

Chapter 12: Limits and Derivatives

NCERT · Class 11 · Maths

Exercise 12.1

Question 1

Evaluate the Given limit. $\lim_{x \rightarrow 3} x + 3$

Solution

Ans. Given,

$$\lim_{x \rightarrow 3} x + 3$$

$$= 3 + 3$$

$$= 6$$

Question 2

Evaluate the Given limit. $\lim_{x \rightarrow \pi} \left(x - \frac{22}{7} \right)$

Solution

Ans. Given,

$$\lim_{x \rightarrow \pi} \left(x - \frac{22}{7} \right)$$

$$=$$

$$\left(\pi - \frac{22}{7} \right)$$

Question 3

Evaluate the Given limit. $\lim_{r \rightarrow 1} \pi r^2$

Solution

Ans. Given,

$$\lim_{r \rightarrow 1} \pi r^2$$

=

$$\pi(1^2)$$

=

$$\pi$$

Question 4

Evaluate the Given limit.

$$\lim_{x \rightarrow 1} \frac{4x + 3}{x - 2}$$

Solution

Ans. Given,

$$\lim_{x \rightarrow 1} \frac{4x + 3}{x - 2}$$

$$= \frac{4(4)+3}{4-2}$$

$$= \frac{16+3}{2}$$

$$= \frac{19}{2}$$

Question 5

Evaluate the Given limit.

$$\lim_{x \rightarrow 1} \frac{x^{10} + x^5 + 1}{x - 1}$$

Solution

Ans. Given,

$$\begin{aligned} & \lim_{x \rightarrow 1} \frac{x^{10} + x^5 + 1}{x - 1} \\ &= \frac{(-1)^{10} + (-1)^5 + 1}{-1 - 1} \\ &= \frac{1 - 1 + 1}{-2} \\ &= -\frac{1}{2} \end{aligned}$$

Question 6

Evaluate the Given limit.

$$\lim_{x \rightarrow 0} \frac{(x + 1)^5 - 1}{x}$$

Solution

Ans. Given,

$$\lim_{x \rightarrow 0} \frac{(x + 1)^5 - 1}{x}$$

Put $x + 1 = y$ So,

$y \rightarrow 1$ as $x \rightarrow 0$

Then,

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{(x + 1)^5 - 1}{x} \\ &= \\ & \lim_{x \rightarrow 1} \frac{(y)^5 - 1}{y - 1} \end{aligned}$$

Using

$$\left[\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1} \right]$$

$$= 5.1^{5-1}$$

$$= 5$$

Question 7

Evaluate the Given limit.

$$\lim_{x \rightarrow 2} \frac{3x^2 - x - 10}{x^2 - 4}$$

Solution

Ans. At $x = 2$

The value of the given rational function takes the form

$$\frac{0}{0}$$

=

$$\lim_{x \rightarrow 2} \frac{3x^2 - x - 10}{x^2 - 4}$$

=

$$\lim_{x \rightarrow 2} \frac{(x - 2)(3x + 5)}{(x - 2)(x + 2)}$$

=

$$\lim_{x \rightarrow 2} \frac{3x + 5}{x + 2}$$

=

$$\frac{3(2) + 5}{2 + 2}$$

$$= \frac{11}{4}$$

Question 8

Evaluate the Given limit.

$$\lim_{x \rightarrow 3} \frac{x^4 - 81}{2x^2 - 5x - 3}$$

Solution

Ans. At $x = 2$

The value of the given rational function takes the form

$$\frac{0}{0}$$

=

$$\lim_{x \rightarrow 3} \frac{x^4 - 81}{2x^2 - 5x - 3}$$

=

$$\lim_{x \rightarrow 3} \frac{(x - 3)(x + 3)(x^2 + 9)}{(x - 3)(2x + 1)}$$

=

$$\lim_{x \rightarrow 3} \frac{(x + 3)(x^2 + 9)}{(2x + 1)}$$

=

$$\frac{(3 + 3)(3^2 + 9)}{(2(3) + 1)}$$

=

$$\frac{6 \times 18}{7}$$

=

$$\frac{108}{7}$$

Question 9

Evaluate the Given limit.

$$\lim_{x \rightarrow 0} \frac{ax + b}{cx + 1}$$

Solution

Ans. Given,

$$\lim_{x \rightarrow 0} \frac{ax + b}{cx + 1}$$

=

$$\frac{a(0) + b}{c(0) + 1}$$

=

$$b$$

Question 10

Evaluate the Given limit.

$$\lim_{z \rightarrow 1} \frac{z^{\frac{1}{3}} - 1}{z^{\frac{1}{6}} - 1}$$

SolutionAns. At $z = 1$

The value of the given rational function takes the form

$$\frac{0}{0}$$

Put $z^{\frac{1}{6}} = x$ So,

$z \rightarrow 1$ as $x \rightarrow 1$

Then,

$$\lim_{z \rightarrow 1} \frac{z^{\frac{1}{3}} - 1}{z^{\frac{1}{6}} - 1}$$

=

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1}$$

Using

$$\left[\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1} \right]$$

$$= 2 \cdot 1^{2-1}$$

$$= 2$$

Question 11

Evaluate the Given limit.

$$\lim_{x \rightarrow 1} \frac{ax^2 + bx + c}{cx^2 + bx + a}$$

,

$$a + b + c \neq 0$$

Solution

Ans. Given,

$$\lim_{x \rightarrow 1} \frac{ax^2 + bx + c}{cx^2 + bx + a}$$

=

$$\frac{a(1)^2 + b(1) + c}{c(1)^2 + b(1) + a}$$

=

$$\frac{a + b + c}{c + b + a}$$

=1

$$(a + b + c \neq 0)$$

Question 12**Evaluate the Given limit.**

$$\lim_{x \rightarrow -2} \frac{\frac{1}{x} + \frac{1}{2}}{x + 2}$$

Solution

Ans. Given,

$$\lim_{x \rightarrow -2} \frac{\frac{1}{x} + \frac{1}{2}}{x + 2}$$

At $x = -2$

The value of the given rational function takes the form

$$\frac{0}{0}$$

$$\lim_{x \rightarrow -2} \frac{\frac{1}{x} + \frac{1}{2}}{x + 2}$$

=

$$\lim_{x \rightarrow -2} \frac{\left(\frac{2+x}{2x}\right)}{x+2}$$

=

$$\lim_{x \rightarrow -2} \frac{1}{2x}$$

=

$$\frac{1}{2(-2)}$$

=

$$-\frac{1}{4}$$

Question 13

Evaluate the Given limit.

$$\lim_{x \rightarrow 0} \frac{\sin ax}{bx}$$

Solution

Ans. Given,

$$\lim_{x \rightarrow 0} \frac{\sin ax}{bx}$$

At $x = 0$

The value of the given rational function takes the form

$$\frac{0}{0}$$

$$\lim_{x \rightarrow 0} \frac{\sin ax}{bx} = \lim_{x \rightarrow 0} \frac{\sin ax}{bx} \times \frac{ax}{ax}$$

=

$$\lim_{x \rightarrow 0} \left(\frac{\sin ax}{bx} \right) \times \frac{a}{b}$$

=

$$\frac{a}{b} \lim_{ax \rightarrow 0} \left(\frac{\sin ax}{bx} \right)$$

$$x \rightarrow 0 \Rightarrow ax \rightarrow 0$$

=

$$\frac{a}{b} \times 1$$

$$\left[\lim_{x \rightarrow 0} \left(\frac{\sin y}{y} \right) \right]$$

=

$$\frac{a}{b}$$

Question 14

Evaluate the Given limit.

$$\lim_{x \rightarrow 0} \frac{\sin ax}{\sin bx}, a, b \neq 0$$

Solution

Ans. Given,

$$\lim_{x \rightarrow 0} \frac{\sin ax}{\sin bx}, a, b \neq 0$$

At $x = 0$

The value of the given rational function takes the form

$$\frac{0}{0}$$

$$\lim_{x \rightarrow 0} \frac{\sin ax}{\sin bx} = \lim_{x \rightarrow 0} \frac{\left(\frac{\sin ax}{ax}\right) \times ax}{\left(\frac{\sin bx}{bx}\right) \times bx}$$

=

$$\frac{a}{b} \times \frac{\lim_{ax \rightarrow 0} \left(\frac{\sin ax}{ax}\right)}{\lim_{bx \rightarrow 0} \left(\frac{\sin bx}{bx}\right)}$$

$$\left[\begin{array}{l} x \rightarrow 0 \Rightarrow ax \rightarrow 0 \\ x \rightarrow 0 \Rightarrow bx \rightarrow 0 \end{array} \right]$$

=

$$\frac{a}{b} \times \frac{1}{1}$$

$$\left[\lim_{x \rightarrow 0} \left(\frac{\sin y}{y}\right) = 1 \right]$$

=

$$\frac{a}{b}$$

Question 15

Evaluate the Given limit.

$$\lim_{x \rightarrow \pi} \frac{\sin(\pi - x)}{\pi(\pi - x)}$$

Solution

Ans. Given,

$$\lim_{x \rightarrow \pi} \frac{\sin(\pi - x)}{\pi(\pi - x)}$$

$$[x \rightarrow \pi \Rightarrow (\pi - x) \rightarrow 0]$$

$$\begin{aligned}
 \lim_{x \rightarrow \pi} \frac{\sin(\pi - x)}{\pi(\pi - x)} &= \frac{1}{\pi} \lim_{(\pi - x) \rightarrow 0} \frac{\sin(\pi - x)}{(\pi - x)} \\
 &= \frac{1}{\pi} \times 1 \quad \left[\lim_{x \rightarrow 0} \left(\frac{\sin y}{y} \right) = 1 \right] \\
 &= \frac{1}{\pi}
 \end{aligned}$$

Question 16

Evaluate the Given limit.

$$\lim_{x \rightarrow 0} \frac{\cos x}{\pi - x}$$

Solution

Ans. Given,

$$\lim_{x \rightarrow 0} \frac{\cos x}{\pi - x}$$

=

$$\frac{\cos 0}{\pi - 0}$$

=

$$\frac{1}{\pi}$$

Question 17

Evaluate the Given limit.

$$\lim_{x \rightarrow 0} \frac{\cos 2x - 1}{\cos x - 1}$$

Solution

Ans. Given,

$$\lim_{x \rightarrow 0} \frac{\cos 2x - 1}{\cos x - 1}$$

At $x = 0$

The value of the given rational function takes the form

$$\frac{0}{0}$$

$$\lim_{x \rightarrow 0} \frac{\cos 2x - 1}{\cos x - 1} = \lim_{x \rightarrow 0} \frac{1 - 2\sin^2 x - 1}{1 - 2\sin^2 \frac{x}{2} - 1}$$

=

$$\lim_{x \rightarrow 0} \frac{\sin^2 x}{\sin^2 \frac{x}{2}} = \lim_{x \rightarrow 0} \frac{\left(\frac{\sin^2 x}{x^2}\right) \times x^2}{\left(\frac{\sin^2 \frac{x}{2}}{\left(\frac{x}{2}\right)^2}\right) \times \frac{x^2}{4}}$$

=

$$4 \frac{\lim_{x \rightarrow 0} \left(\frac{\sin^2 x}{x^2}\right)}{\lim_{x \rightarrow 0} \left(\frac{\sin^2 \frac{x}{2}}{\left(\frac{x}{2}\right)^2}\right)}$$

=

$$4 \frac{\left(\lim_{x \rightarrow 0} \frac{\sin^2 x}{x^2}\right)^2}{\left(\lim_{x \rightarrow 0} \frac{\sin^2 \frac{x}{2}}{\left(\frac{x}{2}\right)^2}\right)^2}$$

$$\left[x \rightarrow 0 \Rightarrow \frac{x}{2} \rightarrow 0\right]$$

=

$$4 \frac{1^2}{1^2}$$

$$\left[\lim_{x \rightarrow 0} \left(\frac{\sin y}{y}\right) = 1\right]$$

=

4

Question 18

Evaluate the Given limit.

$$\lim_{x \rightarrow 0} \frac{ax + x \cos x}{b \sin x}$$

Solution

Ans. Given,

$$\lim_{x \rightarrow 0} \frac{ax + x \cos x}{b \sin x}$$

At $x = 0$

The value of the given rational function takes the form

$$\frac{0}{0}$$

$$\lim_{x \rightarrow 0} \frac{ax + x \cos x}{b \sin x} = \frac{1}{b} \lim_{x \rightarrow 0} \frac{x(a + \cos x)}{\sin x}$$

$$\frac{1}{b} \lim_{x \rightarrow 0} \left(\frac{x}{\sin x} \right) \times \lim_{x \rightarrow 0} (a + \cos x)$$

=

$$\frac{1}{b} \lim_{x \rightarrow 0} \left(\frac{1}{\lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right)} \right) \times \lim_{x \rightarrow 0} (a + \cos x)$$

=

$$\frac{1}{b} \times (a + \cos 0)$$

$$\left[\lim_{y \rightarrow 0} \left(\frac{\sin x}{x} \right) = 1 \right]$$

=

$$\frac{a+1}{b}$$

Question 19

Evaluate the Given limit.

$$\lim_{x \rightarrow 0} x \sec x$$

Solution

Ans. Given,

$$\lim_{x \rightarrow 0} x \sec x$$

=

$$\lim_{x \rightarrow 0} \frac{x}{\cos x}$$

=

$$\frac{0}{\cos 0}$$

=

$$\frac{0}{1}$$

=

$$0$$

Question 20**Evaluate the Given limit.**

$$\lim_{x \rightarrow 0} \frac{\sin ax + bx}{ax + \sin bx}$$

$$a, b, a + b \neq 0$$

SolutionAns. At $x = 0$

The value of the given rational function takes the form

$$\frac{0}{0}$$

$$\lim_{x \rightarrow 0} \frac{\sin ax + bx}{ax + \sin bx}$$

=

$$\lim_{x \rightarrow 0} \frac{\left(\frac{\sin ax}{ax}\right) ax + bx}{ax + bx \left(\frac{\sin bx}{bx}\right)}$$

=

$$\frac{\left(\lim_{x \rightarrow 0} \frac{\sin ax}{ax}\right) \times \lim_{x \rightarrow 0} (ax) + \lim_{x \rightarrow 0} (bx)}{\lim_{x \rightarrow 0} ax + \lim_{x \rightarrow 0} bx \left(\lim_{x \rightarrow 0} \frac{\sin bx}{bx}\right)}$$

$$[x \rightarrow \pi \Rightarrow ax \rightarrow 0]$$

and

$$[bx \rightarrow 0]$$

=

$$\frac{\lim_{x \rightarrow 0} (ax) + \lim_{x \rightarrow 0} (bx)}{\lim_{x \rightarrow 0} ax + \lim_{x \rightarrow 0} bx}$$

$$\left[\lim_{y \rightarrow 0} \left(\frac{\sin x}{x} \right) = 1 \right]$$

=

$$\frac{\lim_{x \rightarrow 0} (ax + bx)}{\lim_{x \rightarrow 0} (ax + bx)}$$

=

$$\lim_{x \rightarrow 0} (1)$$

=

$$1$$

Question 21

Evaluate the Given limit.

$$\lim_{x \rightarrow 0} (\operatorname{cosec} x - \cot x)$$

Solution

Ans. At $x = 0$

The value of the given rational function takes the form

$$\infty \rightarrow \infty$$

$$\lim_{x \rightarrow 0} (\operatorname{cosec} x - \cot x)$$

=

$$\lim_{x \rightarrow 0} \left(\frac{1}{\sin x} - \frac{\cos x}{\sin x} \right)$$

=

$$\lim_{x \rightarrow 0} \left(\frac{1 - \cos x}{\sin x} \right)$$

=

$$\lim_{x \rightarrow 0} \frac{\left(\frac{1 - \cos x}{x} \right)}{\left(\frac{\sin x}{x} \right)}$$

=

$$\frac{\lim_{x \rightarrow 0} \left(\frac{1 - \cos x}{x} \right)}{\lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right)}$$

$$\left[\lim_{y \rightarrow 0} \frac{1 - \cos x}{x} = 0 \right] \text{ and } \left[\lim_{y \rightarrow 0} \left(\frac{\sin x}{x} \right) = 1 \right]$$

=

$$\frac{0}{1}$$

=

$$0$$

Question 22

Evaluate the Given limit.

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{\tan 2x}{x - \frac{\pi}{2}}$$

Solution

Ans. Given,

$$\lim_{x \rightarrow 0} \frac{\tan 2x}{x - \frac{\pi}{2}}$$

At $x = \frac{\pi}{2}$

The value of the given rational function takes the form

$$\frac{0}{0}$$

Put $x - \frac{\pi}{2} = y$

So,

$$\left[x \rightarrow \frac{\pi}{2}, y \rightarrow 0 \right]$$

$$\lim_{x \rightarrow 0} \frac{\tan 2x}{x - \frac{\pi}{2}} = \lim_{y \rightarrow 0} \frac{\tan 2\left(y + \frac{\pi}{2}\right)}{y}$$

=

$$\lim_{y \rightarrow 0} \frac{\tan(\pi + 2y)}{y}$$

=

$$\lim_{y \rightarrow 0} \frac{\tan 2y}{y}$$

$$[\tan(\pi + 2y) = \tan 2y]$$

=

$$\lim_{y \rightarrow 0} \frac{\sin 2y}{y \cos 2y}$$

=

$$\lim_{y \rightarrow 0} \left(\frac{\sin 2y}{2y} \times \frac{2}{\cos 2y} \right)$$

=

$$\lim_{y \rightarrow 0} \left(\frac{\sin 2y}{2y} \right) \times \lim_{y \rightarrow 0} \left(\frac{2}{\cos 2y} \right)$$

$$[y \rightarrow 0 \Rightarrow 2y \rightarrow 0]$$

=

$$1 \times \frac{2}{\cos 0}$$

$$\left[\lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right) = 1 \right]$$

=

$$1 \times \frac{2}{1}$$

=

$$2$$

Question 23

Find $\lim_{x \rightarrow 0} f(x)$ and $\lim_{x \rightarrow 1} f(x)$, where

$$f(x) = \begin{cases} 2x + 3 & x \leq 0 \\ 3(x + 1) & x > 0 \end{cases}$$

$$x \leq 0$$

$$x > 0$$

Solution

Ans. Given.

$$f(x) = \begin{cases} 2x + 3 \\ 3(x + 1) \end{cases}$$

$$x \leq 0$$

$$x > 0$$

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} [2x + 3]$$

=

$$2(0) + 3 = 3$$

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} [3x + 1]$$

=

$$3(0 + 1) = 3$$

Therefore,

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0} f(x) = 3$$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1} (x + 1) = 3(1 + 1) = 6$$

Therefore,

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1} f(x) = 6$$

Question 24

Find $\lim_{x \rightarrow 1} f(x)$, where

$$f(x) = \begin{cases} x^2 - 1 \\ -x - 1 \end{cases}$$

$$x \leq 1$$

$$x > 1$$

Solution

Ans. Given,

$$f(x) = \begin{cases} x^2 - 1 \\ -x - 1 \end{cases}$$

$$\begin{cases} x \leq 1 \\ x > 1 \end{cases}$$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1} (x^2 - 1) = 1^2 - 1 = 1 - 1 = 0$$

So, it is observed that

$$\lim_{x \rightarrow 1^-} f(x) \neq \lim_{x \rightarrow 1^+} f(x)$$

Hence, $\lim_{x \rightarrow 1} f(x)$ does not exist.

Question 25

Find $\lim_{x \rightarrow 0} f(x)$, where

$$f(x) = \begin{cases} \frac{|x|}{x} \\ 0 \end{cases}$$

$$\begin{cases} x \neq 0 \\ x = 0 \end{cases}$$

Solution

Ans. Given

$$f(x) = \begin{cases} \frac{|x|}{x} \\ 0 \end{cases}$$

$$\begin{cases} x \neq 0 \\ x = 0 \end{cases}$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \left[\frac{|x|}{x} \right]$$

=

$$\lim_{x \rightarrow 0} \left(\frac{-x}{x} \right)$$

when x is negative, $|x| = -x$

=

$$\lim_{x \rightarrow 0} (-1)$$

=

$$-1$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \left[\frac{|x|}{x} \right]$$

$$= \lim_{x \rightarrow 0} \left(\frac{x}{x} \right)$$

when x is positive, $|x| = x$

$$= \lim_{x \rightarrow 0} (1)$$

=

$$1$$

$$\lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0^+} f(x)$$

Hence, $\lim_{x \rightarrow 0} f(x)$ does not exist.

Question 26

Find $\lim_{x \rightarrow 0} f(x)$, where

$$f(x) = \begin{cases} \frac{x}{|x|} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$$x \neq 0$$

$$x = 0$$

Solution

Ans. Given

$$f(x) = \begin{cases} \frac{x}{|x|} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$$x \neq 0$$

$$x = 0$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \left[\frac{x}{|x|} \right]$$

=

$$\lim_{x \rightarrow 0} \left(\frac{-x}{x} \right)$$

when x is negative, $|x| = -x$

=

$$\lim_{x \rightarrow 0} (-1)$$

=

$$-1$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \left[\frac{x}{|x|} \right]$$

$$= \lim_{x \rightarrow 0} \left(\frac{x}{x} \right)$$

when x is positive, $|x| = x$

=

$$\lim_{x \rightarrow 0} (1)$$

=

$$1$$

$$\lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0^+} f(x)$$

Hence, $\lim_{x \rightarrow 0} f(x)$ does not exist.

Question 27

Find $\lim_{x \rightarrow 5} f(x)$, where $f(x) = |x| - 5$

Solution

Ans. Given,

$$f(x) = |x| - 5$$

$$\begin{aligned}
 \lim_{x \rightarrow 5^-} f(x) &= \lim_{x \rightarrow 5^-} (|x| - 5) \\
 &= \lim_{x \rightarrow 5} (x - 5) \quad \text{when } x \text{ is positive, } |x| = x \\
 &= 5 - 5 \\
 &= 0
 \end{aligned}$$

$$\begin{aligned}
 \lim_{x \rightarrow 5^+} f(x) &= \lim_{x \rightarrow 5^+} (|x| - 5) \\
 &= \lim_{x \rightarrow 5} (x - 5) \\
 &\text{when } x \text{ is positive, } |x| = x \\
 &= 5 - 5 \\
 &= 0
 \end{aligned}$$

$$\lim_{x \rightarrow 5^-} f(x) = \lim_{x \rightarrow 5^+} f(x)$$

$$\text{Hence, } \lim_{x \rightarrow 5} f(x) = 0$$

Question 28

Suppose $f(x) = \begin{cases} a + bx & x < 0 \\ 4 & x = 1 \\ b - ax & x > 1 \end{cases}$ and if $\lim_{x \rightarrow 1} f(x) = f(1)$ what are possible values of a and b?

Solution

Ans. The given function is

$$f(x) = \begin{cases} a + bx & x < 0 \\ 4 & x = 1 \\ b - ax & x > 1 \end{cases}$$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1} f(ax + bx) = a + b$$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1} f(b - ax) = b - a$$

$$f(1) = 4$$

Given

$$\lim_{x \rightarrow 1} f(x) = f(1)$$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1} f(x) = f(1)$$

$$a + b = 4 \text{ and } b - a = 4$$

On solving, we get

$$a = 0 \text{ and } b = 4$$

Question 29

Let $a_1, a_2, \dots, a_n$ be fixed real number and define a function

$$f(x) = (x - a_1)(x - a_2) \dots (x - a_n)$$

What is $\lim_{x \rightarrow a_1} f(x)$? For some $a \neq a_1, a_2, \dots, a_n$. Compute $\lim_{x \rightarrow a} f(x)$.

Solution

Ans. Given,

$$f(x) = (x - a_1)(x - a_2) \dots (x - a_n)$$

$$\lim_{x \rightarrow a_1} f(x) = \lim_{x \rightarrow a_1} [(x - a_1)(x - a_2) \dots (x - a_n)]$$

=

$$(a_1 - a_1)(a_1 - a_2) \dots (a_1 - a_n) = 0$$

Therefore,

$$\lim_{x \rightarrow a_1} f(x) = 0$$

$$\text{Now, } \lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} [(x - a_1)(x - a_2) \dots (x - a_n)]$$

=

$$(a_1 - a_1)(a_1 - a_2) \dots (a_1 - a_n)$$

Therefore,

$$\lim_{x \rightarrow a} f(x) = (a_1 - a_1)(a_1 - a_2) \dots (a_1 - a_n)$$

Question 30

If

$$f(x) = \begin{cases} |x| + 1 \\ 0 \\ |x| - 1 \end{cases}$$

$$x < 0$$

$$x = 0$$

$$x > 1$$

For what value (s) of does $\lim_{x \rightarrow a} f(x)$ exists?

Solution

Ans. Given,

$$f(x) = \begin{cases} |x| + 1 \\ 0 \\ |x| - 1 \end{cases}$$

$$x < 0$$

$$x = 0$$

$$x > 1$$

When $a = 0$

=

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} (|x| + 1)$$

=

$$\lim_{x \rightarrow 0^-} (-x + 1)$$

when x is negative, $|x| = -x$

$$= 0 + 1$$

$$= 1$$

=

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (|x| + 1)$$

=

$$\lim_{x \rightarrow 0^-} (x - 1)$$

when x is positive, $|x| = x$

$$= 0 - 1$$

$$= -1$$

Here, $\lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0^+} f(x)$

$\lim_{x \rightarrow 0} f(x)$ does not exist.

When

$$\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^+} (|x| + 1)$$

$$a < 0$$

$$\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^-} (|x| + 1)$$

=

$$\lim_{x \rightarrow a^-} (-x + 1)$$

$$[x < a < 0 \Rightarrow |x| = -x]$$

=

$$-a + 1$$

$$\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = -a + 1$$

Thus, limit exists at $x = a$, where

$$a < 0$$

When

$$a > 0$$

$$\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^-} (|x| + 1)$$

$$= \lim_{x \rightarrow a} (-x - 1) \quad [0 < x < a \Rightarrow |x| = x]$$

$$= -a - 1$$

$$\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^+} (|x| + 1)$$

$$= \lim_{x \rightarrow a} (-x - 1) \quad [0 < x < a \Rightarrow |x| = x]$$

$$= a - 1$$

$$\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = a - 1$$

Thus, limit exists at $x = a$, where

$$a > 0$$

$$\text{Thus } \lim_{x \rightarrow a} f(x) \text{ exists for all } a \neq 0 \quad \lim_{x \rightarrow 1} \frac{f(x)-2}{x^2-1} = \pi.$$

Question 31

If function $f(x)$ satisfies,

$$\lim_{x \rightarrow 1} \frac{f(x) - 2}{x^2 - 1} = \pi$$

, evaluate $\lim_{x \rightarrow 1} f(x)$

Solution

Ans. Given,

$$\lim_{x \rightarrow 1} \frac{f(x) - 2}{x^2 - 1} = \pi$$

$$= \frac{\lim_{x \rightarrow 1} (f(x) - 2)}{\lim_{x \rightarrow 1} (x^2 - 1)} = \pi$$

$$= \lim_{x \rightarrow 1} (f(x) - 2) = \pi \lim_{x \rightarrow 1} (x^2 - 1)$$

$$= \lim_{x \rightarrow 1} (f(x) - 2) = \pi(1^2 - 1)$$

$$= \lim_{x \rightarrow 1} (f(x) - 2) = 0$$

$$= \lim_{x \rightarrow 1} f(x) - \lim_{x \rightarrow 1} 2 = 0$$

$$= \lim_{x \rightarrow 1} f(x) - 2 = 0$$

$$\lim_{x \rightarrow 1} f(x) = 2$$

Question 32

$$\text{If } f(x) = \begin{cases} mx^2 + n & x < 0 \\ nx + m & 0 \leq x \leq 1 \\ nx^3 + m & x > 1 \end{cases}$$

For what integers m and n does $\lim_{x \rightarrow 0} f(x)$ and $\lim_{x \rightarrow 1} f(x)$ exists?

Solution

Ans. Given,

$$f(x) = \begin{cases} mx^2 + n & x < 0 \\ nx + m & 0 \leq x \leq 1 \\ nx^3 + m & x > 1 \end{cases}$$

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} (mx^2 + n)$$

$$= m(0)^2 + n$$

$$= n$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0} (nx + m)$$

$$= n(0) + m$$

$$= m$$

Thus, $\lim_{x \rightarrow 0^+} f(x)$ exists if $m=n$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1} (nx + m)$$

$$= n(1) + m$$

$$= m + n$$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1} (nx^3 + m)$$

$$= n(1)^3 + m$$

$$= m + n$$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1} f(x)$$

Thus, $\lim_{x \rightarrow 1} f(x)$ exists for any internal value of m and n

Exercise 12.2

Question 1

Find the derivative of

$$x^2 - 2$$

at $x = 10$

Solution

Ans: Let $f(x) = x^2 - 2$

Accordingly,

$$f'(10) = \lim_{h \rightarrow 0} \frac{f(10+h) - f(10)}{h}$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{[(10+h)^2 - 2] - (10^2 - 2)}{h}$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{10^2 + (2 \times 10 \times h) + h^2 - 2 - 10^2 + 2}{h}$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{20h + h^2}{h}$$

$$\Rightarrow \lim_{h \rightarrow 0} (20 + h) = 20 + 0 = 20$$

Thus, the derivative of $x^2 - 2$ at $x = 10$ is 20

Question 2

Find the derivative of x at $x = 1$

Solution

Ans: Let $f(x) = x$

Accordingly,

$$f'(1) = \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h}$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{(1+h)-1}{h} = \lim_{h \rightarrow 0} \frac{h}{h}$$

$$\Rightarrow \lim_{h \rightarrow 0} (1) = 1$$

Thus, the derivative of x at $x = 1$ is 1

Question 3

Find the derivative of $99x$ at $x = 100$

Solution

Ans: Let $f(x) = 99x$

Accordingly,

$$\begin{aligned} f'(100) &= \lim_{h \rightarrow 0} \frac{f(100+h) - f(100)}{h} \\ \Rightarrow \lim_{h \rightarrow 0} \frac{99(100+h) - (99 \times 100)}{h} &= \lim_{h \rightarrow 0} \frac{(99 \times 100) + 99h - (99 \times 100)}{h} \\ \Rightarrow \lim_{h \rightarrow 0} \frac{99h}{h} &= \lim_{h \rightarrow 0} (99) = 99 \end{aligned}$$

Thus, the derivative of $99x$ at $x = 100$ is 99

Question 4

Find the derivative of the following functions using the first principle.

(i). $x^3 - 27$

(ii). $(x - 1)(x - 2)$

(iii). $\frac{1}{x^2}$

(iv). $\frac{x+1}{x-1}$

Solution

Ans: Let $f(x) = x^3 - 27$

Accordingly, from the first principle,

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &\Rightarrow \lim_{h \rightarrow 0} \frac{[(x+h)^3 - 27] - (x^3 - 27)}{h} \\
 &\Rightarrow \lim_{h \rightarrow 0} \frac{x^3 + h^3 + 3x^2h + 3xh^2 - x^3}{h} = \lim_{h \rightarrow 0} (h^2 + 3x^2 + 3xh) \\
 &\Rightarrow 0 + 3x^2 + 0 = 3x^2
 \end{aligned}$$

Ans: Let $f(x) = (x - 1)(x - 2)$

Accordingly, from the first principle,

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &\Rightarrow \lim_{h \rightarrow 0} \frac{(x+h-1)(x+h-2) - (x-1)(x-2)}{h} \\
 &\Rightarrow \lim_{h \rightarrow 0} \frac{(x^2 + hx - 2x + hx + h^2 - 2h - x - h + 2) - (x^2 - 2x - x + 2)}{h} \\
 &\Rightarrow \lim_{h \rightarrow 0} \frac{(hx + hx + h^2 - 2h - h)}{h} = \lim_{h \rightarrow 0} \frac{2hx + h^2 - 3h}{h} \\
 &\Rightarrow \lim_{h \rightarrow 0} (2x + h - 3) = 2x - 3
 \end{aligned}$$

Ans: Let $f(x) = \frac{1}{x^2}$

Accordingly, from the first principle,

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &\Rightarrow \lim_{h \rightarrow 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h} = \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{x^2 - (x+h)^2}{x^2(x+h)^2} \right] \\
 &\Rightarrow \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{x^2 - x^2 - 2hx - h^2}{x^2(x+h)^2} \right] = \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-h^2 - 2hx}{x^2(x+h)^2} \right] \\
 &\Rightarrow \lim_{h \rightarrow 0} \left[\frac{-h^2 - 2hx}{x^2(x+h)^2} \right] = \frac{0 - 2x}{x^2(x+0)^2} \\
 &\Rightarrow \frac{-2}{x^3}
 \end{aligned}$$

Ans: Let $f(x) = \frac{x+1}{x-1}$

Accordingly, from the first principle,

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &\Rightarrow \lim_{h \rightarrow 0} \frac{\left(\frac{x+h+1}{x+h-1} - \frac{x+1}{x-1} \right)}{h} \\
 &\Rightarrow \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{(x-1)(x+h+1) - (x+1)(x+h-1)}{(x-1)(x+h-1)} \right] \\
 &\Rightarrow \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{(x^2 + hx + x - x - h - 1) - (x^2 + hx - x + x + h - 1)}{(x-1)(x+h-1)} \right] \\
 &\Rightarrow \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-2h}{(x-1)(x+h-1)} \right] \\
 &\Rightarrow \lim_{h \rightarrow 0} \left[\frac{-2}{(x-1)(x+h-1)} \right] = \frac{-2}{(x-1)(x-1)} \\
 &\Rightarrow \frac{-2}{(x-1)^2}
 \end{aligned}$$

Question 5

For the function $f(x) = \frac{x^{100}}{100} + \frac{x^{99}}{99} + \dots + \frac{x^2}{2} + x + 1$ prove that $f'(1) = 100f'(0)$

Solution

Ans: The given function is,

$$f(x) = \frac{x^{100}}{100} + \frac{x^{99}}{99} + \dots + \frac{x^2}{2} + x + 1$$

$$\frac{d}{dx} f(x) = \frac{d}{dx} \left[\frac{x^{100}}{100} + \frac{x^{99}}{99} + \dots + \frac{x^2}{2} + x + 1 \right]$$

$$\frac{d}{dx} f(x) = \frac{d}{dx} \left(\frac{x^{100}}{100} \right) + \frac{d}{dx} \left(\frac{x^{99}}{99} \right) + \dots + \frac{d}{dx} \left(\frac{x^2}{2} \right) + \frac{d}{dx}(x) + \frac{d}{dx}(1)$$

On using theorem $\frac{d}{dx}(nx^{n-1}) = nx^{n-1}$, we obtain

$$\frac{d}{dx} f(x) = \frac{100x^{99}}{100} + \frac{99x^{98}}{99} + \dots + \frac{2x}{2} + 1 + 0$$

$$\Rightarrow x^{99} + x^{98} + \dots + x + 1$$

$$\therefore f'(x) = x^{99} + x^{98} + \dots + x + 1$$

At $x = 0$,

$$f'(0) = 1$$

At $x = 1$,

$$f'(1) = 1^{99} + 1^{98} + \dots + 1 + 1 = [1 + 1 + \dots + 1 + 1]_{100 \text{ terms}}$$

$$\Rightarrow 1 \times 100 = 100$$

$$\text{Thus, } f'(1) = 100f'(0)$$

Question 6

Find the derivative of $x^n + ax^{n-1} + a^2x^{n-2} + \dots + a^{n-1}x + a^n$, where a is some fixed real number.

Solution

$$\text{Ans: Let } f(x) = x^n + ax^{n-1} + a^2x^{n-2} + \dots + a^{n-1}x + a^n$$

$$\frac{d}{dx} f(x) = \frac{d}{dx} (x^n + ax^{n-1} + a^2x^{n-2} + \dots + a^{n-1}x + a^n)$$

$$\Rightarrow \frac{d(x^n)}{dx} + a \frac{d(x^{n-1})}{dx} + a^2 \frac{d(x^{n-2})}{dx} + \dots + a^{n-1} \frac{d(x)}{dx} + a^n \frac{d(1)}{dx}$$

$$\text{On using theorem } \frac{d}{dx}(x^n) = nx^{n-1}$$

So, we obtain

$$f'(x) = nx^{n-1} + a(n-1)x^{n-2} + a^2(n-2)x^{n-3} + \dots + a^{n-1} + a^n(0)$$

$$\therefore f'(x) = nx^{n-1} + a(n-1)x^{n-2} + a^2(n-2)x^{n-3} + \dots + a^{n-1}$$

Question 7

For some constants a and b , find the derivative of the following functions.

(a). $(x - a)(x - b)$

(b). $(ax^2 + b)^2$

(c). $\frac{x-a}{x-b}$

Solution

Ans: Let $f(x) = (x - a)(x - b)$

$$\Rightarrow f(x) = x^2 - (a + b)x + ab$$

$$\therefore f'(x) = \frac{d}{dx}(x^2 - (a + b)x + ab)$$

$$\Rightarrow \frac{d}{dx}(x^2) - (a + b) \frac{d}{dx}(x) + \frac{d}{dx}(ab)$$

On using theorem $\frac{d}{dx}(x^n) = nx^{n-1}$, we obtain

$$f'(x) = 2x - (a + b) + 0 = 2x - a - b$$

Ans: Let, $f(x) = (ax^2 + b)^2$

$$\Rightarrow f(x) = a^2x^4 + 2abx^2 + b^2$$

$$\therefore f'(x) = \frac{d}{dx}(a^2x^4 + 2abx^2 + b^2)$$

$$\Rightarrow a^2 \frac{d}{dx}(x^4) + 2ab \frac{d}{dx}(x^2) + \frac{d}{dx}b^2$$

On using theorem $\frac{d}{dx}(x^n) = nx^{n-1}$, we obtain

$$f'(x) = a^2(4x^3) + 2ab(2x) + b^2(0)$$

$$\Rightarrow 4a^2x^3 + 4abx = 4ax(ax^2 + b)$$

Ans: Let, $f(x) = \frac{x-a}{x-b}$

$$\Rightarrow f'(x) = \frac{d}{dx}\left(\frac{x-a}{x-b}\right)$$

By quotient rule,

$$f'(x) = \frac{(x-b)\frac{d}{dx}(x-a) - (x-a)\frac{d}{dx}(x-b)}{(x-b)^2}$$

$$\Rightarrow \frac{(x-b) \times 1 - (x-a) \times 1}{(x-b)^2}$$

$$\Rightarrow \frac{x-b-x+a}{(x-b)^2} = \frac{a-b}{(x-b)^2}$$

Question 8

Find the derivative of $\frac{x^n - a^n}{x - a}$ for any constant a

Solution

Ans: Let $f(x) = \frac{x^n - a^n}{x - a}$

$$\Rightarrow f'(x) = \frac{d}{dx} \left(\frac{x^n - a^n}{x - a} \right)$$

By quotient rule,

$$f'(x) = \frac{(x-a) \frac{d}{dx}(x^n - a^n) - (x^n - a^n) \frac{d}{dx}(x-a)}{(x-a)^2}$$

$$\Rightarrow \frac{(x-a)(nx^{n-1} - 0) - (x^n - a^n)(1)}{(x-a)^2}$$

$$\Rightarrow \frac{nx^n - anx^{n-1} - x^n + a^n}{(x-a)^2}$$

Question 9

Find the derivative of the following functions

A.

$$. 2x - \frac{3}{4}$$

B.

$$. (5x^3 + 3x - 1)(x - 1)$$

C.

$$. x^{-3}(5 + 3x)$$

D.

$$. x^5(3 - 6x^{-9})$$

Solution

Ans: Let $f(x) = 2x - \frac{3}{4}$

$$f'(x) = \frac{d}{dx} \left(2x - \frac{3}{4} \right)$$

$$\Rightarrow 2 \frac{d}{dx}(x) - \frac{d}{dx}\left(\frac{3}{4}\right)$$

$$\Rightarrow 2 - 0 = 2$$

Ans: Let $f(x) = (5x^3 + 3x - 1)(x - 1)$

By Leibnitz product rule,

$$f'(x) = (5x^3 + 3x - 1) \frac{d}{dx}(x - 1) + (x - 1) \frac{d}{dx}(5x^3 + 3x - 1)$$

$$\Rightarrow ((5x^3 + 3x - 1) \times 1) + (x - 1)(5 \times 3x^2 + 3 - 0)$$

$$\Rightarrow (5x^3 + 3x - 1) + (x - 1)(15x^2 + 3)$$

$$\Rightarrow 5x^3 + 3x - 1 + 15x^3 + 3x - 15x^2 - 3$$

$$\Rightarrow 20x^3 - 15x^2 + 6x - 4$$

Ans: Let, $f(x) = x^{-3}(5 + 3x)$

By Leibnitz product rule,

$$f'(x) = x^{-3} \frac{d}{dx}(5 + 3x) + (5 + 3x) \frac{d}{dx}(x^{-3})$$

$$\Rightarrow x^{-3}(0 + 3) + (5 + 3x)(3x^{-3-1})$$

$$\Rightarrow 3x^{-3} + (5 + 3x)(-3x^{-4}) = 3x^{-3} - 15x^{-4} - 9x^{-3}$$

$$\Rightarrow -6x^{-3} - 15x^{-4} = -3x^{-3}\left(2 + \frac{5}{x}\right) = \frac{-3x^{-3}}{x}(2x + 5)$$

$$\Rightarrow \frac{-3}{x^4}(5 + 2x)$$

Ans: Let, $f(x) = x^5(3 - 6x^{-9})$

By Leibnitz product rule,

$$f'(x) = x^5 \frac{d}{dx}(3 - 6x^{-9}) + (3 - 6x^{-9}) \frac{d}{dx}x^5$$

$$\Rightarrow x^5\{0 - 6(-9)x^{-9-1}\} + (3 - 6x^{-9})(5x^4)$$

$$\Rightarrow x^5(54x^{-10}) + 15x^4 - 30x^{-5} = 54x^{-5} + 15x^4 - 30x^{-5}$$

$$\Rightarrow 24x^{-5} + 15x^4 = 15x^4 + \frac{24}{x^5}$$

(e). $x^{-4} (3 - 4x^{-5})$

Ans: Let $f(x) = x^{-4} (3 - 4x^{-5})$

By Leibnitz product rule,

$$\begin{aligned} f'(x) &= x^{-4} \frac{d}{dx} (3 - 4x^{-5}) + (3 - 4x^{-5}) \frac{d}{dx} (x^{-4}) \\ &\Rightarrow x^{-4} \{0 - 4(-5)x^{-5-1}\} + (3 - 4x^{-5}) (-4)x^{-4-1} \\ &\Rightarrow x^{-4} (20x^{-6}) + (3 - 4x^{-5}) (-4x^{-5}) \\ &\Rightarrow 20x^{-10} - 12x^{-5} + 16x^{-10} = 36x^{-10} - 12x^{-5} \\ &\Rightarrow \frac{36}{x^{10}} - \frac{12}{x^5} \end{aligned}$$

(f). $\frac{2}{x+1} - \frac{x^2}{3x-1}$

Ans: $f(x) = \frac{2}{x+1} - \frac{x^2}{3x-1}$

$$f'(x) = \frac{d}{dx} \left(\frac{2}{x+1} \right) - \frac{d}{dx} \left(\frac{x^2}{3x-1} \right)$$

By quotient rule,

$$\begin{aligned} f'(x) &= \left[\frac{(x+1) \frac{d}{dx} (2) - 2 \frac{d}{dx} (x+1)}{(x+1)^2} \right] - \left[\frac{(3x-1) \frac{d}{dx} (x^2) - x^2 \frac{d}{dx} (3x-1)}{(3x-1)^2} \right] \\ &\Rightarrow \left[\frac{(x+1)(0) - 2(1)}{(x+1)^2} \right] - \left[\frac{(3x-1)(2x) - x^2(3)}{(3x-1)^2} \right] \\ &\Rightarrow \frac{-2}{(x+1)^2} - \left[\frac{6x^2 - 2x - 3x^2}{(3x-1)^2} \right] = \frac{-2}{(x+1)^2} - \left[\frac{3x^2 - 2x}{(3x-1)^2} \right] \\ &\Rightarrow \frac{-2}{(x+1)^2} - \frac{x(3x-2)}{(3x-1)^2} \end{aligned}$$

Question 10

Find the derivative of $\cos x$ from the first principle

Solution

Ans: Let $f(x) = \cos x$

Accordingly from the first principle,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \lim_{h \rightarrow 0} \left[\frac{\cos(x+h) - \cos(x)}{h} \right]$$

$$\Rightarrow \lim_{h \rightarrow 0} \left[\frac{\cos x \cosh - \sin x \sinh - \cos x}{h} \right] = \lim_{h \rightarrow 0} \left[\frac{-\cos x(1 - \cosh)}{h} - \frac{\sin x \sinh}{h} \right]$$

$$\Rightarrow -\cos x \left[\lim_{h \rightarrow 0} \left(\frac{1 - \cosh}{h} \right) \right] - \sin x \left[\lim_{h \rightarrow 0} \left(\frac{\sinh}{h} \right) \right]$$

We know, $\lim_{h \rightarrow 0} \frac{1 - \cosh}{h} = 0$ and $\lim_{h \rightarrow 0} \frac{\sinh}{h} = 1$

$$\Rightarrow f'(x) = -\cos x \times 0 - \sin x \times 1$$

$$\therefore f'(x) = -\sin x$$

Question 11

Find the derivative of the functions below:

(i). $\sin x \cos x$

(ii). $\sec x$

(iii). $5 \sec x + 4 \cos x$

(iv). $\cos ecx$

(v). $3 \cot x + 5 \cos ecx$

(vi). $5 \sin x - 6 \cos x + 7$

(vii). $2 \tan x - 7 \sec x$

Solution

Ans: Let $f(x) = \sin x \cos x$

Accordingly, from the first principle,

$$\begin{aligned}
f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
&\Rightarrow \lim_{h \rightarrow 0} \frac{\sin(x+h) \cos(x+h) - \sin x \cos x}{h} \\
&\Rightarrow \lim_{h \rightarrow 0} \frac{1}{2h} [2 \sin(x+h) \cos(x+h) - 2 \sin x \cos x] \\
&\Rightarrow \lim_{h \rightarrow 0} \frac{1}{2h} [\sin 2(x+h) - \sin 2x] \\
&\Rightarrow \lim_{h \rightarrow 0} \frac{1}{2h} \left[2 \cos \frac{2x+2h+2x}{2} \cdot \sin \frac{2x+2h-2x}{2} \right] \\
&\Rightarrow \lim_{h \rightarrow 0} \frac{1}{2h} \left[2 \cos \frac{4x+2h}{2} \cdot \sin \frac{2h}{2} \right] = \lim_{h \rightarrow 0} \frac{1}{2h} [\cos(2x+h) \sinh] \\
&\Rightarrow \lim_{h \rightarrow 0} \cos(2x+h) \cdot \lim_{h \rightarrow 0} \frac{\sinh}{h} \\
&\Rightarrow \cos(2x+h) \times 1 = \cos 2x
\end{aligned}$$

Ans: Let $f(x) = \sec x$

Accordingly, from the first principle

$$\begin{aligned}
f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
&\Rightarrow \lim_{h \rightarrow 0} \frac{\sec(x+h) - \sec x}{h} \\
&\Rightarrow \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{1}{\cos(x+h)} - \frac{1}{\cos x} \right] = \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\cos x - \cos(x+h)}{\cos x \cos(x+h)} \right] \\
&\Rightarrow \frac{1}{\cos x} \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-2 \sin\left(\frac{x+x+h}{2}\right) \sin\left(\frac{x-x-h}{2}\right)}{\cos(x+h)} \right] \\
&\Rightarrow \frac{1}{\cos x} \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-2 \sin\left(\frac{2x+h}{2}\right) \sin\left(\frac{-h}{2}\right)}{\cos(x+h)} \right] \\
&\Rightarrow \frac{1}{\cos x} \lim_{h \rightarrow 0} \frac{1}{2h} \left[\frac{-2 \sin\left(\frac{2x+h}{2}\right) \frac{\sin\left(\frac{-h}{2}\right)}{\left(\frac{h}{2}\right)}}{\cos(x+h)} \right] \\
&\Rightarrow \frac{1}{\cos x} \lim_{h \rightarrow 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \lim_{h \rightarrow 0} \frac{\sin\left(\frac{2x+h}{2}\right)}{\cos(x+h)} \\
&\Rightarrow \frac{1}{\cos x} \times 1 \times \frac{\sin x}{\cos x} = \sec x \tan x
\end{aligned}$$

Ans: Let,

$$f(x) = 5 \sec x + 4 \cos x$$

Accordingly, from the first principle,

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &\Rightarrow \lim_{h \rightarrow 0} \frac{5 \sec(x+h) + 4 \cos(x+h) - [5 \sec x + 4 \cos x]}{h} \\ &\Rightarrow 5 \lim_{h \rightarrow 0} \frac{[\sec(x+h) - \sec x]}{h} + 4 \lim_{h \rightarrow 0} \frac{[\cos(x+h) - \cos x]}{h} \\ &\Rightarrow 5 \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{1}{\cos(x+h)} - \frac{1}{\cos x} \right] + 4 \lim_{h \rightarrow 0} \frac{1}{h} [\cos(x+h) - \cos x] \\ &\Rightarrow 5 \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\cos x - \cos(x+h)}{\cos x \cos(x+h)} \right] + 4 \lim_{h \rightarrow 0} \frac{1}{h} [\cos x \cosh - \sin x \sinh - \cos x] \\ &\Rightarrow \frac{5}{\cos x} \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-2 \sin\left(\frac{2x+h}{2}\right) \sin\left(\frac{-h}{2}\right)}{\cos(x+h)} \right] + 4 \left[-\cos x \lim_{h \rightarrow 0} \frac{(1 - \cos x)}{h} - \sin x \lim_{h \rightarrow 0} \frac{\sinh}{h} \right] \\ &\Rightarrow \frac{5}{\cos x} \lim_{h \rightarrow 0} \frac{\left[\sin\left(\frac{2x+h}{2}\right) \frac{\sin\left(\frac{-h}{2}\right)}{\left(\frac{h}{2}\right)} \right]}{\cos(x+h)} + 4 [-\cos x (0) - \sin x (1)] \\ &\Rightarrow \frac{5}{\cos x} \left[\lim_{h \rightarrow 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \lim_{h \rightarrow 0} \frac{\sin\left(\frac{2x+h}{2}\right)}{\cos(x+h)} \right] - 4 \sin x \\ &\Rightarrow \frac{5}{\cos x} \times \frac{\sin x}{\cos x} \times 1 - 4 \sin x = 5 \sec x \tan x - 4 \sin x \end{aligned}$$

Ans: Let $f(x) = \operatorname{cosec} x$

Accordingly, from the first principle,

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &\Rightarrow \lim_{h \rightarrow 0} \frac{1}{h} [\operatorname{cosec}(x+h) - \operatorname{cosec} x] = \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{1}{\sin(x+h)} - \frac{1}{\sin x} \right] \\ &\Rightarrow \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin x - \sin(x+h)}{\sin x \sin(x+h)} \right] = \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{2 \cos\left(\frac{x+x+h}{2}\right) \sin\left(\frac{x-x-h}{2}\right)}{\sin x \sin(x+h)} \right] \end{aligned}$$

$$\begin{aligned}
&\Rightarrow \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{2 \cos\left(\frac{2x+h}{2}\right) \sin\left(\frac{-h}{2}\right)}{\sin x \sin(x+h)} \right] \\
&\Rightarrow \lim_{h \rightarrow 0} \frac{\left[-\cos\left(\frac{2x+h}{2}\right) \frac{\sin\left(\frac{-h}{2}\right)}{\left(\frac{h}{2}\right)} \right]}{\sin x \sin(x+h)} \\
&\Rightarrow \lim_{h \rightarrow 0} \left(\frac{-\cos\left(\frac{2x+h}{2}\right)}{\sin x \sin(x+h)} \right) \lim_{\frac{h}{2} \rightarrow 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \\
&\Rightarrow \left(\frac{-\cos x}{\sin x \sin x} \right) \times 1 = -\operatorname{cosec} x \cot x
\end{aligned}$$

Ans: Let $f(x) = 3 \cot x + 5 \operatorname{cosec} x$

Accordingly, from the first principle

$$\begin{aligned}
f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
&\Rightarrow \lim_{h \rightarrow 0} \frac{1}{h} [3 \cot(x+h) + 5 \operatorname{cosec}(x+h) - 3 \cot x - 5 \operatorname{cosec} x] \\
&\Rightarrow 3 \lim_{h \rightarrow 0} \frac{1}{h} [\cot(x+h) - \cot x] + 5 \lim_{h \rightarrow 0} \frac{1}{h} [\operatorname{cosec}(x+h) - \operatorname{cosec} x] \quad \dots (1)
\end{aligned}$$

$$\text{Now, } \lim_{h \rightarrow 0} \frac{1}{h} [\cot(x+h) - \cot x] = \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\cos(x+h)}{\sin(x+h)} - \frac{\cos x}{\sin x} \right]$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\cos(x+h) \sin x - \cos x \sin(x+h)}{\sin x \sin(x+h)} \right] = \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(x-x-h)}{\sin x \sin(x+h)} \right]$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(-h)}{\sin x \sin(x+h)} \right] = \lim_{h \rightarrow 0} \frac{\sin h}{h} \cdot \lim_{h \rightarrow 0} \left[\frac{1}{\sin x \sin(x+h)} \right]$$

$$\Rightarrow -1 \times \frac{1}{\sin x \sin(x+h)} = \frac{-1}{\sin^2 x} = -\operatorname{cosec}^2 x \quad \dots (2)$$

$$\lim_{h \rightarrow 0} \frac{1}{h} [\operatorname{cosec}(x+h) - \operatorname{cosec} x] = \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{1}{\sin(x+h)} - \frac{1}{\sin x} \right]$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin x - \sin(x+h)}{\sin x \sin(x+h)} \right] = \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{2 \cos\left(\frac{x+x+h}{2}\right) \sin\left(\frac{x-x-h}{2}\right)}{\sin x \sin(x+h)} \right]$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{2 \cos\left(\frac{2x+h}{2}\right) \sin\left(\frac{-h}{2}\right)}{\sin x \sin(x+h)} \right] = \lim_{h \rightarrow 0} \frac{-\cos\left(\frac{2x+h}{2}\right) \frac{\sin\left(\frac{-h}{2}\right)}{\left(\frac{h}{2}\right)}}{\sin x \sin(x+h)}$$

$$\Rightarrow \lim_{h \rightarrow 0} \left(\frac{-\cos\left(\frac{2x+h}{2}\right)}{\sin x \sin(x+h)} \right) \lim_{h \rightarrow 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} = \left(\frac{-\cos x}{\sin x \sin x} \right) \cdot 1$$

$$\Rightarrow -\cos x \cot x \dots\dots (3)$$

From (1), (2), and (3), we obtain

$$f'(x) = -3 \cos x \cot x - 5 \cos x \cot x$$

$$\text{Ans: Let } f(x) = 5 \sin x - 6 \cos x + 7$$

Accordingly, from the first principle,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{1}{h} [5 \sin(x+h) - 6 \cos(x+h) + 7 - 5 \sin x + 6 \cos x - 7]$$

$$\Rightarrow 5 \lim_{h \rightarrow 0} \frac{1}{h} [\sin(x+h) - \sin x] - 6 \lim_{h \rightarrow 0} \frac{1}{h} [\cos(x+h) - \cos x]$$

$$\Rightarrow 5 \lim_{h \rightarrow 0} \frac{1}{h} \left[2 \cos\left(\frac{x+h+x}{2}\right) \cdot \sin\left(\frac{x+h-x}{2}\right) \right] - 6 \lim_{h \rightarrow 0} \left[\frac{\cos x \cosh - \sin x \sinh - \cos x}{h} \right]$$

$$\Rightarrow 5 \lim_{h \rightarrow 0} \frac{1}{h} \left[2 \cos\left(\frac{2x+h}{2}\right) \cdot \sin\left(\frac{h}{2}\right) \right] - 6 \lim_{h \rightarrow 0} \left[\frac{-\cos x(1 - \cosh) - \sin x \sinh}{h} \right]$$

$$\Rightarrow 5 \lim_{h \rightarrow 0} \frac{1}{h} \left[\cos\left(\frac{2x+h}{2}\right) \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \right] - 6 \lim_{h \rightarrow 0} \left[\frac{-\cos x(1 - \cosh)}{h} - \frac{\sin x \sinh}{h} \right]$$

$$\Rightarrow 5 \left[\lim_{h \rightarrow 0} \cos\left(\frac{2x+h}{2}\right) \right] \left[\lim_{h \rightarrow 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \right] - 6 \left[-\cos x \left(\lim_{h \rightarrow 0} \frac{1 - \cosh}{h} \right) - \sin x \left(\lim_{h \rightarrow 0} \frac{\sinh}{h} \right) \right]$$

$$\Rightarrow 5 \cos x \cdot 1 - 6 [(-\cos x) \cdot (0) - \sin x \cdot 1]$$

$$\Rightarrow 5 \cos x + 6 \sin x$$

$$\text{Ans: Let } f(x) = 2 \tan x - 7 \sec x$$

Accordingly, from the first principle,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{1}{h} [2 \tan(x+h) - 7 \sec(x+h) - 2 \tan x + 7 \sec x]$$

$$\begin{aligned}
&\Rightarrow 2 \lim_{h \rightarrow 0} \frac{1}{h} [\tan(x+h) - \tan x] - 7 \lim_{h \rightarrow 0} \frac{1}{h} [\sec(x+h) - \sec x] \\
&\Rightarrow 2 \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x} \right] - 7 \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{1}{\cos(x+h)} - \frac{1}{\cos x} \right] \\
&\Rightarrow 2 \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\cos x \sin(x+h) - \sin x \cos(x+h)}{\cos x \cos(x+h)} \right] - 7 \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\cos x - \cos(x+h)}{\cos x \cos(x+h)} \right] \\
&\Rightarrow 2 \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin x + h - x}{\cos x \cos(x+h)} \right] - 7 \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-2 \sin\left(\frac{x+x+h}{2}\right) \sin\left(\frac{x-x-h}{2}\right)}{\cos x \cos(x+h)} \right] \\
&\Rightarrow 2 \left[\lim_{h \rightarrow 0} \left(\frac{\sinh}{h} \right) \frac{1}{\cos x \cos(x+h)} \right] - 7 \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-2 \sin\left(\frac{2x+h}{2}\right) \sin\left(\frac{-h}{2}\right)}{\cos x \cos(x+h)} \right] \\
&\Rightarrow 2 \left(\lim_{h \rightarrow 0} \frac{\sinh}{h} \right) \left[\lim_{h \rightarrow 0} \frac{1}{\cos x \cos(x+h)} \right] - 7 \left(\lim_{h \rightarrow 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \right) \left(\lim_{h \rightarrow 0} \frac{\sin\left(\frac{2x+h}{2}\right)}{\cos x \cos(x+h)} \right) \\
&\Rightarrow 2 \times 1 \times 1 \times \frac{1}{\cos x \cos x} - 7 \left(1 \times \frac{\sin x}{\cos x \cos x} \right) = 2\sec^2 x - 7\sec x \tan x
\end{aligned}$$

Miscellaneous Exercise

Question 1

Find the derivative of the following functions from first principle:

(i) $-x$

(ii) $(-x)^{-1}$

(iii) $\sin(x+1)$

(iv) $\cos\left(x - \frac{\pi}{8}\right)$

Solution

Ans: (i) Let $f(x) = -x$. Accordingly, $f(x+h) = -(x+h)$

By first principle, $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \rightarrow 0} \frac{-(x+h) - (-x)}{h}$$

$$\begin{aligned}
&= \lim_{n \rightarrow 0} \frac{-x - h + x}{h} \\
&= \lim_{n \rightarrow 0} \frac{-h}{h} \\
&= \lim_{h \rightarrow 0} (-1) = -1
\end{aligned}$$

(ii) Let $f(x) = (-x)^{-1} = \frac{1}{-x} = \frac{-1}{x}$. Accordingly, $f(x+h) = \frac{-1}{(x+h)}$

By first principle,

$$\begin{aligned}
f'(x) &= \lim_{n \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-1}{(x+h)} - \left(\frac{-1}{x} \right) \right] \\
&= \lim_{n \rightarrow 0} \frac{1}{h} \left[\frac{-x + (x+h)}{x(x+h)} \right] \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{h}{x(x+h)} \right] \\
&= \lim_{n \rightarrow 0} \frac{1}{x(x+h)} \\
&= \frac{1}{x \cdot X} = \frac{1}{x^2}
\end{aligned}$$

(iii) Let $f(x) = \sin(x+1)$. Accordingly, $f(x+h) = \sin(x+h+1)$

By first principle,

$$\begin{aligned}
f'(x) &= \lim_{n \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{n \rightarrow 0} \frac{1}{h} [\sin(x+h+1) - \sin(x+1)] \\
&= \lim_{n \rightarrow 0} \frac{1}{h} \left[2 \cos \left(\frac{x+h+1+x+1}{2} \right) \sin \left(\frac{x+h+1-x-1}{2} \right) \right] \\
&= \lim_{n \rightarrow 0} \frac{1}{h} \left[2 \cos \left(\frac{2x+h+2}{2} \right) \sin \left(\frac{h}{2} \right) \right]
\end{aligned}$$

$$\begin{aligned}
&= \lim_{n \rightarrow 0} \left[\cos\left(\frac{2x+h+2}{2}\right) \cdot \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \right] \\
&= \lim_{n \rightarrow 0} \frac{1}{h} \cos\left(\frac{2x+h+2}{2}\right) \cdot \lim_{\frac{h}{2} \rightarrow 0} \frac{\sin\left(\frac{h}{2}\right)}{h} \frac{h}{\left(\frac{h}{2}\right)} \quad \left[\text{As } h \rightarrow 0 \Rightarrow \frac{h}{2} \rightarrow 0 \right] \\
&= \cos\left(\frac{2x+0+2}{2}\right) \cdot 1 \quad \left[\lim_{h \rightarrow 0} \frac{\sin x}{x} = 1 \right] \\
&= \cos(x+1)
\end{aligned}$$

(iv) Let $f(x) = \cos\left(x - \frac{\pi}{8}\right)$ Accordingly, $f(x+h) = \cos\left(x+h - \frac{\pi}{8}\right)$

By first principle,

$$\begin{aligned}
f'(x) &= \lim_{n \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left[\cos\left(x+h - \frac{\pi}{8}\right) - \cos\left(x - \frac{\pi}{8}\right) \right] \\
&= \lim_{n \rightarrow 0} \frac{1}{h} \left[-2 \sin \frac{\left(x+h - \frac{\pi}{8} + x - \frac{\pi}{8}\right)}{2} \sin \left(\frac{x+h - \frac{\pi}{8} - x + \frac{\pi}{8}}{2} \right) \right] \\
&= \lim_{n \rightarrow 0} \frac{1}{h} \left[-2 \sin \left(\frac{2x+h - \frac{\pi}{4}}{2} \right) \sin \left(\frac{h}{2} \right) \right] \\
&= \lim_{n \rightarrow 0} \left[-\sin \left(\frac{2x+h - \frac{\pi}{4}}{2} \right) \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \right] \\
&= \lim_{n \rightarrow 0} \left[-\sin \left(\frac{2x+h - \frac{\pi}{4}}{2} \right) \right] \cdot \lim_{\frac{h}{2} \rightarrow 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \quad \left[\text{As } h \rightarrow 0 \Rightarrow \frac{h}{2} \rightarrow 0 \right] \\
&= -\sin \left(\frac{2x+0 - \frac{\pi}{4}}{2} \right) \cdot 1 \\
&= -\sin \left(x - \frac{\pi}{8} \right)
\end{aligned}$$

Question 2

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $(x + a)$

Solution

Ans: Let $f(x) = x + a$. Accordingly, $f(x + h) = x + h + a$ By first principle,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$$

$$\lim_{h \rightarrow 0} \frac{x + h + a - x - a}{h}$$

$$\lim_{h \rightarrow 0} \left(\frac{h}{h} \right)$$

$$= \lim_{h \rightarrow 0} (1)$$

$$= 1$$

Question 3

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $(px + q) \left(\frac{r}{x} + s \right)$

Solution

Ans: Let $f(x) = (px + q) \left(\frac{r}{x} + s \right)$

By Leibnitz product rule.

$$f'(x) = (px + q) \left(\frac{r}{x} + s \right)' + \left(\frac{r}{x} + s \right) (px + q)'$$

$$= (px + q) (rx^{-1} + s)' + \left(\frac{r}{x} + s \right) (p)$$

$$= (px + q) (-rx^{-2}) + \left(\frac{r}{x} + s \right) p$$

$$\begin{aligned}
&= (px + q) \left(\frac{-r}{x^2} \right) + \left(\frac{r}{x} + s \right) p \\
&= \frac{-px}{x} - \frac{qr}{x^2} + \frac{pr}{x} + ps \\
&ps - \frac{qr}{x^2}
\end{aligned}$$

Question 4

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $(ax + b)(cx + d)^2$

Solution

Ans: Let $f'(x) = (ax + b)(cx + d)^2$

By Leibnitz product rule,

$$\begin{aligned}
f'(x) &= (ax + b) \frac{d}{dx} (cx + d)^2 \frac{d}{dx} (ax + b) \\
&= (ax + b) \frac{d}{dx} (c^2x^2 + 2cdx^2) + (cx + d)^2 \frac{d}{dx} (ax + b) \\
&= (ax + b) \left[\frac{d}{dx} (c^2x^2) + \frac{d}{dx} (2cdx) + \frac{d}{dx} d^2 \right] + (cx + d)^2 \left[\frac{d}{dx} ax + \frac{d}{dx} b \right] \\
&= (ax + b) (2c^2x + 2cd) + (cx + d)^2 a \\
&= 2c(ax + b)(cx + d) + a(cx + d)^2
\end{aligned}$$

Question 5

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non zero constants and m and n are integers): $\frac{ax + b}{cx + d}$

Solution

Ans: Let $f(x) = \frac{ax + b}{cx + d}$

By quotient rule,

$$\begin{aligned}
 f(x) &= \frac{(cx + d) \frac{d}{dx}(ax + b) - (ax + b) \frac{d}{dx}(cx + d)}{(cx + d)^2} \\
 &= \frac{(cx + d)(a) - (ax + d)(c)}{(cx + d)^2} \\
 &= \frac{acx + ad - acx - bc}{(cx + d)^2} \\
 &= \frac{ad - bc}{(cx + d)^2}
 \end{aligned}$$

Question 6

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed

non-zero constants and m and n are integers): $\frac{1 + \frac{1}{x}}{1 - \frac{1}{x}}$

Solution

$$\text{Ans: Let } f(x) = \frac{1 + \frac{1}{x}}{1 - \frac{1}{x}} = \frac{\frac{x+1}{x}}{\frac{x-1}{x}} = \frac{x+1}{x-1}, \text{ where } x \neq 0$$

$$\text{By quotient rule, } f'(x) = \frac{(x-1) \frac{d}{dx}(x+1) - (x+1) \frac{d}{dx}(x-1)}{(x-1)^2}, x \neq 0, 1$$

$$= \frac{(x-1)(1) - (x+1)(1)}{(x-1)^2}, x \neq 0, 1$$

$$= \frac{x-1-x-1}{(x-1)^2}, x \neq 0, 1$$

$$= \frac{-2}{(x-1)^2}, x \neq 0, 1$$

Question 7

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers) : $\frac{1}{ax^2 + bx + c}$

Solution

Ans: Let $f(x) = \frac{1}{ax^2 + bx + c}$

By quotient rule,

$$f'(x) = \frac{(ax^2 + bx + c) \frac{d}{dx}(1) - \frac{d}{dx}(ax^2 + bx + c)}{(ax^2 + bx + c)^2}$$

$$\frac{(ax^2 + bx + c)(0) - (2ax + b)}{(ax^2 + bx + c)^2}$$

$$\frac{-(2ax + b)}{(ax^2 + bx + c)^2}$$

Question 8

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{ax + b}{px^2 + qx + r}$

Solution

Ans: Let $f(x) = \frac{ax + b}{px^2 + qx + r}$

By quotient rule,

$$f'(x) = \frac{(px^2 + qx + r) \frac{d}{dx}(ax + b) - (ax + b) \frac{d}{dx}(px^2 + qx + r)}{(px^2 + qx + r)^2}$$

$$= \frac{(px^2 + qx + r)(a) - (ax + b)(2px + q)}{(px^2 + qx + r)^2}$$

$$= \frac{apx^2 + aqx + ar - aqx - 2np x^2 - bq}{(px^2 + qx + r)^2}$$

$$\frac{-apx^2 + 2bpx + ar - bq}{(px^2 + qx + r)^2}$$

Question 9

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{px^2 + qx + r}{ax + b}$

Solution

Ans: Let $f(x) = \frac{px^2 + qx + r}{ax + b}$

By the quotient rule,

$$f'(x) = \frac{(ax + b) \frac{d}{dx}(px^2 + qx + r) - (px^2 + qx + r) \frac{d}{dx}(ax + b)}{(ax + b)^2}$$

$$\frac{(ax + b)(2px + q) - (px^2 + qx + r)(a)}{(ax + b)^2}$$

$$= \frac{2apx^2 + aqx + 2bpx + bq - aqx^2 - aqx - ar}{(ax + b)^2}$$

$$\frac{apx^2 + 2bpx + bq - ar}{(ax + b)^2}$$

Question 10

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{a}{x^4} - \frac{b}{x^2} + \cos x$

Solution

Ans: Let $f(x) = \frac{a}{x^4} - \frac{b}{x^2} + \cos x$

$$f'(x) = \frac{d}{dx} \left(\frac{a}{x^4} \right) - \frac{d}{dx} \left(\frac{b}{x^2} \right) + \frac{d}{dx} (\cos x)$$

$$a \frac{d}{dx} (x^{-4}) - b \frac{d}{dx} (x^2) + \frac{d}{dx} (\cos x)$$

$$-a(-4x^{-5}) - b(2x) + (-\sin x) \quad \left[\frac{d}{dx} (x^n) = nx^{n-1} \text{ and } \frac{d}{dx} (\cos x) = -\sin x \right]$$

$$\frac{-4a}{x^5} + \frac{2b}{x} - \sin x$$

Question 11

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed nonzero constants and m and n are integers): $4\sqrt{x} - 2$

Solution

Ans: Let $f(x) = 4\sqrt{x} - 2$

$$f'(x) = \frac{d}{dx} (4\sqrt{x} - 2) = \frac{d}{dx} (4\sqrt{x}) - \frac{d}{dx} (2)$$

$$= 4 \frac{d}{dx} \left(x^{\frac{1}{2}} \right) - 0 = 4 \left(\frac{1}{2} x^{-\frac{1}{2}} \right)$$

$$= \left(2x^{-\frac{1}{2}} \right) = \frac{2}{\sqrt{x}}$$

Question 12

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $(ax + b)^n$

Solution

Ans: Let $f(x) = (ax + b)^n$. Accordingly, $f(x + h) = \{a(x + h) + b\}^n = (ax + ah + b)^n$

By first principle,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$$

$$\begin{aligned}
&= \lim_{h \rightarrow 0} \frac{(ax + ah + b) - (ax + b)^n}{h} \\
&= \lim_{h \rightarrow 0} \frac{(ax + b)^n \left(1 + \frac{ah}{ax + b}\right)^n - (ax + b)^n}{h} \\
&= (ax + b)^n \lim_{h \rightarrow 0} \frac{1}{h} \left[\left\{ 1 + n \left(\frac{ah}{ax + b} \right) + \frac{n(n-1)}{2} \left(\frac{ah}{ax + b} \right)^2 + \dots \right\} - 1 \right] \quad (\text{using binomial theorem}) \\
&= (ax + b)^n \lim_{h \rightarrow 0} \frac{1}{h} \left[\left(\frac{ah}{ax + b} \right) + \frac{n(n-1)a^2h^2}{2(ax + b)^2} + \dots \text{(Terms containing higher degrees of } h) \right] \\
&= (ax + b)^n \lim_{h \rightarrow 0} \left[\frac{na}{(ax + b)} + \frac{n(n-1)a^2h}{2(ax + b)^2} + \dots \right] \\
&= (ax + b)^n \left[\frac{na}{(ax + b)} + 0 \right] \\
&= na \frac{(ax + b)^n}{ax + b} \\
&= na(ax + b)^{n-1}
\end{aligned}$$

Question 13

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $(ax + b)^n(cx + d)^m$

Solution

Ans: Let $f(x) = (ax + b)^n(cx + d)^m$

By Leibnitz product rule,

$$f'(x) = (ax + b)^n \frac{d}{dx}(cx + d)^m + (cx + d)^m \frac{d}{dx}(ax + b)^n$$

Now let $f_1(x) = (cx + d)^m$

$$f_1(x + h) = (cx + ch + d)^m$$

$$\begin{aligned}
f'_1(x) &= \lim_{h \rightarrow 0} \frac{f_1(x+h) - f_1(x)}{h} \\
&= \lim_{h \rightarrow 0} \frac{(cx + ch + d)^m - (cx + d)^m}{h} \\
&= (cx + d)^m \lim_{h \rightarrow 0} \frac{1}{h} \left[\left(1 + \frac{ch}{cx + d} \right)^m - 1 \right] \\
&= (cx + d)^m \lim_{h \rightarrow 0} \frac{1}{h} \left[\left(1 + \frac{mch}{(cx + d)} + \frac{m(m-1)}{2} \frac{c^2 h^2}{(cx + d)^2} + \dots \right)^m - 1 \right] \\
&= (cx + d)^m \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{mch}{(cx + d)} + \frac{m(m-1)c^2 h^2}{2(cx + d)^2} + \dots \text{(Terms containing higher degree of } h) \right] \\
&= (cx + d)^m \lim_{h \rightarrow 0} \left[\frac{mc}{(cx + d)} + \frac{m(m-1)c^2 h}{2(cx + d)^2} + \dots \right] \\
&= (cx + d)^m \left[\frac{mch}{(cx + d)} + 0 \right] \\
&= \frac{mc(cx + d)^m}{(cx + d)} \\
&= mc(cx + d)^{m-1} \\
\frac{d}{dx}(cx + d)^m &= md(x + d)^{m-1}
\end{aligned}$$

Similarly, $\frac{d}{dx}(ax + b)^n = na(ax + b)^{n-1} \dots (3)$

Therefore, from (1), (2), and (3), we obtain

$$\begin{aligned}
f'(x) &= (ax + b)^n \{ mc(cx + d)^{m-1} \} + (c + d)^m \{ na(ax + b)^{n-1} \} \\
&= (ax + b)^{n-1} (cx + d)^{m-1} [mc(ax + b) + na(cx + d)]
\end{aligned}$$

Question 14

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\sin(x + a)$

Solution

Ans: Let, $f(x) = \sin(x + a)$

$$f(x + h) = \sin(x + h + a)$$

By first principle, $f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$

$$= \lim_{h \rightarrow 0} \frac{\sin(x + h + a) - \sin(x + a)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[2 \cos\left(\frac{x + h + a + x + a}{2}\right) \sin\left(\frac{x + h + a - x - a}{2}\right) \right]$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[2 \cos\left(\frac{2x + 2a + h}{2}\right) \sin\left(\frac{h}{2}\right) \right]$$

$$= \lim_{h \rightarrow 0} \left[\cos\left(\frac{2x + 2a + h}{2}\right) \left[\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \right] \right]$$

$$= \lim_{h \rightarrow 0} \cos\left(\frac{2x + 2a + h}{2}\right) \cdot \lim_{\frac{h}{2} \rightarrow 0} \left[\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \right] \quad \left[\text{As } h \rightarrow 0 \Rightarrow \frac{h}{2} \rightarrow 0 \right]$$

$$= \cos\left(\frac{2x + 2a}{2}\right) \times 1 \quad \left[\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \right]$$

$$= \cos(x + a)$$

Question 15

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\operatorname{cosec} x \cot x$

Solution

Ans: Let $f(x) = \operatorname{cosec} x \cot x$

By Leibnitz product rule,

$$f'(x) = \operatorname{cosec} x (\cot x)' + \cot x (\operatorname{cosec} x)' \dots (1)$$

Let $f_1(x) = \cot x$. Accordingly, $f_1(x + h) = \cot(x + h)$

By first principle,

$$\begin{aligned}
 f''(x) &= \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cot(x + h) - \cot(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{\cos(x + h)}{\sin(x + h)} - \frac{\cos(x)}{\sin x} \right) \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{\sin x \cos(x + h) - \cos x \sin(x + h)}{\sin x \sin(x + h)} \right) \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{\sin(x - x + h)}{\sin x \sin(x + h)} \right) \\
 &= \frac{1}{\sin x} \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(-h)}{\sin(x + h)} \right] \\
 &= \frac{-1}{\sin x} \left(\lim_{h \rightarrow 0} \frac{\sin h}{h} \right) \left(\lim_{h \rightarrow 0} \frac{1}{\sin(x + h)} \right) \\
 &= \frac{-1}{\sin x} \cdot 1 \cdot \left(\lim_{h \rightarrow 0} \frac{1}{\sin(x + 0)} \right) \\
 &= \frac{-1}{\sin^2 x} \\
 &= -\operatorname{cosec}^2 x
 \end{aligned}$$

$$\therefore (\cot x)' = -\operatorname{cosec}^2 x \quad \dots (2)$$

Now, let $f_2(x) = \operatorname{cosec} x$. Accordingly, $f_2(x + h) = \operatorname{cosec}(x + h)$

$$\begin{aligned}
 \text{By first principle, } f_2(x) &= \lim_{h \rightarrow 0} \frac{f_2(x + h) - f_2(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} [\operatorname{cosec}(x + h) - \operatorname{cosec}(x)] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{1}{\sin(x + h)} - \frac{1}{\sin x} \right) \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{\sin x - \sin(x + h)}{\sin x \sin(x + h)} \right)
 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\sin x} \cdot \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{2 \cos\left(\frac{x+x+h}{2}\right) \sin\left(\frac{x-x-h}{2}\right)}{\sin(x+h)} \right] \\
&= \frac{1}{\sin x} \cdot \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{2 \cos\left(\frac{2x+h}{2}\right) \sin\left(\frac{-h}{2}\right)}{\sin(x+h)} \right] \\
&= \frac{1}{\sin x} \cdot \lim_{h \rightarrow 0} \left[\frac{-\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \cdot \frac{\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)} \right] \\
&= \frac{-1}{\sin x} \cdot \lim_{h \rightarrow 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \cdot \lim_{h \rightarrow 0} \frac{\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)} \\
&= \frac{-1}{\sin x} \cdot 1 \cdot \frac{\cos\left(\frac{2x+h}{2}\right)}{\sin(x+0)} \\
&= \frac{-1}{\sin x} \cdot \frac{\cos x}{\sin x} \\
&= -\operatorname{cosec} x \cdot \cot x
\end{aligned}$$

$$\therefore (\operatorname{cosec} x)' = -\operatorname{cosec} x \cdot \cot x$$

From (1), (2), and (3), we obtain

$$\begin{aligned}
f'(x) &= \operatorname{cosec} x (-\operatorname{cosec}^2 x) + \cot x (-\operatorname{cosec} x \cot x) \\
&= -\operatorname{cosec}^3 x - \cot^2 x \operatorname{cosec} x
\end{aligned}$$

Question 16

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{\cos x}{1 + \sin x}$

Solution

Ans: Let $f(x) = \frac{\cos x}{1 + \sin x}$

By the quotient rule,

$$\begin{aligned}
 f'(x) &= \frac{(1 + \sin x) \frac{d}{dx}(\cos x) - (\cos x) \frac{d}{dx}(1 + \sin x)}{(1 + \sin x)^2} \\
 &= \frac{(1 + \sin x)(-\sin x) - (\cos x)(\cos x)}{(1 + \sin x)^2} \\
 &= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1 + \sin x)^2} \\
 &= \frac{-\sin x - (\sin^2 x + \cos^2 x)}{(1 + \sin x)^2} \\
 &= \frac{-\sin x - 1}{(1 + \sin x)^2} \\
 &= \frac{-(1 + \sin x)}{(1 + \sin x)^2} \\
 &= \frac{-1}{(1 + \sin x)}
 \end{aligned}$$

Question 17

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed

non-zero constants and m and n are integers): $\frac{\sin x + \cos x}{\sin x - \cos x}$

Solution

Ans:17: Let $f(x) = \frac{\sin x + \cos x}{\sin x - \cos x}$

By the quotient rule,

$$f''(x) = \frac{(\sin x - \cos x) \frac{d}{dx}(\sin x + \cos x) - (\sin x + \cos x) \frac{d}{dx}(\sin x - \cos x)}{(\sin x - \cos x)^2}$$

$$\begin{aligned}
&= \frac{(\sin x - \cos x)(\cos x - \sin x) - (\sin x + \cos x)(\cos x + \sin x)}{(\sin x + \cos x)^2} \\
&= \frac{-(\sin x - \cos x)^2 - (\sin x + \cos x)^2}{(\sin x + \cos x)^2} \\
&= \frac{-[\sin^2 x + \cos^2 x - 2 \sin x \cos x + \sin^2 x + \cos^2 x + 2 \sin x \cos x]}{(\sin x + \cos x)^2} \\
&= \frac{-[1 + 1]}{(\sin x - \cos x)^2} \\
&= \frac{-2}{(\sin x - \cos x)^2}
\end{aligned}$$

Question 18

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{\sec x - 1}{\sec x + 1}$

Solution

Ans: Let $f(x) = \frac{\sec x - 1}{\sec x + 1}$

$$f(x) = \frac{\frac{1}{\cos x} - 1}{\frac{1}{\cos x} + 1} = \frac{1 - \cos x}{1 + \cos x}$$

By the quotient rule,

$$\begin{aligned}
f'(x) &= \frac{(1 + \cos x) \frac{d}{dx}(1 - \cos x) - (1 - \cos x) \frac{d}{dx}(1 + \cos x)}{(1 + \cos x)^2} \\
&= \frac{(1 + \cos x)(\sin x) - (1 - \cos x)(-\sin x)}{(1 + \cos x)^2} \\
&= \frac{\sin x + \cos x \sin x + \sin x - \sin x \cos x}{(1 + \cos x)^2} \\
&= \frac{2 \sin x}{(1 + \cos x)^2}
\end{aligned}$$

$$\begin{aligned}
&= \frac{2 \sin x}{\left(1 + \frac{1}{\sec x}\right)^2} = \frac{2 \sin x}{\frac{(\sec x + 1)^2}{\sec^2 x}} \\
&= \frac{2 \sin x \sec^2 x}{(\sec x + 1)^2} \\
&= \frac{\frac{2 \sin x}{\cos x} \sec x}{(\sec x + 1)^2} \\
&= \frac{2 \sec x \tan x}{(\sec x + 1)^2}
\end{aligned}$$

Question 19

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\sin^n x$

Solution

Ans: Let $y = \sin^n x$

Accordingly, for $n = 1, y = \sin x$

$$\therefore \frac{dy}{dx} = \cos x, \text{ i.e., } \frac{d}{dx} \sin x = \cos x$$

For $n = 2, y = \sin^2 x$.

$$\therefore \frac{dy}{dx} = \frac{d}{dx} (\sin x \sin x)$$

$$= (\sin x)' (\sin x + \sin x (\sin x)') \quad (\text{By Leibnitz product rule})$$

$$= \cos x \sin x + \sin x \cos x$$

$$= 2 \sin x \cos x$$

$$\dots (1)$$

For $n = 3, y = \sin^3 x$

$$\therefore \frac{dy}{dx} = \frac{d}{dx} (\sin x \sin^2 x)$$

$$= (\sin x)' \sin^2 x + \sin x (\sin x)'$$

(By Leibnitz product rule)

$$= \cos x \sin^2 x + \sin x (2 \sin x \cos x) \quad [\text{Using (1)}]$$

$$= \cos x \sin^2 x + \sin^2 x \cos x$$

$$= 3 \sin^2 x \cos x$$

We assert that $\frac{d}{dx}(\sin^n x) = n \sin^{(n-1)} x \cos x$

Let our assertion be true for $n = k$.

i.e., $\frac{d}{dx}(\sin^k x) = k \sin^{(k-1)} x \cos x \quad \dots (2)$

Consider,

$$\frac{d}{dx}(\sin^{k+1} x) = \frac{d}{dx}(\sin x \sin^{(k)} x)$$

$$= (\sin x)' \sin^k x + \sin x (\sin^k x)'$$

(By Leibnitz product rule)

$$= \cos x \sin^k x + \sin x (k \sin^{k-1} \cos x) \quad [\text{Using (2)}]$$

$$= \cos x \sin^k x + 2 \sin^k x \cos x$$

$$= (k+1) \sin^k x \cos x$$

Thus, our assertion is true for $n = k + 1$.

Hence, by mathematical induction, $\frac{d}{dx}(\sin^n x) = n \sin^{(n-1)} x \cos x$

Question 20

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{a + b \sin x}{c + d \cos x}$

Solution

Ans: Let $f(x) = \frac{a + b \sin x}{c + d \cos x}$

By the quotient rule,

$$\begin{aligned} f'(x) &= \frac{(c + d \cos x) \frac{d}{dx}(a + b \sin x) - (a + b \sin x) \frac{d}{dx}(c + d \cos x)}{(c + d \cos x)^2} \\ &= \frac{(c + d \cos x)(b \cos x) - (a + b \sin x)(-d \sin x)}{(c + d \cos x)^2} \\ &= \frac{cb \cos x + bd \cos^2 x + ad \sin x + bd \sin^2 x}{(c + d \cos x)^2} \\ &= \frac{bc \cos x + ad \sin x + bd (\cos^2 x + \sin^2 x)}{(c + d \cos x)^2} \\ &= \frac{bc \cos x + ad \sin x + bd}{(c + d \cos x)^2} \end{aligned}$$

Question 21

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed

non-zero constants and m and n are integers): $\frac{\sin(x + a)}{\cos x}$

Solution

Ans: Let $f(x) = \frac{\sin(x + a)}{\cos x}$

By the quotient rule,

$$\begin{aligned} f'(x) &= \frac{\cos x \frac{d}{dx}[\sin(x + a)] - \sin(x + a) \frac{d}{dx} \cos x}{\cos^2 x} \\ f'(x) &= \frac{\cos x \frac{d}{dx}[\sin(x + a)] - \sin(x + a) \frac{d}{dx}(-\sin x)}{\cos^2 x} \end{aligned}$$

Let $g(x) = \sin(x + a)$. Accordingly, $g(x + h) = \sin(x + h + a)$

By first principle,

$$\begin{aligned}
 g'(x) &= \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \\
 &= \lim_{n \rightarrow 0} \frac{1}{h} [\sin(x+h+a) - \sin(x+a)] \\
 &= \lim_{n \rightarrow 0} \frac{1}{h} \left[2 \cos\left(\frac{x+h+a+x+a}{2}\right) \sin\left(\frac{x+h+a-x-a}{2}\right) \right] \\
 &= \lim_{n \rightarrow 0} \frac{1}{h} \left[2 \cos\left(\frac{2x+2a+h}{2}\right) \sin\left(\frac{h}{2}\right) \right] \\
 &= \lim_{n \rightarrow 0} \left[\cos\left(\frac{2x+2a+h}{h}\right) \left\{ \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \right\} \right] \\
 &= \lim_{n \rightarrow 0} \cos\left(\frac{2x+2a+h}{h}\right) \cdot \lim_{n \rightarrow 0} \left\{ \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \right\} \quad \left[\text{As } h \rightarrow 0 \Rightarrow \frac{h}{2} \rightarrow 0 \right] \\
 &= \left(\cos \frac{2x+2a}{2} \right) \times 1 \quad \left[\lim_{n \rightarrow 0} \frac{\sin h}{h} = 1 \right] \\
 &= \cos(x+a) \quad \dots \text{(ii)}
 \end{aligned}$$

From (i) and (ii), we obtain $f'(x) = \frac{\cos x \cos(x+a) + \sin x \sin(x+a)}{\cos^2 x}$

$$\begin{aligned}
 &= \frac{\cos(x+a-x)}{\cos^2 x} \\
 &= \frac{\cos a}{\cos^2 x}
 \end{aligned}$$

Question 22

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers) : $x^4(5 \sin x - 3 \cos x)$

Solution

Ans: Let $f(x) = x^4(5 \sin x - 3 \cos x)$

By product rule.

$$\begin{aligned}
 f'(x) &= x^4 \frac{d}{dx} (5 \sin x - 3 \cos x) + (5 \sin x - 3 \cos x) \frac{d}{dx} (x^4) \\
 &= x^4 \left[5 \frac{d}{dx} (\sin x) - 3 \frac{d}{dx} (\cos x) \right] + (5 \sin x - 3 \cos x) \frac{d}{dx} (x^4) \\
 &= x^4 [5 \cos x - 3(-\sin x)] + (5 \sin x - 3 \cos x) (4x^3) \\
 &= x^3 [5x \cos x + 3x \sin x + 20 \sin x - 12 \cos x]
 \end{aligned}$$

Question 23

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $(x^2 + 1) \cos x$

Solution

Ans: Let $f(x) = (x^2 + 1) \cos x$

By product rule.

$$\begin{aligned}
 f'(x) &= (x^2 + 1) \frac{d}{dx} (\cos x) + \cos x \frac{d}{dx} (x^2 + 1) \\
 &= (x^2 + 1) (-\sin x) + \cos x (2x) \\
 &= -x^2 \sin x - \sin x + 2x \cos x
 \end{aligned}$$

Question 24

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $(ax^2 + \sin x) (p + q \cos x)$

Solution

Ans: Let $f(x) = (ax^2 + \sin x) (p + q \cos x)$

By product rule.

$$f'(x) = (ax^2 + \sin x) \frac{d}{dx} (p + q \cos x) + (p + q \cos x) \frac{d}{dx} (ax^2 + \sin x)$$

$$\begin{aligned}
&= (ax^2 + \sin x)(-q \sin x) + (p + q \cos x)(2ax + \cos x) \\
&= -q \sin x (ax^2 + \sin x) + (p + q \cos x)(2ax + \cos x)
\end{aligned}$$

Question 25

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $(x + \cos x)(x - \tan x)$

Solution

Ans: Let $f(x) = (x + \cos x)(x - \tan x)$

By product rule,

$$\begin{aligned}
f'(x) &= (x + \cos x) \frac{d}{dx}(x - \tan x) + (x - \tan x) \frac{d}{dx}(x + \cos x) \\
&= (x + \cos x) \left[\frac{d}{dx}(x) - \frac{d}{dx}(\tan x) \right] + (x - \tan x)(1 - \sin x) \\
&= (x + \cos x) \left[1 - \frac{d}{dx}(\tan x) \right] + (x - \tan x)(1 - \sin x)
\end{aligned}$$

Let $g(x) = \tan x$. Accordingly, $g(x + h) = \tan(x + h)$

By first principle,

$$\begin{aligned}
g''(x) &= \lim_{h \rightarrow 0} \frac{g(x + h) - g(x)}{h} \\
&= \lim_{h \rightarrow 0} \frac{\tan(x + h) - \tan(x)}{h} \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(x + h)}{\cos(x + h)} - \frac{\sin x}{\cos x} \right] \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(x + h) \cos x - \sin x \cos(x + h)}{\cos x \cos(x + h)} \right] \\
&= \frac{1}{\cos x} \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(x + h - x)}{\cos(x + h)} \right] \\
&= \frac{1}{\cos x} \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin h}{\cos(x + h)} \right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\cos x} \left(\lim_{h \rightarrow 0} \frac{\sin h}{h} \right) \left(\lim_{h \rightarrow 0} \frac{1}{\cos(x+h)} \right) \\
&= \frac{1}{\cos x} \cdot \left(\frac{1}{\cos(x+0)} \right) \\
&= \frac{1}{\cos^2 x} \\
&= \sec^2 x \quad \dots \text{(ii)}
\end{aligned}$$

Therefore, from (i) and (ii). We obtain

$$\begin{aligned}
f'(x) &= (x + \cos x)(1 - \sec^2 x) + (x - \tan x)(1 - \sin x) \\
&= (x + \cos x)(-\tan^2 x) + (x - \tan x)(1 - \sin x) \\
&= -\tan^2 x(x + \cos x) + (x - \tan x)(1 - \sin x)
\end{aligned}$$

Question 26

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{4x + 5 \sin x}{3x + 7 \cos x}$

Solution

Ans: Let $f(x) = \frac{4x + 5 \sin x}{3x + 7 \cos x}$

Quotient rule,

$$\begin{aligned}
f'(x) &= \frac{(3x + 7 \cos x) \frac{d}{dx}(4x + 5 \sin x) - (4x + 5 \sin x) \frac{d}{dx}(3x + 7 \cos x)}{(3x + 7 \cos x)^2} \\
&= \frac{(3x + 7 \cos x) \left[4 \frac{d}{dx}(x) + 5 \frac{d}{dx}(\sin x) \right] - (4x + 5 \sin x) \left[3 \frac{d}{dx}(x) + 7 \frac{d}{dx}(\cos x) \right]}{(3x + 7 \cos x)^2} \\
&= \frac{(3x + 7 \cos x)[4x + 5 \cos x] - (4x + 5 \sin x)[3 - 7 \sin x]}{(3x + 7 \cos x)^2} \\
&= \frac{12x + 15x \cos x + 28x \cos x + 35 \cos^2 x - 12x + 28x \sin x - 15 \sin x + 35 (\cos^2 x + \sin^2 x)}{(3x + 7 \cos x)^2}
\end{aligned}$$

$$\begin{aligned}
 &= \frac{15x \cos x + 28 \cos x + 28x \sin x - 15 \sin x + 35 (\cos^2 x + \sin^2 x)}{(3x + 7 \cos x)^2} \\
 &= \frac{35 + 15x \cos x + 28 \cos x + 28x \sin x - 15 \sin x}{(3x + 7 \cos x)^2}
 \end{aligned}$$

Question 27

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed

non-zero constants and m and n are integers): $\frac{x^2 \cos\left(\frac{\pi}{4}\right)}{\sin x}$

Solution

Ans: Let $f(x) = \frac{x^2 \cos\left(\frac{\pi}{4}\right)}{\sin x}$

By quotient rule, $f'(x) = \cos\left(\frac{\pi}{4}\right) \left[\frac{\sin x \frac{d}{dx}(x^2) - x^2 \frac{d}{dx}(\sin x)}{\sin^2 x} \right]$

$$= \cos\left(\frac{\pi}{4}\right) \left[\frac{\sin x(2x) - x^2(\cos x)}{\sin^2 x} \right]$$

$$= \frac{x \cos \frac{\pi}{4} [2 \sin x - x \cos x]}{\sin^2 x}$$

Question 28

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed

non-zero constants and m and n are integers): $\frac{x}{1 + \tan x}$

Solution

Ans: Let $f(x) = \frac{x}{1 + \tan x}$

$$f(x) = \frac{(1 + \tan x) \frac{d}{dx}(x) - (x) \frac{d}{dx}(1 + \tan x)}{(1 + \tan x)^2}$$

$$f'(x) = \frac{(1 + \tan x) - x \frac{d}{dx}(1 + \tan x)}{(1 + \tan x)^2}$$

Let $g(x) = 1 + \tan x$. Accordingly $g(x + h) = 1 + \tan(x + h)$.

By first principle, $\dot{g}(x) = \lim_{h \rightarrow 0} \frac{g(x + h) - g(x)}{h}$

$$\lim_{h \rightarrow 0} \left[\frac{1 + \tan(x + h) - 1 - \tan(x)}{h} \right]$$

$$\lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(x + h)}{\cos(x + h)} - \frac{\sin x}{\cos x} \right]$$

$$\lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(x + h) \cos x - \sin x \cos(x + h)}{\cos(x + h) \cos x} \right]$$

$$\lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(x + h - x)}{\cos(x + h) \cos x} \right]$$

$$\lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sinh}{\cos(x + h) \cos x} \right]$$

$$- \left(\lim_{h \rightarrow 0} \frac{\sinh}{h} \right) \cdot \left(\lim_{h \rightarrow 0} \frac{1}{\cos(x + h) \cos x} \right)$$

$$-1 \times \frac{1}{\cos^2} = \sec^2 x$$

$$\Rightarrow \frac{d}{dx}(1 + \tan^2 x) = \sec^2 x$$

From (i) and (ii), we obtain

$$\dot{f}(x) = \frac{1 + \tan x - x \sec^2 x}{(1 + \tan x)^2}$$

Question 29

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $(x + \sec x)(x - \tan x)$

Solution

Ans: Let $f(x) = (x + \sec x)(x - \tan x)$

By product rule.

$$\begin{aligned} f(x) &= (x + \sec x) \frac{d}{dx}(x - \tan x) + (x - \tan x) \frac{d}{dx}(x + \sec x) \\ &= (x + \sec x) \left[\frac{d}{dx}(x) - \frac{d}{dx} \tan x \right] + (x - \tan x) \left[\frac{d}{dx}(x) + \frac{d}{dx} \sec x \right] \\ &= (x + \sec x) \left[1 - \frac{d}{dx} \tan x \right] + (x - \tan x) \left[1 + \frac{d}{dx} \sec x \right] \\ &\dots \text{(i)} \end{aligned}$$

Let $f_1(x) = \tan x$, $f_2(x) = \sec x$

Accordingly, $f_1(x + h) = \tan(x + h)$ and $f_2(x + h) = \sec(x + h)$

$$\begin{aligned} f_1'(x) &= \lim_{h \rightarrow 0} \left(\frac{f_1(x + h) - f_1(x)}{h} \right) \\ &= \lim_{h \rightarrow 0} \left[\frac{\tan(x + h) - \tan(x)}{h} \right] \end{aligned}$$

$$\begin{aligned}
& - \lim_{n \rightarrow 0} \frac{1}{h} \left[\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x} \right] \\
& - \lim_{n \rightarrow 0} \frac{1}{h} \left[\frac{\sin(x+h) \cos x - \sin x \cos(x+h)}{\cos(x+h) \cos x} \right] \\
& - \lim_{n \rightarrow 0} \frac{1}{h} \left[\frac{\sin(x+h-x)}{\cos(x+h) \cos x} \right] \\
& - \lim_{n \rightarrow 0} \frac{1}{h} \left[\frac{\sin h}{\cos(x+h) \cos x} \right] \\
& - \left(\lim_{n \rightarrow 0} \frac{\sin h}{h} \right) \cdot \left(\lim_{n \rightarrow 0} \frac{1}{\cos(x+h) \cos x} \right) \\
& - 1 \times \frac{1}{\cos^2} = \sec^2 x \\
& \Rightarrow \frac{d}{dx} (1 + \tan^2 x) = \sec^2 x \\
f'_2(x) &= \lim_{n \rightarrow 0} \left(\frac{f_2(x+h) - f_2(x)}{h} \right) \\
&= \lim_{h \rightarrow 0} \left(\frac{\sec(x+h) - \sec(x)}{h} \right) \\
&= \lim_{n \rightarrow 0} \frac{1}{h} \left(\frac{1}{\cos(x+h)} - \frac{1}{\cos x} \right) \\
&= \lim_{n \rightarrow 0} \frac{1}{h} \left(\frac{\cos x - \cos(x+h)}{\cos(x+h) \cos x} \right) \\
&= \frac{1}{\cos x} \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-2 \sin\left(\frac{x+x+h}{2}\right) \cdot \sin\left(\frac{x-x-h}{2}\right)}{\cos(x+h)} \right] \\
&= \frac{1}{\cos x} \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-2 \sin\left(\frac{2x+h}{2}\right) \cdot \sin\left(\frac{-h}{2}\right)}{\cos(x+h)} \right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\cos x} \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin\left(\frac{2x+h}{2}\right) \left\{ \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \right\}}{\cos(x+h)} \right] \\
&= \sec x \frac{\left\{ \lim_{h \rightarrow 0} \sin\left(\frac{2x+h}{2}\right) \right\} \left\{ \lim_{\frac{h}{2} \rightarrow 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \right\}}{\lim_{h \rightarrow 0} \cos(x+h)} \\
&= \sec x \cdot \frac{\sin x \cdot 1}{\cos x} \\
&\Rightarrow \frac{d}{dx} \sec x = \sec x \tan x
\end{aligned}$$

From (i), (ii), and (iii), we obtain

$$f'(x) = (x + \sec x)(1 - \sec^2 x) + (x - \tan x)(1 + \sec x \tan x)$$

Question 30

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{x}{\sin^n x}$

Solution

Ans: Let $f(x) = \frac{x}{\sin^n x}$

By quotient rule, $f'(x) = \frac{\sin^n x \frac{d}{dx} x - x \frac{d}{dx} \sin^n x}{\sin^{2n} x}$

It can be easily shown that $\frac{d}{dx} \sin^n x = n \sin^{n-1} x \cos x$

Therefore,

$$\begin{aligned}f'(x) &= \frac{\sin^n x \times \frac{d}{dx} x - x \frac{d}{dx} \sin^n x}{\sin^{2n} x} \\&= \frac{\sin^n x \cdot 1 - x (n \sin^{n-1} x \cos x)}{\sin^{2n} x} \\&= \frac{\sin^{n-1} x (\sin x - nx \cos x)}{\sin^{2n} x} \\&= \frac{\sin x - nx \cos x}{\sin^{n+1} x}\end{aligned}$$
