

Chapter 10: Conic Sections

NCERT · Class 11 · Maths

Exercise 10.1

Question 1

Find the equation of the circle with centre (0,2) and radius 2

Solution

Ans: The equation of a circle with centre (h,k) and radius (r) is given as

$$(x - h)^2 + (y - k)^2 = r^2$$

It is given that centre $(h, k) = (0, 2)$ and radius $(r) = 2$.

Therefore, the equation of the circle is

$$(x - 0)^2 + (y - 2)^2 = 2^2$$

$$x^2 + y^2 + 4 - 4y = 4$$

$$x^2 + y^2 - 4y = 0$$

Question 2

Find the equation of the circle with centre (-2,3) and radius 4

Solution

Ans: The equation of a circle with centre (h,k) and radius (r) is given as

$$(x - h)^2 + (y - k)^2 = r^2$$

It is given that centre $(h,k)=(-2,3)$ and radius $(r)=4$.

Therefore, the equation of the circle is

$$(x + 2)^2 + (y - 3)^2 = (4)^2$$

$$x^2 + 4x + 4 + y^2 - 6y + 9 = 16$$

$$x^2 + y^2 + 4x - 6y - 3 = 0$$

Question 3

Find the equation of the circle with centre $\left(\frac{1}{2}, \frac{1}{4}\right)$ and radius $\left(\frac{1}{12}\right)$

Solution

Ans: The equation of a circle with centre (h,k) and radius (r) is given as

$$(x - h)^2 + (y - k)^2 = r^2$$

It is given that centre $(h, k) = \left(\frac{1}{2}, \frac{1}{4}\right)$ and radius $(r) = \left(\frac{1}{12}\right)$.

Therefore, the equation of the circle is

$$\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{4}\right)^2 = \left(\frac{1}{12}\right)^2$$

$$x^2 - x + \frac{1}{4} + y^2 - \frac{y}{2} + \frac{1}{16} = \frac{1}{144}$$

$$x^2 - x + \frac{1}{4} + y^2 - \frac{y}{2} + \frac{1}{16} - \frac{1}{144} = 0$$

$$144x^2 - 144x + 36 + 144y^2 - 72y + 9 - 1 = 0 \text{ (Solve by taking LCM)}$$

$$144x^2 - 144x + 144y^2 - 72y + 44 = 0$$

$$36x^2 - 36x + 36y^2 - 18y + 11 = 0$$

$$36x^2 + 36y^2 - 36x - 18y + 11 = 0$$

Question 4

Find the equation of the circle with centre (1,1) and radius $\sqrt{2}$

Solution

Ans: The equation of a circle with centre (h,k) and radius (r) is given as

$$(x - h)^2 + (y - k)^2 = r^2$$

It is given that centre (h,k)=(1,1) and radius $(r) = \sqrt{2}$.

Therefore, the equation of the circle is

$$(x - 1)^2 + (y - 1)^2 = (\sqrt{2})^2$$

$$x^2 - 2x + 1 + y^2 - 2y + 1 = 2$$

$$x^2 + y^2 - 2x - 2y = 0$$

Question 5

Find the equation of the circle with centre (-a,-b) and radius $\sqrt{a^2 - b^2}$

Solution

Ans: The equation of a circle with centre (h,k) and radius (r) is given as

$$(x - h)^2 + (y - k)^2 = r^2$$

It is given that centre (h,k)=(-a,-b) and radius $(r) = \sqrt{a^2 - b^2}$.

Therefore, the equation of the circle is

$$(x + a)^2 + (y + b)^2 = (\sqrt{a^2 - b^2})^2$$

$$x^2 + 2ax + a^2 + y^2 + 2by + b^2 = a^2 - b^2$$

$$x^2 + y^2 + 2ax + 2by + 2b^2 = 0$$

Question 6

Find the centre and radius of the circle $(x + 5)^2 + (y - 3)^2 = 36$

Solution

Ans: The equation of the given circle is $(x + 5)^2 + (y - 3)^2 = 36$

$$(x + 5)^2 + (y - 3)^2 = 36$$

$\Rightarrow \{x - (-5)\}^2 + (y - 3)^2 = (6)^2$, which is the form of $(x - h)^2 + (y - k)^2 = r^2$, where $h = -5, k = 3$, and $r = 6$.

Thus, the centre of the given circle is $(-5, 3)$, while its radius is 6.

Question 7

Find the centre and radius of the circle $x^2 + y^2 - 4x - 8y - 45 = 0$

Solution

Ans: The equation of the given circle is $x^2 + y^2 - 4x - 8y - 45 = 0$

$$\Rightarrow x^2 + y^2 - 4x - 8y - 45 = 0$$

$$\Rightarrow (x^2 - 4x) + (y^2 - 8y) = 45$$

$$\Rightarrow \{x^2 - 2(x)(2) + (2)^2\} + \{y^2 - 2(y)(4) + (4)^2\} - 4 - 16 = 45$$

$$\Rightarrow (x - 2)^2 + (y - 4)^2 = 65$$

$\Rightarrow (x - 2)^2 + (y - 4)^2 = (\sqrt{65})^2$, which is of the form $(x - h)^2 + (y - k)^2 = r^2$, where $h = 2, k = 4$, and $r = \sqrt{65}$

Thus, the centre of the given circle is $(2, 4)$, while its radius is $\sqrt{65}$.

Question 8

Find the centre and radius of the circle $x^2 + y^2 - 8x + 10y - 12 = 0$

Solution

Ans: The equation of the given circle is $x^2 + y^2 - 8x + 10y - 12 = 0$

$$\Rightarrow x^2 + y^2 - 8x + 10y - 12 = 0$$

$$\Rightarrow (x^2 - 8x) + (y^2 + 10y) = 12$$

$$\Rightarrow \{x^2 - 2(x)(4) + (4)^2\} + \{y^2 + 2(y)(5) + (5)^2\} - 16 - 25 = 12$$

$$\Rightarrow (x - 4)^2 + (y + 5)^2 = 53$$

$$\Rightarrow (x - 4)^2 + \{y - (-5)\}^2 = (\sqrt{53})^2, \text{ which is of the form } (x - h)^2 + (y - k)^2 = r^2, \text{ where } h = 4, k = -5, \text{ and } r = \sqrt{53}$$

Thus, the centre of the given circle is $(4, -5)$, while its radius is $\sqrt{53}$.

Question 9

Find the centre and radius of the circle $2x^2 + 2y^2 - x = 0$

Solution

Ans: The equation of the given circle is $2x^2 + 2y^2 - x = 0$

$$\Rightarrow 2x^2 + 2y^2 - x = 0$$

$$\Rightarrow (2x^2 - x) + 2y^2 = 0$$

$$\Rightarrow 2 \left[\left(x^2 - \frac{x}{2} \right) + y^2 \right] = 0$$

$$\Rightarrow \left\{ x^2 - 2.x \left(\frac{1}{4} \right) + \left(\frac{1}{4} \right)^2 \right\} + y^2 - \left(\frac{1}{4} \right)^2 = 0$$

$$\Rightarrow \left(x - \frac{1}{4} \right)^2 + (y - 0)^2 = \left(\frac{1}{4} \right)^2, \text{ which is of the form } (x - h)^2 + (y - k)^2 = r^2, \text{ where } h = \frac{1}{4}, k = 0, \text{ and } r = \frac{1}{4}$$

Thus, the centre of the given circle is $\left(\frac{1}{4}, 0 \right)$, while its radius is $\frac{1}{4}$.

Question 10

Find the equation of the circle passing through the points $(4, 1)$ and $(6, 5)$ and whose centre is on the line $4x + y = 16$

Solution

Ans: Let the equation of the required circle be $(x - h)^2 + (y - k)^2 = r^2$

Since the circle passes through points $(4, 1)$ and $(6, 5)$,

$$(4 - h)^2 + (1 - k)^2 = r^2 \dots\dots\dots(i)$$

$$(6 - h)^2 + (5 - k)^2 = r^2 \dots\dots\dots(ii)$$

Since the centre (h, k) of the circle lies on line $4x + y = 16$,

$$4h + k = 16 \dots\dots\dots(iii)$$

From equations (i) and (ii), we get

$$\Rightarrow (4 - h)^2 + (1 - k)^2 = (6 - h)^2 + (5 - k)^2$$

$$\Rightarrow 16 - 8h + h^2 + 1 - 2k + k^2 = 36 - 12h + h^2 + 25 - 10k + k^2$$

$$\Rightarrow 16 - 8h + 1 - 2k = 36 - 12h + 25 - 10k$$

$$\Rightarrow 4h + 8k = 44$$

$$\Rightarrow h + 2k = 11 \dots\dots\dots(iv)$$

On solving equations (iii) and (iv), we obtain $h = 3$ and $k = 4$

On substituting the values of h and k in equation (i), we obtain

$$(4 - 3)^2 + (1 - 4)^2 = r^2$$

$$\Rightarrow (1)^2 + (-3)^2 = r^2$$

$$\Rightarrow 1 + 9 = r^2$$

$$\Rightarrow r^2 = 10$$

$$\Rightarrow r = \sqrt{10}$$

Thus, the equation of the required circle is

$$(x - 3)^2 + (y - 4)^2 = (\sqrt{10})^2$$

$$x^2 - 6x + 9 + y^2 - 8y + 16 = 10$$

$$x^2 + y^2 - 6x - 8y + 15 = 0$$

Question 11

Find the equation of the circle passing through the points (2,3) and (-1,1) and whose centre is on the line $x - 3y - 11 = 0$

Solution

Ans: Let the equation of the required circle be $(x - h)^2 + (y - k)^2 = r^2$

Since the circle passes through points (2, 3) and (-1, 1),

$$(2 - h)^2 + (3 - k)^2 = r^2 \dots\dots\dots(i)$$

$$(-1 - h)^2 + (1 - k)^2 = r^2 \dots\dots\dots(ii)$$

Since the centre (h,k) of the circle lies on line $x - 3y - 11 = 0$

$$h - 3k = 11 \dots\dots\dots(iii)$$

From equations (i) and (ii), we get

$$\Rightarrow (2 - h)^2 + (3 - k)^2 = (-1 - h)^2 + (1 - k)^2$$

$$\Rightarrow 4 - 4h + h^2 + 9 - 6k + k^2 = 1 + 2h + h^2 + 1 - 2k + k^2$$

$$\Rightarrow 4 - 4h + 9 - 6k = 1 + 2h + 1 - 2k$$

$$\Rightarrow 6h + 4k = 11 \dots\dots\dots(iv)$$

On solving equations (iii) and (iv), we obtain $h = \frac{7}{2}$ and $k = \frac{-5}{2}$

On substituting the values of h and k in equation (i), we obtain

$$\Rightarrow \left(2 - \frac{7}{2}\right)^2 + \left(3 + \frac{5}{2}\right)^2 = r^2$$

$$\Rightarrow \left(\frac{4 - 7}{2}\right)^2 + \left(\frac{6 + 5}{2}\right)^2 = r^2$$

$$\Rightarrow \left(\frac{-3}{2}\right)^2 + \left(\frac{11}{2}\right)^2 = r^2$$

$$\Rightarrow \frac{9}{4} + \frac{121}{4} = r^2$$

$$\Rightarrow \frac{130}{4} = r^2$$

Thus, the equation of the required circle is

$$\left(x - \frac{7}{2}\right)^2 + \left(y + \frac{5}{2}\right)^2 = \frac{130}{4}$$

$$\left(\frac{2x-7}{2}\right)^2 + \left(\frac{2y+5}{2}\right)^2 = \frac{130}{4}$$

$$4x^2 - 28x + 49 + 4y^2 + 20y + 25 = 130$$

$$4x^2 + 4y^2 - 28x + 20y - 56 = 0$$

$$4(x^2 + y^2 - 7x + 5y - 14) = 0$$

$$x^2 + y^2 - 7x + 5y - 14 = 0$$

Question 12

Find the equation of the circle with radius 5 whose centre lies on x-axis and passes through the point (2,3).

Solution

Ans: Let the equation of the required circle be $(x - h)^2 + (y - k)^2 = r^2$

Since the radius of the circle is 5 and its centre lies on the x-axis, $k=0$ and $r=5$.

Now, the equation of the circle passes through point (2, 3).

$$\therefore (2 - h)^2 + 3^2 = 25$$

$$\Rightarrow (2 - h)^2 = 25 - 9$$

$$\Rightarrow (2 - h)^2 = 16$$

$$\Rightarrow (2 - h) = \pm\sqrt{16}$$

$$= \pm 4$$

If $2 - h = 4$, then $h = -2$

If $2 - h = -4$, then $h = 6$

When $h = -2$, the equation of the circle becomes

$$(x + 2)^2 + y^2 = 25$$

$$x^2 + 4x + 4 + y^2 = 25$$

$$x^2 + y^2 + 4x - 21 = 0$$

When $h = 6$, the equation of the circle becomes

$$(x - 6)^2 + y^2 = 25$$

$$x^2 - 12x + 36 + y^2 = 25$$

$$x^2 + y^2 - 12x + 11 = 0$$

So, the equation of the circle can be $x^2 + y^2 + 4x - 21 = 0$ or $x^2 + y^2 - 12x + 11 = 0$

Question 13

Find the equation of the circle passing through (0,0) and making intercepts a and b on the coordinate axes.

Solution

Ans: Let the equation of the required circle be $(x - h)^2 + (y - k)^2 = r^2$

Since the circle passes through (0,0).

$$(0 - h)^2 + (0 - k)^2 = r^2$$

$$\Rightarrow h^2 + k^2 = r^2$$

The equation of the circle now becomes $(x - h)^2 + (y - k)^2 = h^2 + k^2$

It is given that the circle makes intercepts a and b on the coordinate axes. This means that the circle passes through points (a, 0) and (0, b). Therefore,

$$(a - h)^2 + (0 - k)^2 = h^2 + k^2 \dots\dots(1)$$

$$(0 - h)^2 + (b - k)^2 = h^2 + k^2 \dots\dots(2)$$

From equation (1), we'll get

$$a^2 - 2ah + h^2 + k^2 = h^2 + k^2$$

$$\Rightarrow a^2 - 2ah = 0$$

$$\Rightarrow a(a - 2h) = 0$$

$$\Rightarrow a = 0 \text{ or } (a - 2h) = 0$$

$$\text{However, } a \neq 0; \text{ hence, } (a - 2h) = 0 \Rightarrow h = \frac{a}{2}$$

From equation (2), we'll get

$$h^2 + b^2 - 2bk + k^2 = h^2 + k^2$$

$$\Rightarrow b^2 - 2bk = 0$$

$$\Rightarrow b(b - 2k) = 0$$

$$\Rightarrow b = 0 \text{ or } (b - 2k) = 0$$

$$\text{However, } b \neq 0; \text{ hence, } (b - 2k) = 0 \Rightarrow k = \frac{b}{2}$$

Thus, the equation of the required circle is

$$\left(x - \frac{a}{2}\right)^2 + \left(y - \frac{b}{2}\right)^2 = \left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2$$

$$\left(\frac{2x - a}{2}\right)^2 + \left(\frac{2y - b}{2}\right)^2 = \frac{a^2 + b^2}{4}$$

$$\Rightarrow 4x^2 - 4ax + a^2 + 4y^2 - 4by + b^2 = a^2 + b^2$$

$$\Rightarrow 4x^2 + 4y^2 - 4ax - 4by = 0$$

$$\Rightarrow x^2 + y^2 - ax - by = 0$$

Question 14

Find the equation of a circle with centre (2,2) and passes through the point (4,5).

Solution

Ans: The centre of the circle is given as (h,k)=(2,2)

Since the circle passes through point (4,5), the radius (r) of the circle is the distance between the points (2,2) and (4,5).

$$\begin{aligned}
 \therefore r &= \sqrt{(2-4)^2 + (2-5)^2} \\
 &= \sqrt{(-2)^2 + (-3)^2} \\
 &= \sqrt{4+9} \\
 &= \sqrt{13}
 \end{aligned}$$

Thus, the equation of the circle is

$$\begin{aligned}
 (x-h)^2 + (y-k)^2 &= r^2 \\
 (x-2)^2 + (y-2)^2 &= (\sqrt{13})^2 \\
 x^2 - 4x + 4 + y^2 - 4y + 4 &= 13 \\
 x^2 + y^2 - 4x - 4y - 5 &= 0
 \end{aligned}$$

Question 15

Does the point (-2.5, 3.5) lie inside, outside or on the circle $x^2 + y^2 = 25$?

Solution

Ans: The equation of the given circle is $x^2 + y^2 = 25$.

$$x^2 + y^2 = 25$$

$\Rightarrow (x-0)^2 + (y-0)^2 = 5^2$, which is of the form of $(x-h)^2 + (y-k)^2 = r^2$, where $h=0, k=0$ and $r=5$

$\therefore$ Centre = (0,0) and radius = 5

Distance between point (-2.5, 3.5) and centre (0,0)

$$\begin{aligned}
 &= \sqrt{(-2.5-0)^2 + (3.5-0)^2} \\
 &= \sqrt{6.25 + 12.25} \\
 &= \sqrt{18.25} \\
 &= 4.3 \text{ (approx.)} < 5
 \end{aligned}$$

Since the distance between point (-2.5, 3.5) and centre (0,0) of the circle is less than the radius of the circle, point (-2.5, 3.5) lies inside the circle.

Exercise 10.2

Question 1

Find the coordinates of the focus, axis of the parabola, equation of the directrix and length of the latus rectum.

Given: $y^2 = 12x$

Solution

Solution: Compare the given equation with the standard form:

$$y^2 = 4ax$$

So,

$$4a = 12$$

$$a = 3$$

Now,

$$\text{Focus} = (a, 0) = (3, 0)$$

$$\text{Axis} = x\text{-axis } (y = 0)$$

$$\text{Directrix} = x = -a = x = -3$$

$$\text{Length of latus rectum} = 4a = 12$$

Answer:

$$\text{Focus: } (3, 0)$$

Axis: x-axis

$$\text{Directrix: } x = -3$$

Length of latus rectum: 12 units

Question 2**Given:** $x^2 = 6y$ **Solution**

Solution:

Compare with:

$$x^2 = 4ay$$

$$4a = 6$$

$$a = 3/2$$

Therefore,

$$\text{Focus} = (0, 3/2)$$

Axis = y-axis

$$\text{Directrix} = y = -3/2$$

$$\text{Length of latus rectum} = 6$$

Answer:

$$\text{Focus: } (0, 3/2)$$

Axis: y-axis

$$\text{Directrix: } y = -3/2$$

$$\text{Length of latus rectum: } 6 \text{ units}$$

Question 3**Given:** $y^2 = -8x$ **Solution**

Solution:

Compare with:

$$y^2 = -4ax$$

$$4a = 8$$

$$a = 2$$

Hence,

$$\text{Focus} = (-2, 0)$$

Axis = x-axis

$$\text{Directrix} = x = 2$$

$$\text{Length of latus rectum} = 8$$

Answer:

$$\text{Focus: } (-2, 0)$$

Axis: x-axis

$$\text{Directrix: } x = 2$$

$$\text{Length of latus rectum: } 8 \text{ units}$$

Question 4

Given: $x^2 = -16y$

Solution

Solution:

Compare with:

$$x^2 = -4ay$$

$$4a = 16$$

$$a = 4$$

Therefore,

$$\text{Focus} = (0, -4)$$

Axis = y-axis

Directrix = $y = 4$

Length of latus rectum = 16

Answer:

Focus: $(0, -4)$

Axis: y-axis

Directrix: $y = 4$

Length of latus rectum: 16 units

Question 5

Given: $y^2 = 10x$

Solution

Solution:

Compare with:

$$y^2 = 4ax$$

$$4a = 10$$

$$a = 5/2$$

Thus,

$$\text{Focus} = (5/2, 0)$$

Axis = x-axis

$$\text{Directrix} = x = -5/2$$

Length of latus rectum = 10

Answer:

$$\text{Focus: } (5/2, 0)$$

Axis: x-axis

Directrix: $x = -5/2$

Length of latus rectum: 10 units

Question 6

Given: $x^2 = -9y$

Solution

Solution:

Compare with:

$$x^2 = -4ay$$

$$4a = 9$$

$$a = 9/4$$

Hence,

$$\text{Focus} = (0, -9/4)$$

Axis = y-axis

$$\text{Directrix} = y = 9/4$$

$$\text{Length of latus rectum} = 9$$

Answer:

$$\text{Focus: } (0, -9/4)$$

Axis: y-axis

$$\text{Directrix: } y = 9/4$$

$$\text{Length of latus rectum: 9 units}$$

Question 7

Find the equation of the parabola whose focus is (6, 0) and directrix is $x = -6$.

Solution

Solution:

For a parabola opening rightward,

$$\text{Focus} = (a, 0)$$

$$\text{Directrix} = x = -a$$

Comparing,

$$a = 6$$

Using the standard equation:

$$y^2 = 4ax$$

$$y^2 = 4 \times 6 \times x$$

$$y^2 = 24x$$

$$\text{Answer: } y^2 = 24x$$

Question 8

Find the equation of the parabola whose focus is $(0, -3)$ and directrix is $y = 3$.

Solution

Solution:

The parabola opens downward.

Compare with:

$$\text{Focus} = (0, -a)$$

$$\text{Directrix} = y = a$$

Therefore,

$$a = 3$$

Standard equation:

$$x^2 = -4ay$$

$$x^2 = -12y$$

Answer: $x^2 = -12y$

Question 9

Find the equation of the parabola whose vertex is (0, 0) and focus is (3, 0).

Solution

Solution:

$$\text{Focus} = (a, 0)$$

Therefore,

$$a = 3$$

Equation:

$$y^2 = 4ax$$

$$y^2 = 12x$$

Answer: $y^2 = 12x$

Question 10

Find the equation of the parabola whose vertex is (0, 0) and focus is (-2, 0).

Solution

Solution:

$$\text{Focus} = (-a, 0)$$

$$a = 2$$

Equation:

$$y^2 = -4ax$$

$$y^2 = -8x$$

Answer: $y^2 = -8x$

Question 11

Find the equation of the parabola whose vertex is (0, 0), passing through (2, 3) and axis is x-axis.

Solution

Solution:

Since the axis is the x-axis, the equation is

$$y^2 = 4ax$$

Substitute point (2, 3):

$$3^2 = 4a \times 2$$

$$9 = 8a$$

$$a = 9/8$$

Equation:

$$y^2 = 4 \times (9/8)x$$

$$y^2 = (9/2)x$$

Multiplying by 2:

$$2y^2 = 9x$$

Answer: $2y^2 = 9x$

Question 12

Find the equation of the parabola whose vertex is (0, 0), passes through (5, 2) and is symmetric about y-axis.

Solution

Solution:

A parabola symmetric about the y-axis has the equation:

$$x^2 = 4ay$$

Substitute point (5, 2):

$$25 = 4a \times 2$$

$$25 = 8a$$

$$a = 25/8$$

Equation:

$$x^2 = 4 \times (25/8)y$$

$$x^2 = (25/2)y$$

Multiplying by 2:

$$2x^2 = 25y$$

Answer: $2x^2 = 25y$

Exercise 10.3

Question 1

$$x^2/36 + y^2/16 = 1$$

Solution

Ans: The given equation is $x^2/36 + y^2/16 = 1$.

Here, the denominator of $x^2/36$ is greater than the denominator of $y^2/16$. Therefore, the major axis is along the x-axis, and the minor axis is along the y-axis.

Comparing with the standard equation $x^2/a^2 + y^2/b^2 = 1$, we get $a^2 = 36$ and $b^2 = 16$, so $a = 6$ and $b = 4$.

Now, $c = \sqrt{a^2 - b^2} = \sqrt{36 - 16} = \sqrt{20} = 2\sqrt{5}$

- Coordinates of the foci: $(2\sqrt{5}, 0)$ and $(-2\sqrt{5}, 0)$
- Coordinates of the vertices: $(6, 0)$ and $(-6, 0)$
- Length of major axis = $2a = 12$
- Length of minor axis = $2b = 8$
- Eccentricity, $e = c/a = 2\sqrt{5}/6 = \sqrt{5}/3$
- Length of latus rectum = $2b^2/a = (2 \times 16)/6 = 16/3$

Question 2

$$x^2/4 + y^2/25 = 1$$

Solution

Ans: The given equation is $x^2/4 + y^2/25 = 1$.

Here, the denominator of $y^2/25$ is greater than the denominator of $x^2/4$. Therefore, the major axis is along the y-axis, and the minor axis is along the x-axis.

Comparing with $x^2/b^2 + y^2/a^2 = 1$, we get $a^2 = 25$ and $b^2 = 4$, so $a = 5$ and $b = 2$.

$$\text{Now, } c = \sqrt{a^2 - b^2} = \sqrt{25 - 4} = \sqrt{21}$$

- Coordinates of the foci: $(0, \sqrt{21})$ and $(0, -\sqrt{21})$
- Coordinates of the vertices: $(0, 5)$ and $(0, -5)$
- Length of major axis = $2a = 10$
- Length of minor axis = $2b = 4$
- Eccentricity, $e = c/a = \sqrt{21}/5$
- Length of latus rectum = $2b^2/a = (2 \times 4)/5 = 8/5$

Question 3

$$x^2/16 + y^2/9 = 1$$

Solution

Ans: The denominator of $x^2/16$ is greater than that of $y^2/9$, so the major axis is along the x-axis.

Comparing with $x^2/a^2 + y^2/b^2 = 1$, we get $a = 4$ and $b = 3$.

Now, $c = \sqrt{a^2 - b^2} = \sqrt{16 - 9} = \sqrt{7}$

- Coordinates of the foci: $(\sqrt{7}, 0)$ and $(-\sqrt{7}, 0)$
- Coordinates of the vertices: $(4, 0)$ and $(-4, 0)$
- Length of major axis $= 2a = 8$
- Length of minor axis $= 2b = 6$
- Eccentricity, $e = c/a = \sqrt{7}/4$
- Length of latus rectum $= 2b^2/a = (2 \times 9)/4 = 9/2$

Question 4

$$x^2/25 + y^2/100 = 1$$

Solution

Ans: The denominator of $y^2/100$ is greater than that of $x^2/25$, so the major axis is along the y-axis.

Comparing with $x^2/b^2 + y^2/a^2 = 1$, we get $a = 10$ and $b = 5$.

Now, $c = \sqrt{a^2 - b^2} = \sqrt{100 - 25} = \sqrt{75} = 5\sqrt{3}$

- Coordinates of the foci: $(0, 5\sqrt{3})$ and $(0, -5\sqrt{3})$
- Coordinates of the vertices: $(0, 10)$ and $(0, -10)$
- Length of major axis $= 2a = 20$
- Length of minor axis $= 2b = 10$
- Eccentricity, $e = c/a = 5\sqrt{3}/10 = \sqrt{3}/2$
- Length of latus rectum $= 2b^2/a = (2 \times 25)/10 = 5$

Question 5

$$x^2/49 + y^2/36 = 1$$

Solution

Ans: The denominator of $x^2/49$ is greater than that of $y^2/36$, so the major axis is along the x-axis.

Comparing with $x^2/a^2 + y^2/b^2 = 1$, we get $a = 7$ and $b = 6$.

$$\text{Now, } c = \sqrt{a^2 - b^2} = \sqrt{49 - 36} = \sqrt{13}$$

- Coordinates of the foci: $(\sqrt{13}, 0)$ and $(-\sqrt{13}, 0)$
- Coordinates of the vertices: $(7, 0)$ and $(-7, 0)$
- Length of major axis $= 2a = 14$
- Length of minor axis $= 2b = 12$
- Eccentricity, $e = c/a = \sqrt{13}/7$
- Length of latus rectum $= 2b^2/a = (2 \times 36)/7 = 72/7$

Question 6

$$x^2/100 + y^2/400 = 1$$

Solution

Ans: The denominator of $y^2/400$ is greater than that of $x^2/100$, so the major axis is along the y-axis.

Comparing with $x^2/b^2 + y^2/a^2 = 1$, we get $a = 20$ and $b = 10$.

$$\text{Now, } c = \sqrt{a^2 - b^2} = \sqrt{400 - 100} = \sqrt{300} = 10\sqrt{3}$$

- Coordinates of the foci: $(0, 10\sqrt{3})$ and $(0, -10\sqrt{3})$
- Coordinates of the vertices: $(0, 20)$ and $(0, -20)$
- Length of major axis $= 2a = 40$

- Length of minor axis = $2b = 20$
- Eccentricity, $e = c/a = 10\sqrt{3}/20 = \sqrt{3}/2$
- Length of latus rectum = $2b^2/a = (2 \times 100)/20 = 10$

Question 7

$$36x^2 + 4y^2 = 144$$

Solution

Ans: Dividing the entire equation by 144, we get:

$$36x^2/144 + 4y^2/144 = 1, \text{ which gives } x^2/4 + y^2/36 = 1$$

The denominator of $y^2/36$ is greater than that of $x^2/4$, so the major axis is along the y-axis.

Comparing with $x^2/b^2 + y^2/a^2 = 1$, we get $a = 6$ and $b = 2$.

$$\text{Now, } c = \sqrt{a^2 - b^2} = \sqrt{36 - 4} = \sqrt{32} = 4\sqrt{2}$$

- Coordinates of the foci: $(0, 4\sqrt{2})$ and $(0, -4\sqrt{2})$
- Coordinates of the vertices: $(0, 6)$ and $(0, -6)$
- Length of major axis = $2a = 12$
- Length of minor axis = $2b = 4$
- Eccentricity, $e = c/a = 4\sqrt{2}/6 = 2\sqrt{2}/3$
- Length of latus rectum = $2b^2/a = (2 \times 4)/6 = 4/3$

Question 8

$$16x^2 + y^2 = 16$$

Solution

Ans: Dividing the entire equation by 16, we get:

$$x^2/1 + y^2/16 = 1$$

The denominator of $y^2/16$ is greater than that of $x^2/1$, so the major axis is along the y-axis.

Comparing with $x^2/b^2 + y^2/a^2 = 1$, we get $a = 4$ and $b = 1$.

$$\text{Now, } c = \sqrt{a^2 - b^2} = \sqrt{16 - 1} = \sqrt{15}$$

- Coordinates of the foci: $(0, \sqrt{15})$ and $(0, -\sqrt{15})$
- Coordinates of the vertices: $(0, 4)$ and $(0, -4)$
- Length of major axis = $2a = 8$
- Length of minor axis = $2b = 2$
- Eccentricity, $e = c/a = \sqrt{15}/4$
- Length of latus rectum = $2b^2/a = (2 \times 1)/4 = \frac{1}{2}$

Question 9

$$4x^2 + 9y^2 = 36$$

Solution

Ans: Dividing the entire equation by 36, we get:

$$x^2/9 + y^2/4 = 1$$

The denominator of $x^2/9$ is greater than that of $y^2/4$, so the major axis is along the x-axis.

Comparing with $x^2/a^2 + y^2/b^2 = 1$, we get $a = 3$ and $b = 2$.

$$\text{Now, } c = \sqrt{a^2 - b^2} = \sqrt{9 - 4} = \sqrt{5}$$

- Coordinates of the foci: $(\sqrt{5}, 0)$ and $(-\sqrt{5}, 0)$
- Coordinates of the vertices: $(3, 0)$ and $(-3, 0)$
- Length of major axis = $2a = 6$
- Length of minor axis = $2b = 4$
- Eccentricity, $e = c/a = \sqrt{5}/3$

- Length of latus rectum = $2b^2/a = (2 \times 4)/3 = 8/3$

In each of the following Exercises 10 to 20, find the equation for the ellipse that satisfies the given conditions:

Question 10

Vertices $(\pm 5, 0)$, foci $(\pm 4, 0)$

Solution

Ans: Since the vertices are on the x-axis, the major axis is along the x-axis, and the equation of the ellipse is of the form $x^2/a^2 + y^2/b^2 = 1$.

From the vertices $(\pm 5, 0)$, we get $a = 5$. From the foci $(\pm 4, 0)$, we get $c = 4$.

Using the relation $c^2 = a^2 - b^2$: $16 = 25 - b^2$ $b^2 = 25 - 16 = 9$

Therefore, the equation of the ellipse is $x^2/25 + y^2/9 = 1$

Question 11

Vertices $(0, \pm 13)$, foci $(0, \pm 5)$

Solution

Ans: Since the vertices are on the y-axis, the major axis is along the y-axis, and the equation is of the form $x^2/b^2 + y^2/a^2 = 1$.

From the vertices $(0, \pm 13)$, we get $a = 13$. From the foci $(0, \pm 5)$, we get $c = 5$.

Using $c^2 = a^2 - b^2$: $25 = 169 - b^2$ $b^2 = 169 - 25 = 144$

Therefore, the equation of the ellipse is $x^2/144 + y^2/169 = 1$

Question 12

Vertices $(\pm 6, 0)$, foci $(\pm 4, 0)$

Solution

Ans: Since the vertices are on the x-axis, the equation is of the form $x^2/a^2 + y^2/b^2 = 1$.

Here, $a = 6$ and $c = 4$.

Using $c^2 = a^2 - b^2$: $16 = 36 - b^2$ $b^2 = 36 - 16 = 20$

Therefore, the equation of the ellipse is $x^2/36 + y^2/20 = 1$

Question 13

Ends of major axis $(\pm 3, 0)$, ends of minor axis $(0, \pm 2)$

Solution

Ans: Since the ends of the major axis lie on the x-axis, the major axis is along the x-axis, and the equation is of the form $x^2/a^2 + y^2/b^2 = 1$.

From the ends of the major axis $(\pm 3, 0)$, we get $a = 3$. From the ends of the minor axis $(0, \pm 2)$, we get $b = 2$.

Therefore, the equation of the ellipse is $x^2/9 + y^2/4 = 1$

Question 14

Ends of major axis $(0, \pm\sqrt{5})$, ends of minor axis $(\pm 1, 0)$

Solution

Ans: Since the ends of the major axis lie on the y-axis, the major axis is along the y-axis, and the equation is of the form $x^2/b^2 + y^2/a^2 = 1$.

From the ends of the major axis $(0, \pm\sqrt{5})$, we get $a = \sqrt{5}$, so $a^2 = 5$. From the ends of the minor axis $(\pm 1, 0)$, we get $b = 1$, so $b^2 = 1$.

Therefore, the equation of the ellipse is $x^2/1 + y^2/5 = 1$

Question 15**Length of major axis 26, foci $(\pm 5, 0)$** **Solution**

Ans: Since the foci lie on the x-axis, the major axis is along the x-axis, and the equation is of the form $x^2/a^2 + y^2/b^2 = 1$.

Given, length of major axis = $2a = 26$, so $a = 13$. From the foci $(\pm 5, 0)$, we get $c = 5$.

Using $c^2 = a^2 - b^2$: $25 = 169 - b^2$ $b^2 = 144$

Therefore, the equation of the ellipse is $x^2/169 + y^2/144 = 1$

Question 16**Length of minor axis 16, foci $(0, \pm 6)$** **Solution**

Ans: Since the foci lie on the y-axis, the major axis is along the y-axis, and the equation is of the form $x^2/b^2 + y^2/a^2 = 1$.

Given, length of minor axis = $2b = 16$, so $b = 8$. From the foci $(0, \pm 6)$, we get $c = 6$.

Using $c^2 = a^2 - b^2$: $36 = a^2 - 64$ $a^2 = 36 + 64 = 100$

Therefore, the equation of the ellipse is $x^2/64 + y^2/100 = 1$

Question 17**Foci $(\pm 3, 0)$, $a = 4$** **Solution**

Ans: Since the foci lie on the x-axis, the major axis is along the x-axis, and the equation is of the form $x^2/a^2 + y^2/b^2 = 1$.

Given, $c = 3$ and $a = 4$.

Using $c^2 = a^2 - b^2$: $9 = 16 - b^2$ $b^2 = 16 - 9 = 7$

Therefore, the equation of the ellipse is $x^2/16 + y^2/7 = 1$

Question 18

$b = 3$, $c = 4$, centre at the origin; foci on the x-axis

Solution

Ans: Since the foci lie on the x-axis, the major axis is along the x-axis, and the equation is of the form $x^2/a^2 + y^2/b^2 = 1$.

Given, $b = 3$ and $c = 4$.

Using the relation $a^2 = b^2 + c^2$: $a^2 = 9 + 16 = 25$

Therefore, the equation of the ellipse is $x^2/25 + y^2/9 = 1$

Question 19

Centre at $(0, 0)$, major axis on the y-axis, and passes through the points $(3, 2)$ and $(1, 6)$

Solution

Ans: Since the major axis is along the y-axis, the equation of the ellipse is of the form:

$$x^2/b^2 + y^2/a^2 = 1 \dots (i)$$

Since the ellipse passes through the point $(3, 2)$, substituting in equation (i): $9/b^2 + 4/a^2 = 1 \dots (1)$

Since the ellipse passes through the point $(1, 6)$, substituting in equation (i): $1/b^2 + 36/a^2 = 1 \dots (2)$

Let $1/b^2 = p$ and $1/a^2 = q$. Then the equations become: $9p + 4q = 1 \dots (3)$ $p + 36q = 1 \dots (4)$

From equation (4): $p = 1 - 36q$

Substituting in equation (3): $9(1 - 36q) + 4q = 1$ $9 - 324q + 4q = 1$ $-320q = -8$ $q = 1/40$, which gives $a^2 = 40$

Now, $p = 1 - 36 \times (1/40) = 1 - 36/40 = 4/40 = 1/10$, which gives $b^2 = 10$

Therefore, the equation of the ellipse is $x^2/10 + y^2/40 = 1$

Question 20

The major axis on the x-axis and passes through the points (4, 3) and (6, 2)

Solution

Ans: Since the major axis is along the x-axis, the equation of the ellipse is of the form:

$$x^2/a^2 + y^2/b^2 = 1 \dots (i)$$

Since the ellipse passes through the point (4, 3): $16/a^2 + 9/b^2 = 1 \dots (1)$

Since the ellipse passes through the point (6, 2): $36/a^2 + 4/b^2 = 1 \dots (2)$

Let $1/a^2 = p$ and $1/b^2 = q$. Then: $16p + 9q = 1 \dots (3)$ $36p + 4q = 1 \dots (4)$

Multiplying equation (3) by 4 and equation (4) by 9: $64p + 36q = 4 \dots (5)$ $324p + 36q = 9 \dots (6)$

Subtracting equation (5) from equation (6): $260p = 5$ $p = 1/52$, which gives $a^2 = 52$

Substituting $p = 1/52$ in equation (3): $16/52 + 9q = 1$ $9q = 1 - 4/13 = 9/13$ $q = 1/13$, which gives $b^2 = 13$

Therefore, the equation of the ellipse is $x^2/52 + y^2/13 = 1$

Exercise 10.4

Question 1

$$x^2/16 - y^2/9 = 1$$

Solutions:

The given equation has a positive x^2 term, so the transverse axis is along the x-axis.

Comparing with the standard equation $x^2/a^2 - y^2/b^2 = 1$, we get $a^2 = 16$ and $b^2 = 9$, so $a = 4$ and $b = 3$.

Now, $c = \sqrt{a^2 + b^2} = \sqrt{16 + 9} = \sqrt{25} = 5$.

Coordinates of the foci = $(\pm 5, 0)$

Coordinates of the vertices = $(\pm 4, 0)$

Eccentricity, $e = c/a = 5/4$

Length of the latus rectum = $2b^2/a = (2 \times 9)/4 = 9/2$

Solution

Solution coming soon.

Question 2

$$y^2/9 - x^2/27 = 1$$

Solutions:

Here the positive term is y^2 , so the transverse axis is along the y-axis.

Comparing with $y^2/a^2 - x^2/b^2 = 1$, we get $a^2 = 9$ and $b^2 = 27$, so $a = 3$ and $b = 3\sqrt{3}$.

Now, $c = \sqrt{a^2 + b^2} = \sqrt{9 + 27} = \sqrt{36} = 6$.

Coordinates of the foci = $(0, \pm 6)$

Coordinates of the vertices = $(0, \pm 3)$

Eccentricity, $e = c/a = 6/3 = 2$

Length of the latus rectum = $2b^2/a = (2 \times 27)/3 = 18$

Solution

Solution coming soon.

Question 3

$$9y^2 - 4x^2 = 36$$

Solutions:

Dividing both sides by 36: $y^2/4 - x^2/9 = 1$.

The transverse axis is along the y-axis. Here $a^2 = 4$, $b^2 = 9$, so $a = 2$ and $b = 3$.

Now, $c = \sqrt{a^2 + b^2} = \sqrt{4 + 9} = \sqrt{13}$.

Coordinates of the foci = $(0, \pm \sqrt{13})$

Coordinates of the vertices = $(0, \pm 2)$

Eccentricity, $e = c/a = \sqrt{13}/2$

Length of the latus rectum = $2b^2/a = (2 \times 9)/2 = 9$

Solution

Solution coming soon.

Question 4

$$16x^2 - 9y^2 = 576$$

Solutions:

Dividing both sides by 576: $x^2/36 - y^2/64 = 1$.

The transverse axis is along the x-axis. Here $a^2 = 36$, $b^2 = 64$, so $a = 6$ and $b = 8$.

Now, $c = \sqrt{a^2 + b^2} = \sqrt{36 + 64} = \sqrt{100} = 10$.

Coordinates of the foci = $(\pm 10, 0)$

Coordinates of the vertices = $(\pm 6, 0)$

Eccentricity, $e = c/a = 10/6 = 5/3$

Length of the latus rectum = $2b^2/a = (2 \times 64)/6 = 64/3$

Solution

Solution coming soon.

Question 5

$$5y^2 - 9x^2 = 36$$

Solutions:

5. $5y^2 - 9x^2 = 36$

Dividing both sides by 36: $y^2/(36/5) - x^2/4 = 1$.

The transverse axis is along the y-axis. Here $a^2 = 36/5$ and $b^2 = 4$, so $a = 6/\sqrt{5}$.

Now, $c^2 = a^2 + b^2 = 36/5 + 4 = 56/5$, so $c = \sqrt{56/5} = 2\sqrt{70/5}$.

Coordinates of the foci = $(0, \pm 2\sqrt{70/5})$

Coordinates of the vertices = $(0, \pm 6/\sqrt{5})$, i.e., $(0, \pm 6\sqrt{5}/5)$

Eccentricity, $e = c/a = (2\sqrt{70/5}) \div (6/\sqrt{5}) = \sqrt{14/3}$

Length of the latus rectum = $2b^2/a = 8 \div (6/\sqrt{5}) = 8\sqrt{5}/6 = 4\sqrt{5}/3$

Solution

Solution coming soon.

Question 6

$$49y^2 - 16x^2 = 784$$

Solutions:

Dividing both sides by 784: $y^2/16 - x^2/49 = 1$.

The transverse axis is along the y-axis. Here $a^2 = 16$, $b^2 = 49$, so $a = 4$ and $b = 7$.

Now, $c = \sqrt{a^2 + b^2} = \sqrt{16 + 49} = \sqrt{65}$.

Coordinates of the foci = $(0, \pm \sqrt{65})$

Coordinates of the vertices = $(0, \pm 4)$

Eccentricity, $e = c/a = \sqrt{65}/4$

Length of the latus rectum = $2b^2/a = (2 \times 49)/4 = 49/2$

In each of the Exercises 7 to 15, find the equations of the hyperbola satisfying the given conditions.

Solution

Solution coming soon.

Question 7

Vertices $(\pm 2, 0)$, foci $(\pm 3, 0)$

Solutions:

Since the vertices and foci lie on the x-axis, the equation is of the form $x^2/a^2 - y^2/b^2 = 1$.

From the vertices, $a = 2$; from the foci, $c = 3$.

Then $b^2 = c^2 - a^2 = 9 - 4 = 5$.

Required equation: $x^2/4 - y^2/5 = 1$

Solution

Solution coming soon.

Question 8

Vertices $(0, \pm 5)$, foci $(0, \pm 8)$

Solutions:

Since the vertices and foci lie on the y-axis, the equation is of the form $y^2/a^2 - x^2/b^2 = 1$.

Here $a = 5$ and $c = 8$, so $b^2 = c^2 - a^2 = 64 - 25 = 39$.

Required equation: $y^2/25 - x^2/39 = 1$

Solution

Solution coming soon.

Question 9

Vertices $(0, \pm 3)$, foci $(0, \pm 5)$

Solutions:

The foci lie on the y-axis, so the equation is of the form $y^2/a^2 - x^2/b^2 = 1$.

Here $a = 3$ and $c = 5$, so $b^2 = c^2 - a^2 = 25 - 9 = 16$.

Required equation: $y^2/9 - x^2/16 = 1$

Solution

Solution coming soon.

Question 10

Foci $(\pm 5, 0)$, the transverse axis is of length 8

Solutions:

The foci lie on the x-axis, so the equation is of the form $x^2/a^2 - y^2/b^2 = 1$.

Length of the transverse axis $= 2a = 8$, so $a = 4$. Also $c = 5$.

Then $b^2 = c^2 - a^2 = 25 - 16 = 9$.

Required equation: $x^2/16 - y^2/9 = 1$

Solution

Solution coming soon.

Question 11

Foci $(0, \pm 13)$, the conjugate axis is of length 24

Solutions:

The foci lie on the y-axis, so the equation is of the form $y^2/a^2 - x^2/b^2 = 1$.

Length of the conjugate axis $= 2b = 24$, so $b = 12$. Also $c = 13$.

Then $a^2 = c^2 - b^2 = 169 - 144 = 25$.

Required equation: $y^2/25 - x^2/144 = 1$

Solution

Solution coming soon.

Question 12

Foci $(\pm 3\sqrt{5}, 0)$, the latus rectum is of length 8

Solutions:

The foci lie on the x-axis, so the equation is of the form $x^2/a^2 - y^2/b^2 = 1$, with $c = 3\sqrt{5}$.

Length of the latus rectum $= 2b^2/a = 8$, which gives $b^2 = 4a$.

Using $c^2 = a^2 + b^2$: $45 = a^2 + 4a$

$a^2 + 4a - 45 = 0 \implies (a + 9)(a - 5) = 0 \implies a = 5$ (rejecting $a = -9$, as a cannot be negative)

Then $b^2 = 4 \times 5 = 20$.

Required equation: $x^2/25 - y^2/20 = 1$

Solution

Solution coming soon.

Question 13

Foci $(\pm 4, 0)$, the latus rectum is of length 12

Solutions:

The foci lie on the x-axis, so the equation is of the form $x^2/a^2 - y^2/b^2 = 1$, with $c = 4$.

Latus rectum: $2b^2/a = 12$, so $b^2 = 6a$.

Using $c^2 = a^2 + b^2$: $16 = a^2 + 6a$

$a^2 + 6a - 16 = 0 \implies (a + 8)(a - 2) = 0 \implies a = 2$ (rejecting $a = -8$)

Then $b^2 = 6 \times 2 = 12$.

Required equation: $x^2/4 - y^2/12 = 1$

Solution

Solution coming soon.

Question 14

Vertices $(\pm 7, 0)$, $e = 4/3$

Solutions:

The vertices lie on the x-axis, so the equation is of the form $x^2/a^2 - y^2/b^2 = 1$, with $a = 7$.

Since $e = c/a = 4/3$, we get $c = 7 \times 4/3 = 28/3$.

Then $b^2 = c^2 - a^2 = 784/9 - 49 = (784 - 441)/9 = 343/9$.

Required equation: $x^2/49 - y^2/(343/9) = 1$, i.e., $x^2/49 - 9y^2/343 = 1$

Solution

Solution coming soon.

Question 15

Foci $(0, \pm \sqrt{10})$, passing through $(2, 3)$

Solutions:

The foci lie on the y-axis, so the equation is of the form $y^2/a^2 - x^2/b^2 = 1$, with $c = \sqrt{10}$, i.e., $c^2 = 10$, so $b^2 = 10 - a^2$.

Since the hyperbola passes through $(2, 3)$:

$$9/a^2 - 4/(10 - a^2) = 1$$

$$9(10 - a^2) - 4a^2 = a^2(10 - a^2)$$

$$90 - 9a^2 - 4a^2 = 10a^2 - a^4$$

$$a^4 - 23a^2 + 90 = 0$$

Solving as a quadratic in a^2 : $a^2 = [23 \pm \sqrt{(529 - 360)}]/2 = (23 \pm 13)/2$, giving $a^2 = 18$ or $a^2 = 5$.

Since a^2 must be less than $c^2 = 10$, we take $a^2 = 5$, and then $b^2 = 10 - 5 = 5$.

Required equation: $y^2/5 - x^2/5 = 1$, i.e., $y^2 - x^2 = 5$

Solution

Solution coming soon.

Miscellaneous Exercise

Question 1

If a parabolic reflector is

20

cm in diameter and

5

cm deep, find the focus.

Solution

Ans: As we know the origin of the coordinate plane is taken at the vertex of the parabolic reflector, where the axis of the reflector is along the positive x-axis.

The Diagrammatic representation is represented as follows:

As we know the equation of the parabola is of the form of

$$y^2 = 4ax$$

(as it is opening to the right)

Since, the parabola passes through point

$$A(5, 10),$$

$$\Rightarrow y^2 = 4ax$$

$$\Rightarrow 10^2 = 4 \times a \times 5$$

$$\Rightarrow 100 = 20a$$

$$\Rightarrow a = \frac{100}{20}$$

$$\Rightarrow a = 5$$

The focus of the parabola is

$$(a, 0) = (5, 0),$$

which is the midpoint of the diameter.

Hence, the focus of the reflector is at the midpoint of the diameter.

Question 2

An arch is in the form of a parabola with its axis vertical. The arch is 10 m high and 5 m wide at the base. How wide is it 2 m from the vertex of the parabola?

Solution

Ans: As we know the origin of the coordinate plane is taken at the vertex of the arch in such a way that its axis is along the y-axis.

The diagrammatic representation will be as follows:

The equation of the parabola is of the form

$$x^2 = -4ay$$

(as it is opening downwards).

Since the parabola passes through point

$$\left(\frac{5}{2}, -10\right),$$
$$\left(\frac{5}{2}\right)^2 = -4 \times a \times (-10)$$
$$\Rightarrow a = \frac{25}{4 \times 4 \times 10} = \frac{5}{32}$$

Therefore,

The arch is in the form of a parabola whose equation is

$$x^2 = -\frac{5}{8}y$$

When

$$y = -2m, x^2 = -\frac{5}{8} \times (-2)$$

because it's 2m away downward from the vertex, so it will be negative.

$$\Rightarrow x^2 = \frac{5}{4}$$
$$\Rightarrow x = \sqrt{\frac{5}{4}}m$$

The width of the parabola at 2m away from the vertex will be $= 2 \times \sqrt{\frac{5}{4}}m$

$$= 2 \times 1.118m(\text{approx.})$$

$$= 2.23m(\text{approx.})$$

Hence, when the arch is 2 m from the vertex of the parabola, its width is approximately 2.23m.

Question 3

The cable of a uniformly loaded suspension bridge hangs in the form of a parabola. The roadway which is horizontal and 100 m long is supported by vertical wires attached to the cable, the longest wire being 30 m and the shortest being 6 m. Find the length of a supporting wire attached to the roadway 18 m from the middle.

Solution

Ans: The vertex is at the lowest point of the cable. The origin of the coordinate plane is taken as the vertex of the parabola, while its vertical axis is taken along the positive y-axis.

This can be diagrammatically represented as

Here, AB and OC are the longest and shortest wires, respectively, attached to the cable.

DF is the supporting wire attached to the roadway, 18m from the middle.

Here, AB

$$= 30m, OC = 6m, \text{ and } BC = \frac{100}{2} = 50m$$

The equation of the parabola is of the form

$$x^2 = 4ay$$

(as it is opening upwards).

The coordinates of point A are

$$(50, 30 - 6) = (50, 24)$$

.

Since A

$$(50, 24)$$

is a point on the parabola,

$$\begin{aligned} (50)^2 &= 4a(24) \\ \Rightarrow a &= \frac{50 \times 50}{4 \times 24} = \frac{625}{24} \\ &\therefore \end{aligned}$$

Equation of the parabola,

$$x^2 = 4 \times \frac{625}{24} \times y$$

or

$$6x^2 = 625y$$

The x-coordinate of point D is 18.

Hence, at $x = 18$,

$$\begin{aligned} 6(18)^2 &= 625y \\ \Rightarrow y &= \frac{6 \times 18 \times 18}{625} \\ \Rightarrow y &= 3.11(\text{approx.}) \end{aligned}$$

$$\therefore$$

$$DE =$$

$$3.11$$

$$\text{m}$$

$$DF = DE + EF =$$

$$3.11$$

$$\text{m} +$$

$$6$$

$$\text{m} =$$

$$9.11$$

$$\text{m}$$

Thus, the length of the supporting wire attached to the roadway 18 m from the middle is approximately

$$9.11$$

$$\text{m}.$$

Question 4

An arch is in the form of a semi-ellipse. It is 8 m wide and 2 m high at the centre. Find the height of the arch at a point 1.5 m from one end.

Solution

Ans: Since the height and width of the arc from the centre is 2 m and 8 m respectively, it is clear that the length of the major axis is 8 m, while the length of the semi-minor axis is 2 m.

The origin of the coordinate plane is taken as the centre of the ellipse, while the major axis is taken along the x-axis.

Hence, the semi-ellipse can be diagrammatically represented as

The equation of the semi-ellipse will be of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, y \geq 0,$$

where a is the semi-major axis

Accordingly,

$$\begin{aligned} 2a &= 8 \Rightarrow a = 4 \\ b &= 2 \end{aligned}$$

Therefore, the equation of the semi-ellipse is

$$\frac{x^2}{16} + \frac{y^2}{4} = 1, y \geq 0,$$

....(1)

Let A be a point on the major axis such that $AB = 1.5$ m.

Draw $AC \perp OB$.

$$OA = (4 - 1.5) \text{ m} = 2.5 \text{ m}$$

The x -coordinate of point C is 2.5.

On substituting the value of x with 2.5 in equation (1), we'll get

$$\begin{aligned} \frac{(2.5)^2}{16} + \frac{y^2}{4} &= 1 \\ \Rightarrow \frac{6.25}{16} + \frac{y^2}{4} &= 1 \\ \Rightarrow y^2 &= 4 \left(1 - \frac{6.25}{16} \right) \\ \Rightarrow y^2 &= 4 \left(\frac{9.75}{16} \right) \\ \Rightarrow y^2 &= 2.4375 \\ \Rightarrow y &= 1.56(\text{approx.}) \\ \therefore AC &= 1.56 \text{ m} \end{aligned}$$

Thus, the height of the arch at a point 1.5 m from one end is approximately 1.56 m.

Question 5

A rod of length 12 cm moves with its ends always touching the coordinate axes. Determine the equation of the locus of a point P on the rod, which is 3 cm from the end in contact with the x-axis.

Solution

Ans: Let AB be the rod making an angle θ with OX and P (x, y) be the point on it such that AP = 3 cm.

Then, PB = AB – AP = (12 – 3) cm = 9 cm [AB = 12 cm]

From P, draw $PQ \perp OY$ and $PR \perp OX$.

In $\triangle PBQ$,

$$\cos \theta = \frac{PQ}{PB} = \frac{x}{9}$$

In $\triangle PRA$,

$$\sin \theta = \frac{PR}{PA} = \frac{y}{3}$$

Since, $\sin^2 \theta + \cos^2 \theta = 1$,

$$\left(\frac{y}{3}\right)^2 + \left(\frac{x}{9}\right)^2 = 1$$

Or,

$$\frac{x^2}{81} + \frac{y^2}{9} = 1$$

Thus, the equation of the locus of point P on the rod is

$$\frac{x^2}{81} + \frac{y^2}{9} = 1$$

Question 6

Find the area of the triangle formed by the lines joining the vertex of the parabola

$$x^2 = 12y$$

to the ends of its latus rectum.

Solution

Ans: The given parabola is

$$x^2 = 12y$$

.

On comparing this equation with

$$x^2 = 4ay$$

, we'll get

$$4a = 12 \Rightarrow a = 3$$

$\therefore$

The coordinates of foci are S (0, a) = S (0, 3)

Let AB be the latus rectum of the given parabola.

The given parabola can be roughly drawn as

At

$$y = 3,$$

$$x^2 = 12(3) \Rightarrow x^2 = 36 \Rightarrow x = \pm 6$$

$\therefore$

The coordinates of A are (−6, 3), while the coordinates of B are (6, 3).

Therefore, the vertices of ΔOAB are O (0, 0), A (−6, 3), and B (6, 3).

Area of $\triangle OAB$

$$\begin{aligned}
 &= \frac{1}{2} |0(3-3) + (-6)(3-0) + 6(0-3)| \text{ unit}^2 \\
 &= \frac{1}{2} |(-6)(3) + 6(-3)| \text{ unit}^2 \\
 &= \frac{1}{2} |-18 - 18| \text{ unit}^2 \\
 &= \frac{1}{2} |-36| \text{ unit}^2 \\
 &= \frac{1}{2} \times 36 \text{ unit}^2 \\
 &= 18 \\
 &\quad \text{unit}^2
 \end{aligned}$$

Thus, the required area of the triangle is

$$\begin{aligned}
 &18 \\
 &\quad \text{unit}^2
 \end{aligned}$$

Question 7

A man running a racecourse notes that the sum of the distances from the two flag posts from him is always 10 m and the distance between the flag posts is 8 m. Find the equation of the posts traced by the man.

Solution

Ans: Let A and B be the positions of the two flag posts and P(x, y) be the position of the man. Accordingly, $PA + PB = 10$.

We know that if a point moves in a plane in such a way that the sum of its distances from two fixed points is constant, then the path is an ellipse and this constant value is equal to the length of the major axis of the ellipse.

Therefore, the path described by the man is an ellipse where the length of the major axis is 10 m, while points A and B are the foci.

Taking the origin of the coordinate plane as the centre of the ellipse, while taking the major axis along the x-axis, the ellipse can be diagrammatically represented as

The equation of the ellipse will be of the form of

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where a is the semi-major axis.

Accordingly,

$$2a = 10 \Rightarrow a = 5$$

Distance between the foci

$$(2c) = 8 \Rightarrow c = 4$$

On using the relation

$$c = \sqrt{a^2 - b^2},$$

we'll get

$$4 = \sqrt{25 - b^2}$$

$$\Rightarrow 16 = 25 - b^2$$

$$\Rightarrow b^2 = 25 - 16$$

$$\Rightarrow b^2 = 9$$

$$\Rightarrow b = 3$$

Thus, the equation of the path traced by the man is

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

Question 8

An equilateral triangle is inscribed in the parabola

$$y^2 = 4ax$$

, where one vertex is at the vertex of the parabola. Find the length of the side of the triangle.

Solution

Ans: Let OAB be the equilateral triangle inscribed in parabola

$$y^2 = 4ax$$

Let AB intersect the x-axis at point C.

Let OC = k

From the equation of the given parabola, we have

$$y^2 = 4ak \Rightarrow y = \pm 2\sqrt{ak}$$
$$\therefore$$

The respective coordinates of points A and B are

$$(k, -2\sqrt{ak})$$

and

$$(k, 2\sqrt{ak})$$

AB = CA + CB =

$$2\sqrt{ak} + 2\sqrt{ak} = 4\sqrt{ak}$$

Since OAB is an equilateral triangle,

$$OA^2 = AB^2.$$

$$\therefore k^2 + (2\sqrt{ak})^2 = (4\sqrt{ak})^2$$

$$\Rightarrow k^2 + 4ak = 16ak$$

$$\Rightarrow k^2 = 12ak$$

$$\Rightarrow k = 12a$$

$$\therefore AB = 4\sqrt{ak} = 4\sqrt{a \times 12a}$$

$$= 4\sqrt{12a^2}$$

$$= 8\sqrt{3}a$$

Thus, the side of the equilateral triangle inscribed in parabola

$$y^2 = 4ax$$

is

$$8\sqrt{3}a$$