

Chapter 4: Principle of Mathematical Induction Exercise 4.1

NCERT · Class 11 · Maths

Questions

Question 8

$$1.2 + 2.2^2 + 3.2^2 + \dots + n.2^n = (n - 1) 2^{n+1} + 2$$

Solution

We can write the given statement as

$$P(n): 1.2 + 2.2^2 + 3.2^2 + \dots + n.2^n = (n - 1) 2^{n+1} + 2$$

If $n = 1$, we get

$$P(1): 1.2 = 2 = (1 - 1) 2^{1+1} + 2 = 0 + 2 = 2$$

Which is true.

Consider $P(k)$ to be true for some positive integer k .

$$1.2 + 2.2^2 + 3.2^2 + \dots + k.2^k = (k - 1) 2^{k+1} + 2 \dots (i)$$

Now, let us prove that $P(k + 1)$ is true.

Here,

$$\{1.2 + 2.2^2 + 3.2^3 + \dots + k.2^k\} + (k+1) \cdot 2^{k+1}$$

By using equation (i)

$$= (k-1)2^{k+1} + 2 + (k+1)2^{k+1}$$

Taking the common terms out

$$= 2^{k+1} \{(k-1) + (k+1)\} + 2$$

So we get

$$= 2^{k+1} \cdot 2k + 2$$

It can be written as

$$= k \cdot 2^{(k+1)+1} + 2$$

$$= \{(k+1) - 1\} 2^{(k+1)+1} + 2$$

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers, i.e., n.

Question 9

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$$

Solution

We can write the given statement as

$$P(n): \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$$

If $n = 1$ we get

$$P(1): \frac{1}{2} = 1 - \frac{1}{2^1} = \frac{1}{2}$$

Which is true.

Consider $P(k)$ be true for some positive integer k

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^k} = 1 - \frac{1}{2^k} \quad \dots (i)$$

Now let us prove that $P(k+1)$ is true.

Here

$$\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^k} \right) + \frac{1}{2^{k+1}}$$

By using equation (i)

$$= \left(1 - \frac{1}{2^k} \right) + \frac{1}{2^{k+1}}$$

We can write it as

$$= 1 - \frac{1}{2^k} + \frac{1}{2 \cdot 2^k}$$

Taking the common terms out

$$= 1 - \frac{1}{2^k} \left(1 - \frac{1}{2} \right)$$

So we get

$$= 1 - \frac{1}{2^k} \left(\frac{1}{2} \right)$$

It can be written as

$$= 1 - \frac{1}{2^{k+1}}$$

$P(k+1)$ is true whenever $P(k)$ is true.

Therefore, by the principle of mathematical induction, statement $P(n)$ is true for all natural numbers, i.e., n .

Question 10

$$\frac{1}{2.5} + \frac{1}{5.8} + \frac{1}{8.11} + \dots + \frac{1}{(3n-1)(3n+2)} = \frac{n}{(6n+4)}$$

Solution

We can write the given statement as

$$P(n): \frac{1}{2.5} + \frac{1}{5.8} + \frac{1}{8.11} + \dots + \frac{1}{(3n-1)(3n+2)} = \frac{n}{(6n+4)}$$

If $n = 1$ we get

$$P(1) = \frac{1}{2.5} = \frac{1}{10} = \frac{1}{6 \cdot 1 + 4} = \frac{1}{10}$$

Which is true.

Consider $P(k)$ be true for some positive integer k

$$\frac{1}{2.5} + \frac{1}{5.8} + \frac{1}{8.11} + \dots + \frac{1}{(3k-1)(3k+2)} = \frac{k}{6k+4} \quad \dots (i)$$

Now let us prove that $P(k+1)$ is true.

Here

$$\frac{1}{2.5} + \frac{1}{5.8} + \frac{1}{8.11} + \dots + \frac{1}{(3k-1)(3k+2)} + \frac{1}{\{3(k+1)-1\}\{3(k+1)+2\}}$$

By using equation (i)

$$= \frac{k}{6k+4} + \frac{1}{(3k+3-1)(3k+3+2)}$$

By simplification of terms

$$= \frac{k}{6k+4} + \frac{1}{(3k+2)(3k+5)}$$

Taking 2 as common

$$= \frac{k}{2(3k+2)} + \frac{1}{(3k+2)(3k+5)}$$

Taking the common terms out

$$= \frac{1}{(3k+2)} \left(\frac{k}{2} + \frac{1}{3k+5} \right)$$

Taking LCM

$$= \frac{1}{(3k+2)} \left(\frac{k(3k+5)+2}{2(3k+5)} \right)$$

By multiplication

$$= \frac{1}{(3k+2)} \left(\frac{3k^2+5k+2}{2(3k+5)} \right)$$

Separating the terms

$$= \frac{1}{(3k+2)} \left(\frac{(3k+2)(k+1)}{2(3k+5)} \right)$$

By further calculation

$$= \frac{(k+1)}{6k+10}$$

So we get

$$= \frac{(k+1)}{6(k+1)+4}$$

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers, i.e., n.

Question 11

$$\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \dots + \frac{1}{n(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$$

Solution

We can write the given statement as

$$P(n): \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{3 \cdot 4 \cdot 5} + \dots + \frac{1}{n(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$$

If $n = 1$ we get

$$P(1): \frac{1}{1 \cdot 2 \cdot 3} = \frac{1 \cdot (1+3)}{4(1+1)(1+2)} = \frac{1 \cdot 4}{4 \cdot 2 \cdot 3} = \frac{1}{1 \cdot 2 \cdot 3}$$

Which is true.

Consider $P(k)$ be true for some positive integer k

$$\frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{3 \cdot 4 \cdot 5} + \dots + \frac{1}{k(k+1)(k+2)} = \frac{k(k+3)}{4(k+1)(k+2)} \quad \dots (i)$$

Now let us prove that $P(k+1)$ is true.

Here

$$\left[\frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{3 \cdot 4 \cdot 5} + \dots + \frac{1}{k(k+1)(k+2)} \right] + \frac{1}{(k+1)(k+2)(k+3)}$$

By using equation (i)

$$= \frac{k(k+3)}{4(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)}$$

Taking out the common terms

$$= \frac{1}{(k+1)(k+2)} \left\{ \frac{k(k+3)}{4} + \frac{1}{k+3} \right\}$$

Taking LCM

$$= \frac{1}{(k+1)(k+2)} \left\{ \frac{k(k+3)^2 + 4}{4(k+3)} \right\}$$

Expanding using formula

$$= \frac{1}{(k+1)(k+2)} \left\{ \frac{k(k^2 + 6k + 9) + 4}{4(k+3)} \right\}$$

By further calculation

$$= \frac{1}{(k+1)(k+2)} \left\{ \frac{k^3 + 6k^2 + 9k + 4}{4(k+3)} \right\}$$

We can write it as

$$= \frac{1}{(k+1)(k+2)} \left\{ \frac{k^3 + 2k^2 + k + 4k^2 + 8k + 4}{4(k+3)} \right\}$$

Taking the common terms

$$= \frac{1}{(k+1)(k+2)} \left\{ \frac{k(k^2 + 2k + 1) + 4(k^2 + 2k + 1)}{4(k+3)} \right\}$$

We get

$$= \frac{1}{(k+1)(k+2)} \left\{ \frac{k(k+1)^2 + 4(k+1)^2}{4(k+3)} \right\}$$

Here

$$\begin{aligned} &= \frac{(k+1)^2 (k+4)}{4(k+1)(k+2)(k+3)} \\ &= \frac{(k+1)\{(k+1)+3\}}{4\{(k+1)+1\}\{(k+1)+2\}} \end{aligned}$$

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers, i.e., n.

Question 12

$$a + ar + ar^2 + \dots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1}$$

Solution

We can write the given statement as

$$P(n): a + ar + ar^2 + \dots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1}$$

If $n = 1$ we get

$$P(1): a = \frac{a(r^1 - 1)}{(r - 1)} = a$$

Which is true.

Consider $P(k)$ be true for some positive integer k

$$a + ar + ar^2 + \dots + ar^{k-1} = \frac{a(r^k - 1)}{r - 1} \quad \dots (i)$$

Now let us prove that $P(k + 1)$ is true.

Here

$$\{a + ar + ar^2 + \dots + ar^{k-1}\} + ar^{(k+1)-1}$$

By using equation (i)

$$= \frac{a(r^k - 1)}{r - 1} + ar^k$$

Taking LCM

$$= \frac{a(r^k - 1) + ar^k(r - 1)}{r - 1}$$

Multiplying the terms

$$= \frac{a(r^k - 1) + ar^{k+1} - ar^k}{r - 1}$$

So we get

$$= \frac{ar^k - a + ar^{k+1} - ar^k}{r - 1}$$

By further simplification

$$= \frac{ar^{k+1} - a}{r - 1}$$

Taking the common terms out

$$= \frac{a(r^{k+1} - 1)}{r - 1}$$

$P(k + 1)$ is true whenever $P(k)$ is true.

Therefore, by the principle of mathematical induction, statement $P(n)$ is true for all natural numbers, i.e., n .

Question 13

$$\left(1 + \frac{3}{1}\right)\left(1 + \frac{5}{4}\right)\left(1 + \frac{7}{9}\right) \dots \left(1 + \frac{(2n+1)}{n^2}\right) = (n+1)^2$$

Solution

We can write the given statement as

$$P(n) : \left(1 + \frac{3}{1}\right)\left(1 + \frac{5}{4}\right)\left(1 + \frac{7}{9}\right) \dots \left(1 + \frac{(2n+1)}{n^2}\right) = (n+1)^2$$

If $n = 1$ we get

$$P(1) : \left(1 + \frac{3}{1}\right) = 4 = (1+1)^2 = 2^2 = 4,$$

Which is true.

Consider $P(k)$ be true for some positive integer k

$$\left(1 + \frac{3}{1}\right)\left(1 + \frac{5}{4}\right)\left(1 + \frac{7}{9}\right) \dots \left(1 + \frac{(2k+1)}{k^2}\right) = (k+1)^2 \quad \dots (1)$$

Now let us prove that $P(k + 1)$ is true.

Here

$$\left[\left(1 + \frac{3}{1}\right) \left(1 + \frac{5}{4}\right) \left(1 + \frac{7}{9}\right) \dots \left(1 + \frac{(2k+1)}{k^2}\right) \right] \left[1 + \frac{\{2(k+1)+1\}}{(k+1)^2} \right]$$

By using equation (i)

$$= (k+1)^2 \left(1 + \frac{2(k+1)+1}{(k+1)^2} \right)$$

Taking LCM

$$= (k+1)^2 \left[\frac{(k+1)^2 + 2(k+1) + 1}{(k+1)^2} \right]$$

So we get

$$= (k+1)^2 + 2(k+1) + 1$$

By further simplification

$$= \{(k+1)+1\}^2$$

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers, i.e., n.

Question 14

$$\left(1 + \frac{1}{1}\right) \left(1 + \frac{1}{2}\right) \left(1 + \frac{1}{3}\right) \dots \left(1 + \frac{1}{n}\right) = (n+1)$$

Solution

We can write the given statement as

$$P(n): \left(1 + \frac{1}{1}\right) \left(1 + \frac{1}{2}\right) \left(1 + \frac{1}{3}\right) \dots \left(1 + \frac{1}{n}\right) = (n+1)$$

If $n = 1$ we get

$$P(1): \left(1 + \frac{1}{1}\right) = 2 = (1+1)$$

Which is true.

Consider $P(k)$ be true for some positive integer k

$$P(k): \left(1 + \frac{1}{1}\right) \left(1 + \frac{1}{2}\right) \left(1 + \frac{1}{3}\right) \dots \left(1 + \frac{1}{k}\right) = (k+1) \quad \dots (1)$$

Now let us prove that $P(k+1)$ is true.

Here

$$\left[\left(1 + \frac{1}{1}\right) \left(1 + \frac{1}{2}\right) \left(1 + \frac{1}{3}\right) \dots \left(1 + \frac{1}{k}\right) \right] \left(1 + \frac{1}{k+1}\right)$$

By using equation (i)

$$= (k+1) \left(1 + \frac{1}{k+1}\right)$$

Taking LCM

$$= (k+1) \left(\frac{(k+1)+1}{(k+1)} \right)$$

By further simplification,

$$= (k+1) + 1$$

$P(k+1)$ is true whenever $P(k)$ is true.

Therefore, by the principle of mathematical induction, statement $P(n)$ is true for all natural numbers, i.e., n .

Question 15

$$1^2 + 3^2 + 5^2 + \dots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

Solution

We can write the given statement as

$$P(n) = 1^2 + 3^2 + 5^2 + \dots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

If $n = 1$ we get

$$P(1) = 1^2 = 1 = \frac{1(2 \cdot 1 - 1)(2 \cdot 1 + 1)}{3} = \frac{1 \cdot 1 \cdot 3}{3} = 1,$$

Which is true.

Consider $P(k)$ be true for some positive integer k

$$P(k) = 1^2 + 3^2 + 5^2 + \dots + (2k-1)^2 = \frac{k(2k-1)(2k+1)}{3} \quad \dots (1)$$

Now let us prove that $P(k+1)$ is true.

Here

$$\{1^2 + 3^2 + 5^2 + \dots + (2k-1)^2\} + \{2(k+1)-1\}^2$$

By using equation (i)

$$= \frac{k(2k-1)(2k+1)}{3} + (2k+2-1)^2$$

So we get

$$= \frac{k(2k-1)(2k+1)}{3} + (2k+1)^2$$

Taking LCM

$$= \frac{k(2k-1)(2k+1) + 3(2k+1)^2}{3}$$

Taking the common terms out

$$= \frac{(2k+1)\{k(2k-1) + 3(2k+1)\}}{3}$$

By further simplification

$$= \frac{(2k+1)\{2k^2 - k + 6k + 3\}}{3}$$

So we get

$$= \frac{(2k+1)\{2k^2 + 5k + 3\}}{3}$$

We can write it as

$$= \frac{(2k+1)\{2k^2 + 2k + 3k + 3\}}{3}$$

Splitting the terms

$$= \frac{(2k+1)\{2k(k+1) + 3(k+1)\}}{3}$$

We get

$$= \frac{(2k+1)(k+1)(2k+3)}{3}$$

$$= \frac{(k+1)\{2(k+1)-1\}\{2(k+1)+1\}}{3}$$

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers, i.e., n.

Question 16

$$\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \dots + \frac{1}{(3n-2)(3n+1)} = \frac{n}{(3n+1)}$$

Solution

We can write the given statement as

$$P(n): \frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \dots + \frac{1}{(3n-2)(3n+1)} = \frac{n}{(3n+1)}$$

If $n = 1$ we get

$$P(1) = \frac{1}{1.4} = \frac{1}{3.1+1} = \frac{1}{4} = \frac{1}{1.4}$$

Which is true.

Consider $P(k)$ be true for some positive integer k

$$P(k) = \frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \dots + \frac{1}{(3k-2)(3k+1)} = \frac{k}{3k+1} \quad \dots (1)$$

Now let us prove that $P(k+1)$ is true.

Here

$$\left\{ \frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \dots + \frac{1}{(3k-2)(3k+1)} \right\} + \frac{1}{\{3(k+1)-2\}\{3(k+1)+1\}}$$

By using equation (i)

$$= \frac{k}{3k+1} + \frac{1}{(3k+1)(3k+4)}$$

So we get

$$= \frac{1}{(3k+1)} \left\{ k + \frac{1}{(3k+4)} \right\}$$

Taking LCM

$$= \frac{1}{(3k+1)} \left\{ \frac{k(3k+4)+1}{(3k+4)} \right\}$$

Multiplying the terms

$$= \frac{1}{(3k+1)} \left\{ \frac{3k^2 + 4k + 1}{(3k+4)} \right\}$$

It can be written as

$$= \frac{1}{(3k+1)} \left\{ \frac{3k^2 + 3k + k + 1}{(3k+4)} \right\}$$

Separating the terms

$$= \frac{(3k+1)(k+1)}{(3k+1)(3k+4)}$$

By further calculation

$$= \frac{(k+1)}{3(k+1)+1}$$

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers, i.e., n.

Question 17

$$\frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9} + \dots + \frac{1}{(2n+1)(2n+3)} = \frac{n}{3(2n+3)}$$

Solution

We can write the given statement as

$$P(n): \frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9} + \dots + \frac{1}{(2n+1)(2n+3)} = \frac{n}{3(2n+3)}$$

If $n = 1$ we get

$$P(1): \frac{1}{3.5} = \frac{1}{3(2.1+3)} = \frac{1}{3.5}$$

Which is true.

Consider $P(k)$ be true for some positive integer k

$$P(k): \frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9} + \dots + \frac{1}{(2k+1)(2k+3)} = \frac{k}{3(2k+3)} \quad \dots (1)$$

Now let us prove that $P(k+1)$ is true.

Here

$$\left[\frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9} + \dots + \frac{1}{(2k+1)(2k+3)} \right] + \frac{1}{\{2(k+1)+1\}\{2(k+1)+3\}}$$

By using equation (i)

$$= \frac{k}{3(2k+3)} + \frac{1}{(2k+3)(2k+5)}$$

So we get

$$= \frac{1}{(2k+3)} \left[\frac{k}{3} + \frac{1}{(2k+5)} \right]$$

Taking LCM

$$= \frac{1}{(2k+3)} \left[\frac{k(2k+5)+3}{3(2k+5)} \right]$$

Multiplying the terms

$$= \frac{1}{(2k+3)} \left[\frac{2k^2 + 5k + 3}{3(2k+5)} \right]$$

It can be written as

$$= \frac{1}{(2k+3)} \left[\frac{2k^2 + 2k + 3k + 3}{3(2k+5)} \right]$$

Separating the terms

$$= \frac{1}{(2k+3)} \left[\frac{2k(k+1)+3(k+1)}{3(2k+5)} \right]$$

By further calculation

$$= \frac{(k+1)(2k+3)}{3(2k+3)(2k+5)}$$
$$= \frac{(k+1)}{3\{2(k+1)+3\}}$$

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers, i.e., n.

Question 18

$$1+2+3+\dots+n < \frac{1}{8}(2n+1)^2$$

Solution

We can write the given statement as

$$P(n): 1 + 2 + 3 + \dots + n < \frac{1}{8}(2n+1)^2$$

If $n = 1$ we get

$$1 < \frac{1}{8}(2 \cdot 1 + 1)^2 = \frac{9}{8}$$

Which is true.

Consider $P(k)$ be true for some positive integer k

$$1 + 2 + \dots + k < \frac{1}{8}(2k+1)^2 \quad \dots (1)$$

Now let us prove that $P(k+1)$ is true.

Here

$$(1 + 2 + \dots + k) + (k+1) < \frac{1}{8}(2k+1)^2 + (k+1)$$

By using equation (i)

$$< \frac{1}{8}\{(2k+1)^2 + 8(k+1)\}$$

Expanding terms using formula

$$< \frac{1}{8}\{4k^2 + 4k + 1 + 8k + 8\}$$

By further calculation

$$< \frac{1}{8}\{4k^2 + 12k + 9\}$$

So we get

$$< \frac{1}{8}(2k+3)^2$$

$$< \frac{1}{8}\{2(k+1)+1\}^2$$

$$(1 + 2 + 3 + \dots + k) + (k+1) < \frac{1}{8}(2k+1)^2 + (k+1)$$

$P(k+1)$ is true whenever $P(k)$ is true.

Therefore, by the principle of mathematical induction, statement $P(n)$ is true for all natural numbers, i.e., n .

Question 19

$n(n + 1)(n + 5)$ is a multiple of 3.

Solution

We can write the given statement as

$P(n)$: $n(n + 1)(n + 5)$, which is a multiple of 3.

If $n = 1$, we get

$1(1 + 1)(1 + 5) = 12$, which is a multiple of 3.

Which is true.

Consider $P(k)$ to be true for some positive integer k .

$k(k + 1)(k + 5)$ is a multiple of 3

$k(k + 1)(k + 5) = 3m$, where $m \in \mathbb{N} \dots\dots (1)$

Now, let us prove that $P(k + 1)$ is true.

Here,

$(k + 1)\{(k + 1) + 1\}\{(k + 1) + 5\}$

We can write it as

$= (k + 1)(k + 2)\{(k + 5) + 1\}$

By multiplying the terms,

$= (k + 1)(k + 2)(k + 5) + (k + 1)(k + 2)$

So, we get

$= \{k(k + 1)(k + 5) + 2(k + 1)(k + 5)\} + (k + 1)(k + 2)$

Substituting equation (1),

$= 3m + (k + 1)\{2(k + 5) + (k + 2)\}$

By multiplication,

$= 3m + (k + 1)\{2k + 10 + k + 2\}$

On further calculation,

$$= 3m + (k + 1) (3k + 12)$$

Taking 3 as common,

$$= 3m + 3 (k + 1) (k + 4)$$

We get

$$= 3 \{m + (k + 1) (k + 4)\}$$

$= 3 \times q$ where $q = \{m + (k + 1) (k + 4)\}$ is some natural number

$(k + 1) \{(k + 1) + 1\} \{(k + 1) + 5\}$ is a multiple of 3

$P (k + 1)$ is true whenever $P (k)$ is true.

Therefore, by the principle of mathematical induction, statement $P (n)$ is true for all natural numbers, i.e., n .

Question 20

$10^{2n-1} + 1$ is divisible by 11.

Solution

We can write the given statement as

$P (n)$: $10^{2n-1} + 1$ is divisible by 11.

If $n = 1$, we get

$$P (1) = 10^{2 \cdot 1 - 1} + 1 = 11, \text{ which is divisible by 11.}$$

Which is true.

Consider $P (k)$ to be true for some positive integer k .

$10^{2k-1} + 1$ is divisible by 11

$$10^{2k-1} + 1 = 11m, \text{ where } m \in \mathbb{N} \dots\dots (1)$$

Now, let us prove that $P (k + 1)$ is true.

Here,

$$10^{2(k+1)-1} + 1$$

We can write it as

$$= 10^{2k+2-1} + 1$$

$$= 10^{2k+1} + 1$$

By addition and subtraction of 1,

$$= 10^2 (10^{2k-1} + 1 - 1) + 1$$

We get

$$= 10^2 (10^{2k-1} + 1) - 10^2 + 1$$

Using equation 1, we get

$$= 10^2 \cdot 11m - 100 + 1$$

$$= 100 \times 11m - 99$$

Taking out the common terms,

$$= 11 (100m - 9)$$

$$= 11 r, \text{ where } r = (100m - 9) \text{ is some natural number}$$

$$10^{2(k+1)-1} + 1 \text{ is divisible by } 11$$

$P(k+1)$ is true whenever $P(k)$ is true.

Therefore, by the principle of mathematical induction, statement $P(n)$ is true for all natural numbers, i.e., n .

Question 21

$x^{2n} - y^{2n}$ is divisible by $x + y$.

Solution

We can write the given statement as

$$P(n): x^{2n} - y^{2n} \text{ is divisible by } x + y$$

If $n = 1$, we get

$$P(1) = x^{2 \times 1} - y^{2 \times 1} = x^2 - y^2 = (x + y)(x - y), \text{ which is divisible by } (x + y)$$

Which is true.

Consider $P(k)$ to be true for some positive integer k .

$x^{2k} - y^{2k}$ is divisible by $x + y$

$$x^{2k} - y^{2k} = m(x + y), \text{ where } m \in \mathbb{N} \dots\dots (1)$$

Now, let us prove that $P(k + 1)$ is true.

Here,

$$x^{2(k+1)} - y^{2(k+1)}$$

We can write it as

$$= x^{2k} \cdot x^2 - y^{2k} \cdot y^2$$

By adding and subtracting y^{2k} , we get

$$= x^2 (x^{2k} - y^{2k} + y^{2k}) - y^{2k} \cdot y^2$$

From equation (1), we get

$$= x^2 \{m(x + y) + y^{2k}\} - y^{2k} \cdot y^2$$

By multiplying the terms,

$$= m(x + y)x^2 + y^{2k} \cdot x^2 - y^{2k} \cdot y^2$$

Taking out the common terms,

$$= m(x + y)x^2 + y^{2k}(x^2 - y^2)$$

Expanding using formula,

$$= m(x + y)x^2 + y^{2k}(x + y)(x - y)$$

So, we get

$$= (x + y)\{mx^2 + y^{2k}(x - y)\}, \text{ which is a factor of } (x + y)$$

$P(k + 1)$ is true whenever $P(k)$ is true.

Therefore, by the principle of mathematical induction, statement $P(n)$ is true for all natural numbers, i.e., n .

Question 22

$3^{2n+2} - 8n - 9$ is divisible by 8.

Solution

We can write the given statement as

$P(n)$: $3^{2n+2} - 8n - 9$ is divisible by 8.

If $n = 1$, we get

$$P(1) = 3^{2 \times 1 + 2} - 8 \times 1 - 9 = 64, \text{ which is divisible by 8.}$$

Which is true.

Consider $P(k)$ to be true for some positive integer k .

$3^{2k+2} - 8k - 9$ is divisible by 8

$$3^{2k+2} - 8k - 9 = 8m, \text{ where } m \in \mathbb{N} \dots\dots (1)$$

Now, let us prove that $P(k+1)$ is true.

Here,

$$3^{2(k+1)+2} - 8(k+1) - 9$$

We can write it as

$$= 3^{2k+2} \cdot 3^2 - 8k - 8 - 9$$

By adding and subtracting $8k$ and 9 , we get

$$= 3^2 (3^{2k+2} - 8k - 9 + 8k + 9) - 8k - 17$$

On further simplification,

$$= 3^2 (3^{2k+2} - 8k - 9) + 3^2 (8k + 9) - 8k - 17$$

From equation (1), we get

$$= 9 \cdot 8m + 9 (8k + 9) - 8k - 17$$

By multiplying the terms,

$$= 9 \cdot 8m + 72k + 81 - 8k - 17$$

So, we get

$$= 9 \cdot 8m + 64k + 64$$

By taking out the common terms,

$$= 8 (9m + 8k + 8)$$

$$= 8r, \text{ where } r = (9m + 8k + 8) \text{ is a natural number}$$

So, $3^{2(k+1)+2} - 8(k+1) - 9$ is divisible by 8.

$P(k+1)$ is true whenever $P(k)$ is true.

Therefore, by the principle of mathematical induction, statement $P(n)$ is true for all natural numbers, i.e., n .

Question 23

$41^n - 14^n$ is a multiple of 27.

Solution

We can write the given statement as

$P(n): 41^n - 14^n$ is a multiple of 27.

If $n = 1$, we get

$$P(1) = 41^1 - 14^1 = 27, \text{ which is a multiple by 27.}$$

Which is true.

Consider $P(k)$ to be true for some positive integer k .

$41^k - 14^k$ is a multiple of 27

$$41^k - 14^k = 27m, \text{ where } m \in \mathbb{N} \dots\dots (1)$$

Now, let us prove that $P(k+1)$ is true.

Here,

$$41^{k+1} - 14^{k+1}$$

We can write it as

$$= 41^k \cdot 41 - 14^k \cdot 14$$

By adding and subtracting 14^k , we get

$$= 41 (41^k - 14^k + 14^k) - 14^k \cdot 14$$

On further simplification,

$$= 41 (41^k - 14^k) + 41 \cdot 14^k - 14^k \cdot 14$$

From equation (1), we get

$$= 41 \cdot 27m + 14^k (41 - 14)$$

By multiplying the terms,

$$= 41 \cdot 27m + 27 \cdot 14^k$$

By taking out the common terms,

$$= 27 (41m - 14^k)$$

$$= 27r, \text{ where } r = (41m - 14^k) \text{ is a natural number}$$

So, $41^{k+1} - 14^{k+1}$ is a multiple of 27.

$P(k+1)$ is true whenever $P(k)$ is true.

Therefore, by the principle of mathematical induction, statement $P(n)$ is true for all natural numbers, i.e., n .

Question 24

$$(2n+7) < (n+3)^2$$

Solution

We can write the given statement as

$$P(n): (2n+7) < (n+3)^2$$

If $n = 1$, we get

$$2 \cdot 1 + 7 = 9 < (1+3)^2 = 16$$

Which is true.

Consider $P(k)$ to be true for some positive integer k .

$$(2k + 7) < (k + 3)^2 \dots (1)$$

Now, let us prove that $P(k + 1)$ is true.

Here,

$$\{2(k + 1) + 7\} = (2k + 7) + 2$$

We can write it as

$$= \{2(k + 1) + 7\}$$

From equation (1), we get

$$(2k + 7) + 2 < (k + 3)^2 + 2$$

By expanding the terms,

$$2(k + 1) + 7 < k^2 + 6k + 9 + 2$$

On further calculation,

$$2(k + 1) + 7 < k^2 + 6k + 11$$

$$\text{Here, } k^2 + 6k + 11 < k^2 + 8k + 16$$

We can write it as

$$2(k + 1) + 7 < (k + 4)^2$$

$$2(k + 1) + 7 < \{(k + 1) + 3\}^2$$

$P(k + 1)$ is true whenever $P(k)$ is true.

Therefore, by the principle of mathematical induction, statement $P(n)$ is true for all natural numbers, i.e., n .