrow a &= A + \sqrt{(A + G)(A - G)} \quad \dots\dots (vi) \end{aligned}$$

Thus, the equations (iii) and (vi) gives

$$\begin{aligned} b &= 2A - A - \sqrt{(A+G)(A-G)} \\ &= A - \sqrt{(A+G)(A-G)}. \end{aligned}$$

Hence, the two positive numbers are $A \pm \sqrt{(A+G)(A-G)}$.

Question 30

The number of bacteria in a certain culture doubles every hour. If there were 30 bacteria present in the culture originally, how many bacteria will be present at the end of 2nd hour, 4th hour, and nth hour?

Solution

Ans:

Since, the number of bacteria is increasing by two times in each hour, so the problem makes a geometric progression.

It is given that, originally there were 30 bacteria.

So, the first term of the G.P. is $a = 30$ and the common ratio is $r = 2$.

Thus, the number of bacteria will present at the end of 2nd hour is given by

$$a_3 = ar^{3-1} = (30)(2)^2 = 120.$$

The number of bacteria will present at the end of 4th hour is given by

$$a_5 = ar^{5-1} = (30)(2)^4 = 480.$$

Also, the number of bacteria will present at the end of nth hour is given by $a_{n+1} = ar^n = (30)2^n$.

Question 31

What will Rs. 500 amounts to 10 years after its deposit in a bank which pays an annual interest rate of 10 compounded annually?

Solution

Ans:

The deposited amount to the bank is Rs. 500.

The amount obtained after the end of the first year

$$= \text{Rs. } 500 \left(1 + \frac{1}{10} \right) = \text{Rs. } 500 (1.1).$$

The amount obtained after the end of the second year,

$$= \text{Rs. } 500 (1.1) (1.1).$$

The amount obtained after the end of third year,

$$= \text{Rs. } 500 (1.1) (1.1) (1.1) \text{ and ...so on}$$

Therefore, after the end of 10 years, the amount = $\text{Rs. } 500 (1.1) (1.1) \dots (10 \text{ times})$

$$= \text{Rs. } 500 (1.1)^{10}.$$

Question 32

If A.M. and G.M. of roots of a quadratic equation are 8 and 5, respectively, then obtain the quadratic equation.

Solution

Ans:

First suppose that a and b are the roots of the quadratic equation.

Then, A.M. and G.M. of the roots a and b are such that

$$\text{A.M.} = \frac{a+b}{2} \text{ and}$$

$$\text{G.M.} = \sqrt{ab}.$$

Therefore, by the condition given to us,

$$\frac{a+b}{2} = 8$$

$$\Rightarrow a+b = 16 \quad \dots\dots (i)$$

$$\text{and } \sqrt{ab} = 5$$

$$\Rightarrow ab = 25 \quad \dots\dots (ii)$$

Now, we know that, in a quadratic equation of x ,

$$x^2 - x(\text{sum of roots}) + (\text{product of the roots}) = 0$$

$$\Rightarrow x^2 - (a + b)x + (ab) = 0$$

$$\Rightarrow x^2 - 16x + 25 = 0, \text{ substituting the values of } a + b \text{ and } ab.$$

Hence, the needed quadratic equation is $x^2 - 16x + 25 = 0$.

Miscellaneous Exercise

Question 1

If f is a function satisfying

$$f(x + y) = f(x) \cdot f(y)$$

for all

$$x, y \in \mathbb{N}$$

, such that

$$f(1) = 3$$

and

$$\sum_{x=1}^n f(x) = 120$$

find the value of

$$n$$

.

Solution

Ans: According to the given conditions in the question,

$$f(x + y) = f(x) \times f(y)$$

for all

$$x, y, \in N$$

$$f(1) = 3$$

Let

$$x = y = 1$$

.

Then,

$$f(1 + 1) = f(1 + 2) = f(1) f(2) = 3 \times 3 = 9$$

We can also write

$$f(1 + 1 + 1) = f(3) = f(1 + 2) = f(1) f(2) = 3 \times 9 = 27$$

$$f(4) = f(1 + 4) = f(1) f(3) = 3 \times 27 = 81$$

Both the first term and common ratio of

$$f(1), f(2), f(3), \dots,$$

that is

$$3, 9, 27, \dots,$$

that forms s G.P. is equal to

$$3$$

We know that,

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

Given that,

$$\sum_{k=1}^n f(x) = 120$$

Then,

$$120 = \frac{3(3^n - 1)}{3 - 1}$$

$$\Rightarrow 120 = \frac{3}{2}(3^n - 1)$$

$$\Rightarrow 3^n - 1 = 80$$

$$\Rightarrow 3^n = 80 = 3^4$$

$$\Rightarrow 3^n - 1 = 80$$

$$n = 4$$

Therefore,

$$4$$

is the value of

$$n$$

.

Question 2

The sum of some terms of G.P. is

$$315$$

whose first term and the common ratio are

$$5$$

and

$$2$$

, respectively. Find the last term and the number of terms.

Solution

Ans: Let

$$315$$

be the sum of

$$n$$

terms of the G.P.

We know that,

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

The first term

$$a$$

of the A.P. is

$$5$$

and the common difference

$$r$$

is

$$2$$

.

Substitute the values of

$$a$$

and

$$r$$

in the equation

$$315 = \frac{5(2^n - 1)}{2 - 1}$$

$$\Rightarrow 2^n - 1 = 63$$

$$\Rightarrow 2^n = 63 = (2)^6$$

$$\Rightarrow n = 6$$

Therefore, the

$$6^{th}$$

term is the last term of the G.P.

$$6^{th}$$

term

$$= ar^{6-1} = (5)(2)^5 = (5)(32) = 160$$

Therefore,

$$160$$

is the last term of the G.P and the number of terms is

$$6$$

.

Question 3

The first term of a G.P. is

$$1$$

. The sum of the third term and fifth term is

$$90$$

. Find the common ratio of G.P.

Solution

Ans: Let the first term of the G.P. be

$$a$$

and the common ratio be

$$r$$

.

Then,

$$a = 1$$

$$a_3 = ar^2 = r^2$$

$$a_5 = ar^4 = r^4$$

Therefore,

$$r^2 + r^4 = 90$$

$$\Rightarrow r^4 + r^2 - 90 = 0$$

$$\Rightarrow r^2 = \frac{-1 + \sqrt{1 + 360}}{2}$$

$$= \frac{-1 + \sqrt{361}}{2}$$

$$= -10$$

or

$$9$$

$$\Rightarrow r = \pm 3$$

Therefore,

$$\pm 3$$

is the common ratio of the G.P.

Question 4

The sum of the three numbers in G.P. is

$$56$$

. If we subtract

$$1, 7, 21$$

from these numbers in that order, we obtain an arithmetic progression. Find the numbers.

Solution

Ans: Let

$$a, ar$$

and

$$ar^2$$

be the three numbers in G.P.

According to the conditions given in the question,

$$\begin{aligned}a + ar + ar^2 &= 56 \\ \Rightarrow a(1 + r + r^2) &= 56\end{aligned}$$

...(1)

An A.P. is formed by

$$a - 1, ar - 7, ar^2 - 21$$

Therefore,

$$\begin{aligned}(ar - 7) - (a - 1) &= (ar^2 - 21) - (ar - 7)b \\ \Rightarrow ar - a - 6 &= ar^2 - ar - 14 \\ \Rightarrow ar^2 - 2ar + a &= 8 \\ \Rightarrow ar^2 - ar - ar + a &= 8 \\ \Rightarrow a(r^2 + 1 - 2r) &= 8 \\ \Rightarrow a(r^2 - 1)^2 &= 8\end{aligned}$$

...(2)

Equating (1) and (2), we get

$$\begin{aligned}\Rightarrow 7(r^2 - 2r + 1) &= 1 + r + r^2 \\ \Rightarrow 7r^2 - 14r + 7 - 1 - r - r^2 & \\ \Rightarrow 6r^2 - 15r + 6 &= 0 \\ \Rightarrow 6r^2 - 12r - 3r + 6 &= 0 \\ \Rightarrow 6(r - 2) - 3(r - 2) &= 0 \\ \Rightarrow (6r - 3)(r - 2) &= 0\end{aligned}$$

Then,

$$8, 16$$

and

$$32$$

are the three numbers when

$$r = 2$$

and

$$32, 16$$

and

$$8$$

are the numbers when

$$r = \frac{1}{2}$$

.

Therefore,

$$8, 16$$

and

$$32$$

are the three required numbers in either case.

Question 5

A G.P. consists of an even number of terms. If the sum of all the terms is

$$5$$

times the sum of terms occupying odd places, then find its common ratio.

Solution

Ans: Let

$$T_1, T_2, T_3, T_4, \dots, T_{2n}$$

be the G.P.

$$2n$$

is the number of terms.

According to the conditions given in the question,

$$\begin{aligned} T_1 + T_2 + T_3 + \dots + T_{2n} &= 5 [T_1 + T_3 + \dots + T_{2n-1}] \\ \Rightarrow T_1 + T_2 + T_3 + \dots + T_{2n} - 5 [T_1 + T_3 + \dots + T_{2n-1}] &= 0 \\ \Rightarrow T_2 + T_4 + \dots + T_{2n} &= 4 [T_1 + T_3 + \dots + T_{2n-1}] \end{aligned}$$

Let

$$a, ar, ar^2, ar^3$$

be the G.P.

Therefore,

$$\begin{aligned} \frac{ar(r^n - 1)}{r - 1} &= \frac{4 \times a(r^n - 1)}{r - 1} \\ \Rightarrow ar &= 4a \\ \Rightarrow r &= 4 \end{aligned}$$

Therefore,

$$4$$

is the common ratio of the G.P.

Question 6

If

$$\frac{a + bx}{a - bx} = \frac{b + cx}{b - cx} = \frac{c + dx}{c - dx} \quad (x \neq 0)$$

then show that

$$a, b, c$$

and

$$d$$

are in G.P.

Solution

Ans: Given,

$$\frac{a + bx}{a - bx} = \frac{b + cx}{b - cx}$$

$$\Rightarrow (a + bx)(b - cx) = (b + cx)(a - bx)$$

$$\Rightarrow ab - acx + b^2x - bcx^2 = ab - b^2x + -acx - bcx^2$$

$$\Rightarrow 2b^2x = 2acx$$

$$\Rightarrow b^2 = ac$$

$$\Rightarrow \frac{b}{a} = \frac{c}{b}$$

It is also given that,

$$\frac{b + cx}{b - cx} = \frac{c + dx}{c - dx}$$

$$\Rightarrow (b + cx)(c - dx) = (b - cx)(c + dx)$$

$$\Rightarrow bc - bdx + c^2x - cdx^2 = bc + bdx - c^2x - cdx^2$$

$$\Rightarrow 2c^2x = 2bdx$$

$$\Rightarrow c^2 = bd$$

$$\Rightarrow \frac{c}{d} = \frac{d}{c}$$

Equating both results, we get

$$\frac{b}{a} = \frac{c}{b} = \frac{d}{c}$$

Therefore, it is proved that

$$a, b, c$$

and

$$d$$

are in G.P.

Question 7

Let

$$S$$

be the sum,

$$P$$

the product and

$$R$$

the sum of reciprocals of n terms in a G.P. Prove that

$$P^2 R^n = S^n$$

.

Solution

Ans: Let

$$a, ar, ar^2, ar^3 \dots ar^{n-1}$$

be the G.P.

According to the conditions given in the question,

$$S = \frac{a(r^n - 1)}{r - 1}$$

$$P = a^n \times r^{1+2+\dots+n-1}$$

Since the sum of first

$$n$$

natural numbers is

$$n \frac{(n+1)}{2}$$

$$\Rightarrow P = a^n r^{\frac{n(n-1)}{2}}$$

$$R = \frac{1}{a} + \frac{1}{ar} + \dots + \frac{1}{ar^{n-1}}$$

$$= \frac{r^{n-1} + r^{n-2} + \dots + r + 1}{ar^{n-1}}$$

Since

$$1, r, \dots, r^{n-1}$$

forms a G.P.,

$$\Rightarrow R = \frac{1(r^n - 1)}{(r - 1)} \times \frac{1}{ar^{n-1}}$$

$$= \frac{r^n - 1}{ar^{n-1}(r - 1)}$$

Then,

$$P^2 R^n = a^{2n} r^{n(n-1)} \frac{(r^n - 1)^n}{a^n r^{n(n-1)} (r - 1)^n}$$

$$= \frac{a^n (r^n - 1)^n}{(r - 1)^n}$$

$$= \left[\frac{a(r^n - 1)}{(r - 1)} \right]^n$$

$$= S^n$$

Therefore,

$$P^2 R^n = S^n$$

.

Question 8

If

$$a, b, c, d$$

are in G.P., prove that

$$(a^n + b^n), (b^n + c^n), (c^n + d^n)$$

are in G.P

Solution

Ans: Given:

$$a, b, c$$

and

$$d$$

are in G.P.

Therefore,

$$b^2 = ac$$

$$c^2 = bd$$

$$ad = bc$$

To prove:

$$(a^n + b^n), (b^n + c^n), (c^n + d^n)$$

are in G.P.

That is,

$$(b^n + c^n)^2 = (a^n + b^n)(c^n + d^n)$$

Then,

L.H.S

$$\begin{aligned} &= (b^n + c^n)^2 \\ &= b^{2n} + 2b^n c^n + c^{2n} \\ &= (b^2)^n + 2b^n c^n + (c^2)^n \\ &= (ac)^n + 2b^n c^n + (bd)^n \\ &= a^n c^n + b^n c^n + b^n c^n + b^n d^n \\ &= a^n c^n + b^n c^n + a^n d^n + b^n d^n \\ &= c^n (a^n + b^n) + d^n (a^n + b^n) \\ &= (a^n + b^n)(a^n + d^n) \\ &= \end{aligned}$$

R.H.S

Therefore,

$$(b^n + c^n)^2 = (a^n + b^n)(c^n + d^n)$$

Therefore,

$$(b^n + c^n), (b^n + c^n)$$

and

$$(c^n + d^n)$$

are in G.P.

Question 9

If

$$a$$

and

$$b$$

are the roots of

$$x^2 - 3x + p = 0$$

and

$$c, d$$

are roots of

$$x^2 - 12x + q = 0$$

, where

$$a, b, c, d$$

form a G.P. Prove that

$$(q + p) : (q - p) = 17 : 15$$

.

Solution

Ans: Given:

$$a$$

and

$$b$$

are the roots of

$$x^2 - 3x + p = 0$$

.

Therefore,

$$a + b = 3$$

and

$$ab = p$$

...(1)

We also know that

$$c$$

and

$$d$$

are the roots of

$$x^2 - 12x + q = 0$$

.

Therefore,

$$c + d = 12$$

and

$$cd = q$$

...(2)

Also,

$$a, b, c, d$$

are in G.P.

Let us take

$$a = x, b = xr, c = xr^2$$

and

$$d = xr^3$$

.

We get from (1) and (2) that,

$$x + xr = 3$$

$$\Rightarrow x(1 + r) = 3$$

Also,

$$\begin{aligned}xr^2 + xr^3 &= 12 \\ \Rightarrow xr^2 + (1 + r) &= 12\end{aligned}$$

Divide both the equations obtained.

$$\begin{aligned}\frac{xr^2(1+r)}{x(1+r)} &= \frac{12}{3} \\ \Rightarrow r^2 &= 4 \\ \Rightarrow r &= \pm 2 \\ x &= \frac{3}{1+2} = \frac{3}{3} = 1\end{aligned}$$

, when

$$r = 2$$

and

$$x = \frac{3}{1-2} = \frac{3}{-1} = -3$$

, when

$$r = -2$$

.

Case I:

$$ab = x^2r = 2$$

,

$$cd = x^2r^5 = 32$$

when

$$r = 2$$

and

$$x = 1$$

.

Therefore,

$$\begin{aligned}\frac{q+p}{q-p} &= \frac{32+2}{32-2} = \frac{34}{30} = \frac{17}{15} \\ \Rightarrow (q+p) : (q-p) &= 17 : 15\end{aligned}$$

Case II:

$$ab = x^2r = 18$$

,

$$cd = x^2r^5 = -288$$

when

$$r = -2$$

and

$$x = -3$$

.

Therefore,

$$\begin{aligned}\frac{q+p}{q-p} &= \frac{-288-18}{-288+18} = \frac{-306}{-270} = \frac{17}{15} \\ \Rightarrow (q+p) : (q-p) &= 17 : 15\end{aligned}$$

Therefore, it is proved that

$$(q+p) : (q-p) = 17 : 15$$

as we obtain the same for both cases.

Question 10

The ratio of the A.M and G.M. of two positive numbers

$$a$$

and

$$b$$

is

$$m : n$$

. Show that

$$a : b = \left(m + \sqrt{m^2 - n^2} \right) : \left(m - \sqrt{m^2 - n^2} \right)$$

.

Solution

Ans: Let

$$a$$

and

$$b$$

be the two numbers.

The arithmetic mean, A.M

$$= \frac{a + b}{2}$$

and the geometric mean, G.M

$$= \sqrt{ab}$$

According to the conditions given in the question,

$$\frac{a + b}{2\sqrt{ab}} = \frac{m}{n}$$

$$\Rightarrow \frac{(a + b)^2}{4(ab)} = \frac{m^2}{n^2}$$

$$\Rightarrow (a + b) = \frac{4abm^2}{n^2}$$

$$\Rightarrow (a + b) = \frac{2\sqrt{ab}m}{n}$$

...(1)

Using the above equation in the identity

$$(a - b)^2 = (a + b)^2 - 4ab$$

, we obtain

$$\begin{aligned}(a - b)^2 &= \frac{4abm^2}{n^2} - 4ab = \frac{4ab(m^2 - n^2)}{n^2} \\ \Rightarrow (a - b) &= \frac{2\sqrt{ab}\sqrt{m^2 - n^2}}{n}\end{aligned}$$

...(2)

Add equation (1) and (2)

$$\begin{aligned}2a &= \frac{2\sqrt{ab}}{n} \left(m + \sqrt{m^2 - n^2} \right) \\ \Rightarrow a &= \frac{\sqrt{ab}}{n} \left(m + \sqrt{m^2 - n^2} \right)\end{aligned}$$

Substitute in (1) the value of

$$a$$

.

$$\begin{aligned}b &= \frac{2\sqrt{ab}}{n}m - \frac{\sqrt{ab}}{n} \left(m + \sqrt{m^2 - n^2} \right) \\ &= \frac{\sqrt{ab}}{n}m - \frac{\sqrt{ab}}{n} \sqrt{m^2 - n^2} \\ &= \frac{\sqrt{ab}}{n} \left(m - \sqrt{m^2 - n^2} \right)\end{aligned}$$

Therefore,

$$a : b = \frac{a}{b} = \frac{\frac{\sqrt{ab}}{n} (m + \sqrt{m^2 - n^2})}{\frac{\sqrt{ab}}{n} (m - \sqrt{m^2 - n^2})} = \frac{(m + \sqrt{m^2 - n^2})}{(m - \sqrt{m^2 - n^2})}$$

Therefore, it is proved that

$$a : b = (m + \sqrt{m^2 - n^2}) : (m - \sqrt{m^2 - n^2})$$

.

Question 11

Find the sum of the following series up to

n

terms:

(i)

$$5 + 55 + 555 + \dots$$

(ii)

$$.6 + .66 + .666 + \dots$$

Solution

Ans: Let

$$S_n = 5 + 55 + 555 + \dots$$

to

n

terms.

$$= \frac{5}{9}$$

(9+99+999+... to n terms.)

$$= \frac{5}{9}((10 - 1) + (10^2 - 1) + (10^3 - 1) + \dots$$

to n terms)

$$= \frac{5}{9}((10 + 10^2 + 10^3 \dots$$

to n terms) - $(1 + 1 + \dots$ to n terms)

$$= \frac{5}{9} \left[\frac{10(10^n - 1)}{10 - 1} - n \right]$$

$$= \frac{5}{9} \left[\frac{10(10^n - 1)}{9} - n \right]$$

$$= \frac{50}{81}(10^n - 1) - \frac{5n}{9}$$

Therefore, the sum of

$$n$$

terms of the given series is

$$\frac{50}{81}(10^n - 1) - \frac{5n}{9}$$

.

Ans: Let

$$S_n = 0.6 + 0.66 + 0.666 +$$

to

$$n$$

terms.

$$= 6$$

$(0.1 + 0.11 + 0.111 + \dots$ to

$$n$$

terms)

$$= \frac{6}{9}$$

(0.9+0.99+0.999+... to

n

terms)

$$= \frac{6}{9} \left(\left(1 - \frac{1}{10}\right) + \left(1 - \frac{1}{10^2}\right) + \left(1 - \frac{1}{10^3}\right) \right) + \dots$$

to n terms

$$= \frac{2}{3}$$

((

$$1 + 1 + \dots$$

to

n

terms)

—

$$\frac{1}{10}$$

(

$$1 + \frac{1}{10} + \frac{1}{10^2}$$

to

n

terms))

$$= \frac{2}{3} \left(n - \frac{1}{10} \left(\frac{1 - \left(\frac{1}{10}\right)^n}{1 - \frac{1}{10}} \right) \right)$$

$$= \frac{2}{3}n - \frac{2}{30} \times \frac{10}{9} (1 - 10^{-n})$$

$$= \frac{2}{3}n - \frac{2}{27}(1 - 10^{-n})$$

Therefore, the sum of

$$n$$

terms of the given series is

$$\frac{2}{3}n - \frac{2}{27}(1 - 10^{-n})$$

.

Question 12

Find the

$$20^{th}$$

term of the series

$$2 \times 4 + 4 \times 6 + 6 \times 8 + \dots + n$$

terms.

Solution

Ans:

$$2 \times 4 + 4 \times 6 + 6 \times 8 + \dots + n$$

is the given series,

Therefore the

$$n^{th}$$

term

$$a_n = 2n \times (2n + 2) = 4n^2 + 4n$$

Then,

$$a_{20} = 4(20)^2 + 4(20)$$

$$\begin{aligned}
 &= 4(400) + 80 \\
 &= 1600 + 80 \\
 &= 1680
 \end{aligned}$$

Therefore,

$$1680$$

is the

$$20^{th}$$

term of the series.

Question 13

A farmer buys a used tractor for Rs.

$$12000$$

. He pays Rs.

$$6000$$

cash and agrees to pay the balance in annual instalments of Rs.

$$500$$

plus

$$12$$

interest on the unpaid amount. How much will the tractor cost him?

Solution

Ans: It is given that Rs.

$$6000$$

is paid in cash by the farmer.

Therefore, the unpaid amount is given by

Rs.

$$12000 -$$

Rs.

$$6000 =$$

Rs.

$$6000$$

According to the conditions given in the question, the interest to be paid annually by the farmer is

$$12$$

of

$$6000$$

,

$$12$$

of

$$5500$$

,

$$12$$

of

$$5000 \dots 12$$

of

$$500$$

Therefore, the total interest to be paid by the farmer

$$= 12$$

of

$$6000 + 12$$

of

$$5500 + 12$$

of

$$5000 + \dots + 12$$

of

$$\begin{aligned} & 500 \\ & = 12 \end{aligned}$$

of

$$\begin{aligned} & (6000 + 5500 + 5000 + \dots + 500) \\ & = 12 \end{aligned}$$

of

$$(500 + 1000 + 1500 + \dots + 6000)$$

With both the first term and common difference equal to

$$500$$

, the series

$$500, 1000, 1500 \dots 6000$$

is an A.P.

Let

$$n$$

be the number of terms of the A.P.

Therefore,

$$\begin{aligned} 6000 &= 500 + (n - 1) 500 \\ \Rightarrow 1 + (n - 1) &= 12 \\ \Rightarrow n &= 12 \end{aligned}$$

Therefore, the sum of the given A.P.

$$\begin{aligned} &= \frac{12}{2} [2 (500) + (12 - 1) (500)] \\ &= 6 [1000 + 5500] \\ &= 6 (6500) \\ &= 39000 \end{aligned}$$

Therefore, the total interest to be paid by the farmer

$$= 12$$

of

$$(500 + 1000 + 1500 + \dots + 6000)$$

$$= 12$$

of Rs.

$$39000$$

$$=$$

Rs.

$$4680$$

Therefore, the total cost of a tractor

$$=$$

(Rs.

$$12000 +$$

Rs.

$$4680$$

)

$$=$$

Rs.

$$16680$$

Therefore, the total cost of the tractor is Rs.

$$16680$$

.

Question 14

Shamshad Ali buys a scooter for Rs.

22000

. He pays Rs.

4000

cash and agrees to pay the balance in an annual instalment of Rs.

1000

plus

10

interest on the unpaid amount. How much will the scooter cost him?

Solution

Ans: It is given that for Rs.

22000

Shamshad Ali buys a scooter and Rs.

4000

is paid in cash.

Therefore, the unpaid amount is given by

Rs.

22000—

Rs.

4000 =

Rs.

18000

According to the conditions given in the question, the interest to be paid annually

is

10

of

$$18000$$

,

$$10$$

of

$$17000$$

,

$$10$$

of

$$16000 \dots 10$$

of

$$1000$$

Therefore, the total interest to be paid by the farmer

$$= 10$$

of

$$18000 + 10$$

of

$$17000 + 10$$

of

$$16000 + \dots + 10$$

of

$$1000$$

$$= 10$$

of

$$(18000 + 17000 + 16000 + \dots + 1000)$$

$$= 10$$

of

$$(1000 + 2000 + 3000 + \dots + 18000)$$

With both the first term and common difference equal to

$$1000$$

, the series

$$1000, 2000, 3000 \dots 18000$$

is an A.P.

Let

$$n$$

be the number of terms of the A.P.

Therefore,

$$18000 = 1000 + (n - 1) 1000$$

$$\Rightarrow 1 + (n - 1) = 18$$

$$\Rightarrow n = 18$$

Therefore, the sum of the given A.P.

$$= \frac{18}{2} [2 (1000) + (18 - 1) (1000)]$$

$$= 9 [2000 + 17000]$$

$$= 9 (19000)$$

$$= 171000$$

Therefore, the total interest to be paid

$$= 10$$

of

$$(18000 + 17000 + 16000 + \dots + 1000)$$

$$= 10$$

of Rs.

$$171000$$

=

Rs.

17100

Therefore, the total cost of the scooter

=

(Rs.

22000+

Rs.

17100

)

=

Rs.

39100

Therefore, the total cost of the scooter is Rs.

39100

.

Question 15

A person writes a letter to four of his friends. He asks each one of them to copy the letter and mail it to four different persons with the instruction that they move the chain similarly. Assuming that the chain is not broken and that it costs

50

paise to mail one letter. Find the amount spent on the postage when

 8^{th}

set of letters is mailed.

Solution

Ans:

$$4, 4^2, \dots, 4^8$$

is the number of letters mailed and it forms a G.P.

The first term

$$a = 4$$

, the common ratio

$$r = 4$$

and the number of terms

$$n = 8$$

of the G.P.

We know that the sum of

$$n$$

terms of a G.P. is

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

Therefore,

$$\begin{aligned} S_8 &= \frac{4(4^8 - 1)}{4 - 1} \\ &= \frac{4(65536 - 1)}{3} \\ &= \frac{4(65535)}{3} \\ &= 4(21845) \\ &= 87380 \\ &\quad 50 \end{aligned}$$

paisa is the cost of mailing one letter.

Therefore,

Cost of mailing

$$87380$$

letters

$$=$$

Rs.

$$87380 \times \frac{50}{100}$$
$$=$$

Rs.

$$43690$$

Therefore, Rs.

$$43690$$

is the amount spent when

$$8^{th}$$

set of letter is mailed.

Question 16

A man deposited Rs.

$$10000$$

in a bank at the rate of

$$5$$

simple interest annually. Find the amount in

$$15^{th}$$

year since he deposited the amount and also calculate the total amount after

$$20$$

years.

Solution

Ans: Rs.

$$10000$$

is deposited by the man in a bank at the rate of

$$5$$

simple interest annually

$$= \frac{5}{100} \times$$

Rs.

$$10000 =$$

Rs.

$$500$$

Therefore,

$$10000 + 500 + 500 + \dots + 500$$

is the interest in

$$15^{th}$$

year. (

$$500$$

is

$$14$$

added times)

Therefore, the amount in

$$15^{th}$$

year

$$=$$

Rs.

$$10000 + 14 \times$$

Rs.

$$500$$

$$=$$

Rs.

$$10000 +$$

Rs.

$$7000$$

$$=$$

Rs.

$$17000$$

Rs.

$$10000 + 500 + 500 + \dots + 500$$

is the amount after

$$20$$

years. (

$$500$$

is

$$20$$

added times)

Therefore, the amount after

$$20$$

years

$$=$$

Rs.

$$10000 + 20 \times$$

Rs.

$$500$$

$$=$$

Rs.

$$10000 +$$

Rs.

$$10000$$

$$=$$

Rs.

$$20000$$

The total amount after

$$20$$

years is Rs.

$$20000$$

.

Question 17

A manufacturer reckons that the value of a machine, which costs him Rs.

$$15625$$

, will depreciate each year by

$$20$$

. Find the estimated value at the end of

$$5$$

years.

Solution

Ans: The cost of the machine is Rs.

$$15625$$

Every year machine depreciates by

$$20$$

Therefore,

$$80$$

of the original cost,i.e.,

$$\frac{4}{5}$$

of the original cost is its value after every year.

Therefore, the value at the end of

$$5$$

years

$$\begin{aligned} &= 15626 \times \frac{4}{5} \times \frac{4}{5} \times \dots \times \frac{4}{5} \\ &= 5 \times 1024 \\ &= 5120 \end{aligned}$$

Therefore, Rs.

$$5120$$

is the value of the machine at the end of

$$5$$

years.

Question 18

$$150$$

workers were engaged to finish a job in a certain number of days.

4

workers dropped out on the second day,

4

more workers dropped out on the third day and so on. It took

8

more days to finish the work. Find the number of days in which the work was completed.

Solution

Ans: Let the number of days in which

150

workers finish the work be

 x

.

According to the conditions given in the question,

$$150x = 150 + 146 + 142 + \dots (x + 8)$$

terms

With first term

$$a = 146$$

, common difference

$$d = -4$$

and number of terms as

$$(x + 8)$$

, the series

$$150 + 146 + 142 + \dots (x + 8)$$

terms is an A.P.

$$\Rightarrow 150x = \frac{(x + 8)}{2} [2(150) + (x + 8 - 1)(-4)]$$

$$\begin{aligned}\Rightarrow 150x &= (x + 8) [150 + (x + 7) (-2)] \\ \Rightarrow 150x &= (x + 8) (150 - 2x - 14) \\ \Rightarrow 150x &= (x + 8) (136 - 2x) \\ \Rightarrow 75x &= (x + 8) (68 - x) \\ \Rightarrow 75x &= 68x - x^2 + 544 - 8x \\ \Rightarrow x^2 + 75x - 60x - 544 &= 0 \\ \Rightarrow x^2 + 15x - 544 &= 0 \\ \Rightarrow x^2 + 32x - 17x - 544 &= 0 \\ \Rightarrow x(x + 32) - 17(x + 32) &= 0 \\ \Rightarrow (x - 17)(x + 32) &= 0 \\ \Rightarrow x &= 17\end{aligned}$$

or

$$x = -32$$

We know that

$$x$$

cannot be negative.

So,

$$x = 17$$

.

Therefore,

$$17$$

is the number of days in which the work was completed. Then the required number of days

$$= (17 + 8) = 25$$

.