

Chapter 8: Binomial Theorem Miscellaneous Exercise

NCERT · Class 11 · Maths

Questions

Question 1

Find a , b and n in the expansion of $(a + b)^n$ if the first three terms of the expansion are 729, 7290 and 30375, respectively.

Solution

We know that $(r + 1)^{\text{th}}$ term, (T_{r+1}) in the binomial expansion of $(a + b)^n$ is given by

$$T_{r+1} = {}^nC_r a^{n-r} b^r$$

The first three terms of the expansion are given as 729, 7290 and 30375, respectively. Then we have,

$$T_1 = {}^nC_0 a^{n-0} b^0 = a^n = 729 \dots 1$$

$$T_2 = {}^nC_1 a^{n-1} b^1 = na^{n-1} b = 7290 \dots 2$$

$$T_3 = {}^nC_2 a^{n-2} b^2 = \frac{n(n-1)}{2} a^{n-2} b^2 = 30375 \dots 3$$

Dividing 2 by 1, we get

$$\frac{na^{n-1}b}{a^n} = \frac{7290}{729}$$

$$\frac{nb}{a} = 10$$

Dividing 3 by 2, we get

$$\frac{n(n-1)a^{n-2}b^2}{2na^{n-1}b} = \frac{30375}{7290}$$

$$\frac{(n-1)b}{2a} = \frac{30375}{7290}$$

$$\frac{(n-1)b}{a} = \frac{30375}{7290} \times 2 = \frac{25}{3}$$

$$\frac{(nb)}{a} - \frac{b}{a} = \frac{25}{3}$$

$$10 - \frac{b}{a} = \frac{25}{3}$$

$$\frac{b}{a} = 10 - \frac{25}{3} = \frac{5}{3}$$

From 4 and 5, we have

$$n \cdot \frac{5}{3} = 10$$

$$n = 6$$

Substituting $n = 6$ in 1, we get

$$a^6 = 729$$

$$a = 3$$

From 5, we have $b/3 = 5/3$

$$b = 5$$

Thus, $a = 3$, $b = 5$ and $n = 76$

Question 2

Find a if the coefficients of x^2 and x^3 in the expansion of $(3 + ax)^9$ are equal.

Solution

We know that general term of expansion $(a + b)^n$ is

$$T_{r+1} = \binom{n}{r} a^{n-r} b^r$$

For $(3+ax)^9$

Putting $a = 3$, $b = ax$ & $n = 9$

General term of $(3+ax)^9$ is

$$T_{r+1} = \binom{9}{r} 3^{n-r} (ax)^r$$

$$T_{r+1} = \binom{9}{r} 3^{n-r} a^r x^r$$

Since we need to find the coefficients of x^2 and x^3 , therefore

For $r = 2$

$$T_{2+1} = \binom{9}{2} 3^{n-2} a^2 x^2$$

Thus, the coefficient of $x^2 = \binom{9}{2} 3^{n-2} a^2$

For $r = 3$

$$T_{3+1} = \binom{9}{3} 3^{n-3} a^3 x^3$$

Thus, the coefficient of $x^3 = \binom{9}{3} 3^{n-3} a^3$

Given that coefficient of $x^2 =$ Coefficient of x^3

$$\Rightarrow \binom{9}{2} 3^{n-2} a^2 = \binom{9}{3} 3^{n-3} a^3$$

$$\Rightarrow \frac{9!}{2!(9-2)!} \times 3^{n-2} a^2 = \frac{9!}{3!(9-3)!} \times 3^{n-3} a^3$$

$$\Rightarrow \frac{3^{n-2} a^2}{3^{n-3} a^3} = \frac{2!(9-2)!}{3!(9-3)!}$$

$$\Rightarrow \frac{3^{(n-2)-(n-3)}}{a} = \frac{2!7!}{3!6!}$$

$$\Rightarrow \frac{3}{a} = \frac{7}{3}$$

$$\therefore a = 9/7$$

Hence, $a = 9/7$

Question 3

Find the coefficient of x^5 in the product $(1 + 2x)^6 (1 - x)^7$ using binomial theorem.

Solution

$$(1 + 2x)^6 = {}^6C_0 + {}^6C_1 (2x) + {}^6C_2 (2x)^2 + {}^6C_3 (2x)^3 + {}^6C_4 (2x)^4 + {}^6C_5 (2x)^5 + {}^6C_6 (2x)^6$$

$$= 1 + 6(2x) + 15(2x)^2 + 20(2x)^3 + 15(2x)^4 + 6(2x)^5 + (2x)^6$$

$$= 1 + 12x + 60x^2 + 160x^3 + 240x^4 + 192x^5 + 64x^6$$

$$(1 - x)^7 = {}^7C_0 - {}^7C_1 (x) + {}^7C_2 (x)^2 - {}^7C_3 (x)^3 + {}^7C_4 (x)^4 - {}^7C_5 (x)^5 + {}^7C_6 (x)^6 - {}^7C_7 (x)^7$$

$$= 1 - 7x + 21x^2 - 35x^3 + 35x^4 - 21x^5 + 7x^6 - x^7$$

$$(1 + 2x)^6 (1 - x)^7 = (1 + 12x + 60x^2 + 160x^3 + 240x^4 + 192x^5 + 64x^6) (1 - 7x + 21x^2 - 35x^3 + 35x^4 - 21x^5 + 7x^6 - x^7)$$

$$192 - 21 = 171$$

Thus, the coefficient of x^5 in the expression $(1+2x)^6(1-x)^7$ is 171.

Question 4

If a and b are distinct integers, prove that $a - b$ is a factor of $a^n - b^n$, whenever n is a positive integer.

[Hint: Write $a^n = (a - b + b)^n$ and expand]

Solution

In order to prove that $(a - b)$ is a factor of $(a^n - b^n)$, it has to be proved that

$a^n - b^n = k(a - b)$, where k is some natural number.

a can be written as $a = a - b + b$

$$a^n = (a - b + b)^n = [(a - b) + b]^n$$

$$= {}^nC_0 (a - b)^n + {}^nC_1 (a - b)^{n-1} b + \dots + {}^nC_n b^n$$

$$a^n - b^n = (a - b) [(a - b)^{n-1} + {}^nC_1 (a - b)^{n-2} b + \dots + {}^nC_{n-1} b^{n-1}]$$

$$a^n - b^n = (a - b) k$$

Where $k = [(a - b)^{n-1} + {}^nC_1 (a - b)^{n-2} b + \dots + {}^nC_{n-1} b^{n-1}]$ is a natural number

This shows that $(a - b)$ is a factor of $(a^n - b^n)$, where n is a positive integer.

Question 5

Evaluate

$$(\sqrt{3} + \sqrt{2})^6 - (\sqrt{3} - \sqrt{2})^6$$

Solution

Using the binomial theorem, the expression $(a + b)^6$ and $(a - b)^6$ can be expanded

$$(a + b)^6 = {}^6C_0 a^6 + {}^6C_1 a^5 b + {}^6C_2 a^4 b^2 + {}^6C_3 a^3 b^3 + {}^6C_4 a^2 b^4 + {}^6C_5 a b^5 + {}^6C_6 b^6$$

$$(a - b)^6 = {}^6C_0 a^6 - {}^6C_1 a^5 b + {}^6C_2 a^4 b^2 - {}^6C_3 a^3 b^3 + {}^6C_4 a^2 b^4 - {}^6C_5 a b^5 + {}^6C_6 b^6$$

$$\begin{aligned} \text{Now } (a + b)^6 - (a - b)^6 &= {}^6C_0 a^6 + {}^6C_1 a^5 b + {}^6C_2 a^4 b^2 + {}^6C_3 a^3 b^3 + {}^6C_4 a^2 b^4 + {}^6C_5 a b^5 + {}^6C_6 b^6 - \\ &[{}^6C_0 a^6 - {}^6C_1 a^5 b + {}^6C_2 a^4 b^2 - {}^6C_3 a^3 b^3 + {}^6C_4 a^2 b^4 - {}^6C_5 a b^5 + {}^6C_6 b^6] \end{aligned}$$

Now, by substituting $a = \sqrt{3}$ and $b = \sqrt{2}$, we get

$$(\sqrt{3} + \sqrt{2})^6 - (\sqrt{3} - \sqrt{2})^6 = 2 [6 (\sqrt{3})^5 (\sqrt{2}) + 20 (\sqrt{3})^3 (\sqrt{2})^3 + 6 (\sqrt{3}) (\sqrt{2})^5]$$

$$= 2 [54\sqrt{6} + 120\sqrt{6} + 24\sqrt{6}]$$

$$= 2 (\sqrt{6}) (198)$$

$$= 396 \sqrt{6}$$

Question 6

Find the value of

$$\left(a^2 + \sqrt{a^2 - 1}\right)^4 + \left(a^2 - \sqrt{a^2 - 1}\right)^4$$

Solution

Firstly the expression $(x + y)^4 + (x - y)^4$ is simplified by using binomial theorem

$$(x + y)^4 = {}^4C_0x^4 + {}^4C_1x^3y + {}^4C_2x^2y^2 + {}^4C_3xy^3 + {}^4C_4y^4$$

$$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

$$(x - y)^4 = {}^4C_0x^4 - {}^4C_1x^3y + {}^4C_2x^2y^2 - {}^4C_3xy^3 + {}^4C_4y^4$$

$$= x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$$

$$\therefore (x + y)^4 + (x - y)^4 = 2(x^4 + 6x^2y^2 + y^4)$$

Putting $x = a^2$ and $y = \sqrt{a^2 - 1}$, we obtain

$$\begin{aligned} & \left(a^2 + \sqrt{a^2 - 1}\right)^4 + \left(a^2 - \sqrt{a^2 - 1}\right)^4 \\ &= 2 \left[(a^2)^4 + 6(a^2)^2 (\sqrt{a^2 - 1})^2 + (\sqrt{a^2 - 1})^4 \right] \\ &= 2 \left[a^8 + 6a^4 (a^2 - 1) + (a^2 - 1)^2 \right] \\ &= 2 \left[a^8 + 6a^6 - 6a^4 + a^4 - 2a^2 + 1 \right] \\ &= 2 \left[a^8 + 6a^6 - 5a^4 - 2a^2 + 1 \right] \\ &= 2a^8 + 12a^6 - 10a^4 - 4a^2 + 2 \end{aligned}$$

Question 7

Find an approximation of $(0.99)^5$ using the first three terms of its expansion.

Solution

0.99 can be written as

$$0.99 = 1 - 0.01$$

Now, by applying the binomial theorem, we get

$$(0.99)^5 = (1 - 0.01)^5$$

$$= {}^5C_0 (1)^5 - {}^5C_1 (1)^4 (0.01) + {}^5C_2 (1)^3 (0.01)^2$$

$$= 1 - 5(0.01) + 10(0.01)^2$$

$$= 1 - 0.05 + 0.001$$

$$= 0.951$$

Question 8

Find n, if the ratio of the fifth term from the beginning to the fifth term from the end of the expansion of

$$\left(\sqrt[4]{2} + \frac{1}{\sqrt[4]{3}} \right)^n \text{ is } \sqrt{6}:1$$

Solution

In the expansion $(a + b)^n$, if n is even then the middle term is $(n/2 + 1)^{\text{th}}$ term

$${}^nC_4 (\sqrt[4]{2})^{n-1} \left(\frac{1}{\sqrt[4]{3}} \right)^4 = {}^nC_4 \frac{(\sqrt[4]{2})^n}{(\sqrt[4]{2})^4} \cdot \frac{1}{3} = {}^nC_4 \frac{(\sqrt[4]{2})^n}{2} \cdot \frac{1}{3} = \frac{n!}{6 \cdot 4! (n-4)!} (\sqrt[4]{2})^n$$

$${}^nC_{n-4} (\sqrt[4]{2})^4 \left(\frac{1}{\sqrt[4]{3}} \right)^{n-4} = {}^nC_{n-1} \cdot 2 \cdot \frac{(\sqrt[4]{3})^4}{(\sqrt[4]{3})^n} = {}^nC_{n-1} \cdot 2 \cdot \frac{3}{(\sqrt[4]{3})^n} = \frac{6n!}{(n-4)! 4!} \cdot \frac{1}{(\sqrt[4]{3})^n}$$

$$\begin{aligned}
& \frac{n!}{6 \cdot 4!(n-4)!} (\sqrt[4]{2})^n : \frac{6n!}{(n-4)!!4!} \cdot \frac{1}{(\sqrt[4]{3})^n} = \sqrt{6} : 1 \\
& \Rightarrow \frac{(\sqrt[4]{2})^n}{6} : \frac{6}{(\sqrt[4]{3})^n} = \sqrt{6} : 1 \\
& \Rightarrow \frac{(\sqrt[4]{2})^n}{6} \times \frac{(\sqrt[4]{3})^n}{6} = \sqrt{6} \\
& \Rightarrow (\sqrt[4]{6})^n = 36\sqrt{6} \\
& \Rightarrow 6^{\frac{n}{4}} = 6^{\frac{5}{2}} \\
& \Rightarrow \frac{n}{4} = \frac{5}{2} \\
& \Rightarrow n = 4 \times \frac{5}{2} = 10
\end{aligned}$$

Thus the value of $n = 10$

Question 9

Expand using the Binomial Theorem.

$$\left(1 + \frac{x}{2} - \frac{2}{x}\right)^4, x \neq 0$$

Solution

Using the binomial theorem, the given expression can be expanded as

$$\begin{aligned}
& \left[\left(1 + \frac{x}{2}\right) - \frac{2}{x} \right]^4 \\
& = {}^4C_0 \left(1 + \frac{x}{2}\right)^4 - {}^4C_1 \left(1 + \frac{x}{2}\right)^3 \left(\frac{2}{x}\right) + {}^4C_2 \left(1 + \frac{x}{2}\right)^2 \left(\frac{2}{x}\right)^2 - {}^4C_3 \left(1 + \frac{x}{2}\right) \left(\frac{2}{x}\right)^3 + {}^4C_4 \left(\frac{2}{x}\right)^4 \\
& = \left(1 + \frac{x}{2}\right)^4 - 4 \left(1 + \frac{x}{2}\right)^3 \left(\frac{2}{x}\right) + 6 \left(1 + \frac{x}{2}\right)^2 \left(\frac{4}{x^2}\right) - 4 \left(1 + \frac{x}{2}\right) \left(\frac{8}{x^3}\right) + \frac{16}{x^4} \\
& = \left(1 + \frac{x}{2}\right)^4 - \frac{8}{x} \left(1 + \frac{x}{2}\right)^3 + \frac{24}{x^2} + \frac{24}{x} + 6 - \frac{32}{x^3} - \frac{16}{x^2} + \frac{16}{x^4} \\
& = \left(1 + \frac{x}{2}\right)^4 - \frac{8}{x} \left(1 + \frac{x}{2}\right)^3 + \frac{8}{x^2} + \frac{24}{x} + 6 - \frac{32}{x^3} + \frac{16}{x^4} \quad \dots(1)
\end{aligned}$$

Again, by using the binomial theorem to expand the above terms, we get

$$\begin{aligned}
\left(1 + \frac{x}{2}\right)^4 &= {}^4C_0(1)^4 + {}^4C_1(1)^3\left(\frac{x}{2}\right) + {}^4C_2(1)^2\left(\frac{x}{2}\right)^2 + {}^4C_3(1)\left(\frac{x}{2}\right)^3 + {}^4C_4\left(\frac{x}{2}\right)^4 \\
&= 1 + 4 \times \frac{x}{2} + 6 \times \frac{x^2}{4} + 4 \times \frac{x^3}{8} + \frac{x^4}{16} \\
&= 1 + 2x + \frac{3x^2}{2} + \frac{x^3}{2} + \frac{x^4}{16} \quad \dots(2) \\
\left(1 + \frac{x}{2}\right)^3 &= {}^3C_0(1)^3 + {}^3C_1(1)^2\left(\frac{x}{2}\right) + {}^3C_2(1)\left(\frac{x}{2}\right)^2 + {}^3C_3\left(\frac{x}{2}\right)^3 \\
&= 1 + \frac{3x}{2} + \frac{3x^2}{4} + \frac{x^3}{8} \quad \dots(3)
\end{aligned}$$

From equations 1, 2 and 3, we get

$$\begin{aligned}
&\left[\left(1 + \frac{x}{2}\right) - \frac{2}{x}\right]^4 \\
&= 1 + 2x + \frac{3x^2}{2} + \frac{x^3}{2} + \frac{x^4}{16} - \frac{8}{x}\left(1 + \frac{3x}{2} + \frac{3x^2}{4} + \frac{x^3}{8}\right) + \frac{8}{x^2} + \frac{24}{x} + 6 - \frac{32}{x^3} + \frac{16}{x^4} \\
&= 1 + 2x + \frac{3}{2}x^2 + \frac{x^3}{2} + \frac{x^4}{16} - \frac{8}{x} - 12 - 6x - x^2 + \frac{8}{x^2} + \frac{24}{x} + 6 - \frac{32}{x^3} + \frac{16}{x^4} \\
&= \frac{16}{x} + \frac{8}{x^2} - \frac{32}{x^3} + \frac{16}{x^4} - 4x + \frac{x^2}{2} + \frac{x^3}{2} + \frac{x^4}{16} - 5
\end{aligned}$$

Question 10

Find the expansion of $(3x^2 - 2ax + 3a^2)^3$ using binomial theorem.

Solution

We know that $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

Putting $a = 3x^2$ & $b = -a(2x-3a)$, we get

$$\begin{aligned}
&23 \\
&= (3x^2)^3 + 3(3x^2)^2(-a(2x-3a)) + 3(3x^2)(-a(2x-3a))^2 + (-a(2x-3a))^3 \\
&= 27x^6 - 27ax^4(2x-3a) + 9a^2x^2(2x-3a)^2 - a^3(2x-3a)^3 \\
&= 27x^6 - 54ax^5 + 81a^2x^4 + 9a^2x^2(4x^2 - 12ax + 9a^2) - a^3[(2x)^3 - (3a)^3 - 3(2x)^2(3a) + 3(2x)(3a)^2] \\
&= 27x^6 - 54ax^5 + 81a^2x^4 + 36a^2x^4 - 108a^3x^3 + 81a^4x^2 - 8a^3x^3 + 27a^6 + 36a^4x^2 - 54a^5x
\end{aligned}$$

$$= 27x^6 - 54ax^5 + 117a^2x^4 - 116a^3x^3 + 117a^4x^2 - 54a^5x + 27a^6$$

Thus, $(3x^2 - 2ax + 3a^2)^3$

$$= 27x^6 - 54ax^5 + 117a^2x^4 - 116a^3x^3 + 117a^4x^2 - 54a^5x + 27a^6$$

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